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THE LARGEST AND ALL SUBSEQUENT NUMBERS OF CONGRUENCES OF n-ELEMENT LATTICES[1][2]

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Abstract: For a positive integer n, let $\mathrm{SCL}(n) = \{|\mathrm{Con}(L)| : L \text{ is an } n\text{-element lattice}\}$ stand for the set of Sizes of the Congruence Lattices of n-element lattices. The k-th Largest Number of Congruences of n-element lattices, denoted by $\mathrm{lnc}(n, k)$, is the k-th largest member of $\mathrm{SCL}(n)$. Let $(n_1, \ldots, n_6) := (1, 4, 5, 6, 6, 7)$, and let $n_k := k$ for $k \geq 7$. In 1997, R. Freese proved that for $n \geq n_1 = 1$, $\mathrm{lnc}(n, 1) = 2^{n-1}$. For $n \geq n_2$, the present author gave $\mathrm{lnc}(n, 2)$. For $k = 3, 4, 5$ and $n \geq n_k$, C. Mureşan and J. Kulin determined $\mathrm{lnc}(n, k)$ in their 2020 paper. For $k \leq 5$ and $n \geq n_k$, the above-mentioned authors described the n-element lattices witnessing $\mathrm{lnc}(n, k)$, too. For all positive integers k and $n \geq n_k$, this paper determines $\mathrm{lnc}(n, k)$ and presents the lattices that witness it. It turns out that, for each fixed k, the quotient $\mathrm{lcd}(k) := \mathrm{lnc}(n, k)/\mathrm{lnc}(n, 1)$ does not depend on $n \geq n_k$. Furthermore, $\mathrm{lcd}(k)$ converges to $1/8$ as k tends to infinity.

Keywords: Number of lattice congruences, Size of the congruence lattice of a finite lattice, Lattice with many congruences, Congruence density.

1. Introduction and stating the results

If a finite lattice or a semilattice L has many congruences, then the number $|\mathrm{Con}(L)|$ of the congruences of L together with the number $|L|$ of the elements of L gives some insight into the structure of L; this is exemplified by Czédli [3, 5, 6], and by Mureşan and Kulin [13]. There are analogous results for lattices with many sublattices, too; we mention only Ahmed and Horváth [1] and Czédli [4].

Our goal is to prove a new result, Theorem 2, on finite lattices with many congruences. We fix the following notation. For $n \in \mathbb{N}^+ := \{1, 2, 3, \ldots\}$, let $\mathrm{Lat}(n)$ denote the set of n-element lattices. (We say "set" rather than "class", since we do not differentiate between isomorphic lattices.) For $L \in \mathrm{Lat}(n)$, $\mathrm{Con}(L)$ and $|\mathrm{Con}(L)|$ stand for the *congruence lattice* of L and the number of congruences of L, respectively. We use the notation

$$\mathrm{SCL}(n) := \{|\mathrm{Con}(L)| : L \in \mathrm{Lat}(n)\}$$

for the set of *Sizes of Congruence Lattices* of n-element lattices, where the capitalization explains the acronym SCL. For $k, n \in \mathbb{N}^+$, the kth largest member of $\mathrm{SCL}(n)$ is the kth *Largest Number of Congruences* of n-element lattices; we denote this number by $\mathrm{lnc}(n, k)$, which is defined only if $k \leq |\mathrm{SCL}(n)|$. For $n \in \mathbb{N}^+$ and $L \in \mathrm{Lat}(n)$, we call the quotient $|\mathrm{Con}(L)|/2^{n-1}$ the *Congruence Density* of L, and we denote it by

$$\mathrm{cd}(L);\ \text{ so } \mathrm{cd}(L) := |\mathrm{Con}(L)|/2^{|L|-1}. \tag{1.1}$$

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[2]*This paper is dedicated to Professor Sándor Radeleczki, an esteemed coauthor of mine, on his sixty-fifth birthday*

This terminology is motivated by the fact that the denominator is $2^{n-1} = \mathrm{lnc}(n,1)$ by Freese [8]. The kth largest member of $\{\mathrm{cd}(L) : L \in \mathrm{Lat}(n)\}$ is the kth *Largest Congruence Density* of n-element lattices; it will be denoted by $\mathrm{lcd}_n(k)$. Clearly, for a finite lattice L,

$$\text{if } |L| = n, \quad \text{then} \quad |\mathrm{Con}(L)| = 2^{n-1} \cdot \mathrm{cd}(L). \tag{1.2}$$

Hence, the study of $\mathrm{lnc}(n,k)$ and that of $\mathrm{lcd}_n(k)$ are equivalent. However, we will mostly work only with the latter, since it has nice properties established by many forthcoming lemmas. Define

$$n_1 := 1, \quad n_2 := 4, \quad n_3 := 5, \quad n_4 := 6, \quad n_5 := 6, \quad n_6 := 7, \quad \text{and} \quad n_k := k, \quad \text{for} \quad k \geq 7. \tag{1.3}$$

For $k = 1$, $k = 2$, and $k \in \{3, 4, 5\}$, Freese [8], Czédli [3], and Mureşan and Kulin [13], respectively, determined $\mathrm{lnc}(n,k)$ and the lattices witnessing it, provided that $n \geq n_k$. Equivalently by (1.2), they determined $\mathrm{lcd}_n(k)$ and their witnesses for $k \leq 5$ and $n \geq n_k$; see Theorem 1 later for details. Before presenting a new result, Theorem 2, we continue introducing some notations and terminology.

Every lattice in this paper *is assumed to be finite.* (However, sometimes we repeat this convention.) The n-element chain will be denoted by C_n. For a lattice L and $u \in L$, u is *join-irreducible* if it covers exactly one element, and it is *join-reducible* if it has at least two lower covers. *Meet-irreducible* and *meet-reducible* elements are defined dually. Let $\mathrm{Ji}(L)$, $\mathrm{Mi}(L)$, $\mathrm{Jr}(L)$, and $\mathrm{Mr}(L)$ stand for the set of join-irreducible elements, that of meet-irreducible elements, that of join-reducible elements, and that of meet-reducible elements of L, respectively. Note that

$$|\mathrm{Ji}(L)| + |\mathrm{Jr}(L)| = |L| - 1 = |\mathrm{Mi}(L)| + |\mathrm{Mr}(L)|.$$

An element $u \in L$ is said to be a *narrows* if it is comparable with every other element of L; the set of all narrows of L will be denoted by $\mathrm{Nar}\,(L)$. A subset U of L is *convex* if for every $u_1, u_2 \in U$ and $x \in L$, $u_1 \leq x \leq u_2$ implies that $x \in U$. The maximal subsets of $\mathrm{Nar}\,(L)$ that are convex subsets of L are called the *narrow chain components* of L.

To form the *glued sum* $L_1 \dotplus L_2$ of finite lattices L_1 and L_2, we first put the (diagram) of L_2 atop L_1, and then identify 1_{L_1} (the top element of L_1) with 0_{L_2} (the bottom element of L_2). For example, $H_{5,3}$ in Fig. 5 is $\mathsf{B}_4 \dotplus \mathsf{B}_4$; here and later, B_4 denotes the 4-element Boolean lattice (see Fig. 1). We can write

$$L_1 \dotplus L_2 \dotplus \cdots \dotplus L_t \quad \text{or} \quad \dot{\sum}_{1 \leq i \leq t} L_i$$

without parentheses since forming glued sums is an associative operation. Clearly, 0_{L_1}, 1_{L_t}, and $1_{L_{i-1}} = 0_{L_i}$ for $i \in \{2, \dots, t\}$ are in $\mathrm{Nar}\,(L_1 \dotplus \cdots \dotplus L_t)$.

Conversely, assume that L is a finite lattice but not a chain. Then there are uniquely determined narrows $0_L = u_0 < u_1 < \cdots < u_t = 1_L$ such that for each $i \in \{1, \dots, t\}$, either the interval $[u_{i-1}, u_i]$ is a narrow chain component of L or this interval has at least four elements and

$$[u_{i-1}, u_i] \cap \mathrm{Nar}\,(L) = \{u_{i-1}, u_i\}.$$

Then L decomposes to the glued sum of the just-mentioned intervals as follows:

$$L = [u_0, u_1] \dotplus [u_1, u_2] \dotplus \cdots \dotplus [u_{t-1}, u_t], \tag{1.4}$$

which we call the *canonical glued sum decomposition* of L. We call the glued sum of the non-chain summands in (1.4) the *core* of L; we denote it by $\mathrm{Cor}(L)$. In other words, denoting the set of at least four-element intervals of L by $\mathrm{Intv}_4(L)$,

$$\mathrm{Cor}(L) = \dot{\sum}_{I \text{ is maximal in } \mathrm{Intv}_4(L) \text{ such that } I \cap \mathrm{Nar}\,(L) = \{0_I, 1_I\}} I. \tag{1.5}$$

For a finite chain $C = \mathsf{C}_k$, let $\mathrm{Cor}(C)$ be the singleton lattice C_1. If L is not a chain, then its core is an at least four-element lattice. As $L = \mathsf{B}_4 \dotplus \mathsf{C}_2 \dotplus \mathsf{B}_4$ exemplifies, $\mathrm{Cor}(L)$ need not be a sublattice of L. To justify the notation used in (1.5), let us agree that $\mathrm{Cor}(L)$ is defined only up to the order of its summands. For example, $\mathsf{B}_4 \dotplus \mathsf{N}_5$ and $\mathsf{N}_5 \dotplus \mathsf{C}_7 \dotplus \mathsf{B}_4$ have the same cores; here and later, N_5 is the five-element non-modular lattice (see Fig. 1). For our purposes,

$$\text{cores describe some sets of finite lattices very conveniently.} \tag{1.6}$$

We can illustrate this by rewriting

$$\{\mathsf{C}_i \dotplus \mathsf{B}_4 \dotplus \mathsf{C}_m \dotplus \mathsf{B}_4 \dotplus \mathsf{C}_{n-k-i-4} : i, m \in \mathbb{N}^+ \text{ and } k + i \le n - 5\},$$

taken from Mureşan and Kulin [13], into $\{L : \mathrm{Cor}(L) = \mathsf{B}_4 \dotplus \mathsf{B}_4\}$. The forthcoming Lemma 3 offers another motivation to use $\mathrm{Cor}(L)$.

For $n \ge 4$, let $\mathrm{Circ}(n)$ be the set of lattices whose covering graphs are n-element circles. That is,

$$\mathrm{Circ}(n) := \{L \in \mathrm{Lat}(n) : \text{ the covering graph of } L \text{ is a circle}\}. \tag{1.7}$$

For example, $\mathrm{Circ}(4) = \{\mathsf{B}_4\}$, $\mathrm{Circ}(5) = \{\mathsf{N}_5\}$, and $|\mathrm{Circ}(6)| = 2$.

Next, using (1.2), (1.6), and the notations, conventions, and concepts introduced so far, we recall the previously known related results. To improve readability, we present these results both in their original form and in our new terminology based on congruence density. Furthermore, to facilitate faster comparisons among the fractions $\mathrm{lcd}_n(k)$, we scale up most of them to have denominators 64.

Theorem 1 (Proved in [3, 8, 13]). (A) (Freese [8]) *For $n \in \mathbb{N}^+$, the largest number of congruences of n-element lattices is 2^{n-1}; equivalently, $\mathrm{lcd}_n(1) = 64/64 = 1$. Furthermore, an n-element finite lattice L has exactly 2^{n-1} congruences (that is, $\mathrm{cd}(L) = 1$) if and only if L is a chain.*

(B) (Czédli [3]) *Let $4 \le n \in \mathbb{N}^+$. Then the second largest number of congruences of n-element lattices is 2^{n-2}; equivalently, $\mathrm{lcd}_n(2) = 32/64 = 1/2$. Furthermore, an n-element lattice L has exactly 2^{n-2} congruences (that is, $\mathrm{cd}(L) = 32/64$) if and only if $\mathrm{Cor}(L) = \mathsf{B}_4$.*

(C) (Mureşan and Kulin [13]) *Let $5 \le n \in \mathbb{N}^+$. Then the third largest number of congruences of n-element lattices is $5 \cdot 2^{n-5}$; equivalently, $\mathrm{lcd}_n(3) = 20/64 = 5/16$. Furthermore, an n-element lattice L has exactly $5 \cdot 2^{n-5}$ congruences (that is, $\mathrm{cd}(L) = 20/64$) if and only if $\mathrm{Cor}(L) = \mathsf{N}_5$.*

(D) (Mureşan and Kulin [13]) *Let $6 \le n \in \mathbb{N}^+$. Then the fourth largest number of congruences of n-element lattices is 2^{n-3}; equivalently, $\mathrm{lcd}_n(4) = 16/64 = 1/4$. Furthermore, an n-element lattice L has exactly 2^{n-3} congruences (that is, $\mathrm{cd}(L) = 16/64$) if and only if $\mathrm{Cor}(L) = \mathsf{C}_2 \times \mathsf{C}_3$ or $\mathrm{Cor}(L) = \mathsf{B}_4 \dotplus \mathsf{B}_4$.*

(E) (Mureşan and Kulin [13]) *Let $6 \le n \in \mathbb{N}^+$. Then the fifth largest number of congruences of n-element lattices is $7 \cdot 2^{n-6}$; equivalently, $\mathrm{lcd}_n(5) = 14/64 = 7/32$. Furthermore, an n-element lattice L has exactly $7 \cdot 2^{n-6}$ congruences (that is, $\mathrm{cd}(L) = 14/64$) if and only if $\mathrm{Cor}(L) \in \mathrm{Circ}(6)$.*

There are two ways to obtain a Hall–Dilworth gluing[3] L of a finite lattice K and B_4. First, if $|K \cap \mathsf{B}_4| = 1$, then L is the glued sum $K \dotplus \mathsf{B}_4$ or $\mathsf{B}_4 \dotplus K$. Second, if $|K \cap \mathsf{B}_4| = 2$, then L is called an *edge gluing* of K and B_4 (or B_4 and K). For example, $\mathsf{C}_3 \times \mathsf{C}_2$ is an edge-gluing of two copies of B_4, and $H_{5,7}$ in Fig. 6 is an edge-gluing of N_5 and B_4. We emphasize: A Hall–Dilworth gluing of B_4 and another lattice K is either a glued sum or an edge-gluing, since *we do not allow trivial gluings*

[3] See, e.g., Grätzer [9, Lemma 298], where "Hall–Dilworth" is dropped.

where one of B_4 and K is a sublattice of the other one. Moreover, the term "a Hall–Dilworth gluing of K and B_4" (with an indefinite article) will express that K an B_4 can be taken in either arrangement: "K and B_4" or "B_4 and K".

At this stage, based on the notations and concepts introduced in (1.1), (1.3), (1.5), (1.6), and (1.7), we are in a position to state the sole result of the paper. While the theorem below is formulated using the congruence density approach, the subsequent corollary presents the same result in terms of the numbers of congruences.

Theorem 2 (Main Theorem). *As in (1.3), let $n_6 := 7$ and, for $7 \le k \in \mathbb{N}^+$, let $n_k := k$.*

(i) *Assume that $n_6 \le n \in \mathbb{N}^+$. Then $\mathrm{lcd}_n(6) = 11/64$. Furthermore, for every n-element lattice L, $\mathrm{cd}(L) = 11/64$ if and only if $\mathrm{Cor}(L) \in \mathrm{Circ}(7)$.*

(ii) *Assume that $n_7 \le n \in \mathbb{N}^+$. Then $\mathrm{lcd}_n(7) = 10/64 = 5/32$. Furthermore, for every n-element lattice L, $\mathrm{cd}(L) = 10/64$ if and only if $\mathrm{Cor}(L)$ is a Hall–Dilworth gluing of N_5 and B_4 (that is, if $\mathrm{Cor}(L)$ is $\mathsf{N}_5 \dotplus \mathsf{B}_4$, $\mathsf{B}_4 \dotplus \mathsf{N}_5$, or an edge-gluing of N_5 and B_4 in either arrangement).*

(iii) *Assume that $8 \le k \in \mathbb{N}^+$ and $n_k \le n \in \mathbb{N}^+$. Then*

$$\mathrm{lcd}_n(k) = \frac{8 + 3/2^{k-7}}{64} = \frac{1}{8} + \frac{3}{2^{k-1}}.$$

Furthermore, for every n-element lattice L, $\mathrm{cd}(L) = (8 + 3/2^{k-7})/64$ if and only if $\mathrm{Cor}(L) \in \mathrm{Circ}(k)$.

In part (ii) above, $\mathrm{Cor}(L) = \mathsf{N}_5 \dotplus \mathsf{B}_4$ can occur only when $n \ge 8 = n_7 + 1$. Below, we present Corollary 1, which aligns with the title of the paper and follows directly from Theorem 2 and (1.2).

Corollary 1. (i) *Assume that $7 \le n \in \mathbb{N}^+$. Then the sixth largest number of congruences of n-element lattices is $\mathrm{lnc}(n, 6) = 11 \cdot 2^{n-7}$. Furthermore, an n-element lattice L has exactly $11 \cdot 2^{n-7}$ congruences if and only if $\mathrm{Cor}(L) \in \mathrm{Circ}(7)$.*

(ii) *Assume that $7 \le n \in \mathbb{N}^+$. Then the seventh largest number of congruences of n-element lattices is $\mathrm{lnc}(n, 7) = 10 \cdot 2^{n-7} = 5 \cdot 2^{n-6}$. Furthermore, an n-element lattice L has exactly $10 \cdot 2^{n-7}$ congruences if and only if $\mathrm{Cor}(L)$ is a Hall–Dilworth gluing of N_5 and B_4 (in either arrangement).*

(iii) *Assume that $8 \le k \le n \in \mathbb{N}^+$. Then the kth largest number of congruences of n-element lattices is $\mathrm{lnc}(n, k) = (8 + 3/2^{k-7}) \cdot 2^{n-7}$. Furthermore, an n-element lattice L has exactly $(8 + 3/2^{k-7}) \cdot 2^{n-7}$ congruences if and only if $\mathrm{Cor}(L) \in \mathrm{Circ}(k)$.*

The remainder of the paper proves Theorem 2 and, as a byproduct, presents a new proof of Theorem 1.

2. Facts about lattice congruences

For a poset (= partially ordered set) P and $u \in P$, the *principal ideal* $\{x \in P : x \le u\}$ and the *principal filter* $\{x \in P : x \ge u\}$ will be denoted by $\mathrm{idl}(u)$ and $\mathrm{fil}(u)$, respectively. A subset X of P is an *order ideal* of P if for every $u \in X$, $\mathrm{idl}(u) \subseteq X$. The set of order ideals of P will be denoted by $\mathrm{Idl}(P)$. Note that $\mathrm{Idl}(P) = (\mathrm{Idl}(P); \cap, \cup)$ is a distributive lattice. For a finite lattice L, $\mathrm{Ji}(L)$ is a subposet of L with respect to the order inherited from L. It is well known that $D \cong \mathrm{Idl}(\mathrm{Ji}(D))$ for each finite distributive lattice D; see, e.g., Grätzer [9, Theorem 107]. In particular, since the congruence lattice of any lattice is well known to be distributive, see [9, Theorem 149],

$$\text{for each finite lattice } L, \quad \mathrm{Con}(L) \cong \mathrm{Idl}(\mathrm{Ji}(\mathrm{Con}(L))). \tag{2.1}$$

Let L be a finite lattice. As the blocks of each $\Theta \in \mathrm{Con}(L)$ are convex sublattices, Θ is determined by

$$\{(a,b) : (a,b) \in \Theta \text{ and } a \le b\}.$$

Thus, when dealing with lattice congruences, we consider only the *comparable* pairs they collapse. For such a pair (a,b), so $a \le b$, $\mathrm{con}(a,b)$ will stand for the *smallest congruence containing* (a,b). It is well known (see the first sentence in Grätzer [11]) and it is easy to prove that

$$\mathrm{Ji}(\mathrm{Con}(L)) = \{\mathrm{con}(a,b) : a \prec b\}; \tag{2.2}$$

that is, the join-irreducible congruences and the congruences generated by edges (of the diagram of L) are the same. (Note the terminological nuance: for $a \prec b$, $(a,b) \in L^2$ is an *edge*, but $[a,b] = \{a,b\} \subseteq L$ is a *prime interval*.) The "pentagon lattice" N_5 has five edges. With the notation of the first diagram in Fig. 1, (d,i) and (o,d) are the (upper and lower) *long edges*, (v_2,i) and (o,u_2) are the (upper and lower) *short edges*, and (u_2,v_2) is the *monolith edge* of N_5. In B_4, there are two *upper edges*, (u_1,i) and (v_2,i) on the right of Fig. 1, and two *lower edges*.

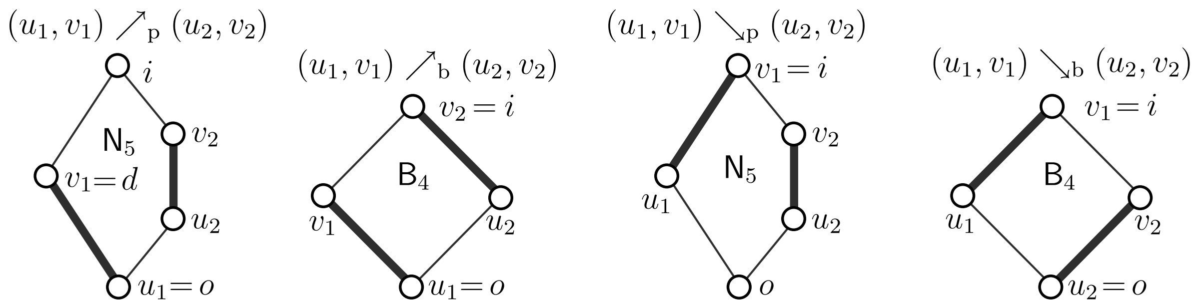

Figure 1. Visualizing $\nearrow_{\mathrm{p}}$, $\nearrow_{\mathrm{b}}$, $\searrow_{\mathrm{p}}$, and $\searrow_{\mathrm{b}}$.

Definition 1. *On the* set $\mathrm{Edge}(L)$ of edges *of our finite lattice L, we define the following eight relations; some of them are visualized in Fig. 1, where the* **bold edges** *denote coverings in L. For* distinct *edges $(u_1,v_1),(u_2,v_2) \in \mathrm{Edge}(L)$, the following definitions apply*:

- $(u_1,v_1) \nearrow_{\mathrm{p}} (u_2,v_2) \overset{\text{def}}{\iff} (u_1,v_1)$ *is the long lower edge and* (u_2,v_2) *is the monolith edge of an* N_5 *sublattice*;
- $(u_1,v_1) \searrow_{\mathrm{p}} (u_2,v_2) \overset{\text{def}}{\iff} (u_1,v_1)$ *is the long upper edge and* (u_2,v_2) *is the monolith edge of an* N_5 *sublattice*;
- $(u_1,v_1) \to_{\mathrm{p}} (u_2,v_2) \overset{\text{def}}{\iff} (u_1,v_1) \nearrow_{\mathrm{p}} (u_2,v_2)$ *or* $(u_1,v_1) \searrow_{\mathrm{p}} (u_2,v_2)$;
- $(u_1,v_1) \nearrow_{\mathrm{b}} (u_2,v_2) \overset{\text{def}}{\iff} (u_1,v_1)$ *is a lower edge and* (u_2,v_2) *is the opposite upper edge of a* B_4 *sublattice*;
- $(u_1,v_1) \searrow_{\mathrm{b}} (u_2,v_2) \overset{\text{def}}{\iff} (u_1,v_1)$ *is an upper edge and* (u_2,v_2) *is the opposite lower edge of a* B_4 *sublattice*;
- $(u_1,v_1) \to_{\mathrm{b}} (u_2,v_2) \overset{\text{def}}{\iff} (u_1,v_1) \nearrow_{\mathrm{b}} (u_2,v_2)$ *or* $(u_1,v_1) \searrow_{\mathrm{b}} (u_2,v_2)$;
- $(u_1,v_1) \to_{\mathrm{bp}} (u_2,v_2) \overset{\text{def}}{\iff} (u_1,v_1) \to_{\mathrm{b}} (u_2,v_2)$ *or* $(u_1,v_1) \to_{\mathrm{p}} (u_2,v_2)$;
- $(u_1,v_1) \to^{*} (u_2,v_2) \overset{\text{def}}{\iff} \big((u_1,v_1),(u_2,v_2)\big)$ *is in the reflexive-transitive closure of the relation* $\to_{\mathrm{bp}}$.

For (a,b), $(c,d) \in \mathrm{Edge}(L)$, Grätzer [11, Lemma 1.4 and Note 1.2] proved that

$$\mathrm{con}(a,b) \ge \mathrm{con}(c,d) \iff (a,b) \to^{*} (c,d). \tag{2.3}$$

Let the Strict Order

$$\{(x,y) : x,y \in L \text{ and } x < y\} \quad \text{be denoted by } \mathrm{SO}(L).$$

The relations given in Definition 1 are meaningful even for $(u_1,v_1),(u_2,v_2) \in \mathrm{SO}(L)$. Clearly, see also Grätzer [11], for any $(a,b),(c,d) \in \mathrm{SO}(L)$,

$$(a,b) \to^* (c,d) \quad \text{implies that} \quad \mathrm{con}(a,b) \geq \mathrm{con}(c,d). \tag{2.4}$$

Since $\to_{\mathrm{b}}$, which is called the *perspectivity relation*, is a symmetric, (2.4) implies that for (x_1,y_1), $(x_2,y_2) \in \mathrm{SO}(L)$,

$$\text{if } (x_1,y_1) \to_{\mathrm{b}} (x_2,y_2), \quad \text{then} \quad \mathrm{con}(x_1,y_1) = \mathrm{con}(x_2,y_2). \tag{2.5}$$

Let us emphasize: While $\to^*$ is a reflexive relation on $\mathrm{Edge}(L)$, each of $\searrow_{\mathrm{b}}$, $\nearrow_{\mathrm{b}}$, $\searrow_{\mathrm{p}}$, $\nearrow_{\mathrm{p}}$, $\to_{\mathrm{p}}$, $\to_{\mathrm{b}}$, and $\to_{\mathrm{bp}}$ is irreflexive. For $a \in \mathrm{Ji}(L)$, the *unique lower cover* of a will be denoted by a_*. Dually, for $a \in \mathrm{Mi}(L)$, we denote the *unique* (upper) *cover* of a by a^*. Day [7, Page 71] strengthened (2.2) to the following equality:

$$\mathrm{Ji}(\mathrm{Con}(L)) = \{\mathrm{con}(a_*,a) : a \in \mathrm{Ji}(L)\}. \tag{2.6}$$

Although the "C-relation" in Day [7] (which was called the "D-relation" later) combines (2.3) and (2.6) to describe $\mathrm{Con}(L)$, we stick to (2.3), which is visual and easier to apply (especially when it is used together with its dual).

In (2.8), which will be formulated soon, we make only minor additions to Theorem 3.10 from Grätzer [10]. The reader could be interested in how to derive (2.8) from the widely known (2.1) in a straightforward way, so we present some details. (In spirit, we follow both Grätzer [10] and Czédli [2, Page 317], though the reader need not consult these sources.) A *quasiorder* is a reflexive and transitive relation. For a quasiorder τ, $x \leq_\tau y$ will stand for $(x,y) \in \tau$, and we define

$$x \equiv_\tau y \overset{\text{def}}{\iff} (x \leq_\tau y \quad \text{and} \quad y \leq_\tau x).$$

For a *quasiordered set* $(A;\nu)$, a (possibly empty) subset X of A is an *ideal* if for all $x \in X$ and $y \in A$, $y \leq_\nu x$ implies that $y \in X$. With respect to the subset inclusion "$\subseteq$", the ideals of $(A;\nu)$ form a lattice, which we denote by $\mathrm{Idl}(A;\nu)$. The *canonical equivalence* $\theta := \nu \cap \nu^{-1}$ is the same as $\equiv_\tau$. For $u \in A$, u/θ will stand for the θ-block $\{x \in A : (x,u) \in \theta\}$ of u. For $a,b \in A$, let $a/\theta \leq_{\nu/\theta} b/\theta$ mean that $a \leq_\nu b$; the choice of a and b in their θ-blocks is irrelevant. Clearly, $(A/\theta;\nu/\theta)$ is a poset, the so called *canonical poset associated with* $(A;\nu)$, and it is easy to see that the function

$$\mathrm{Idl}(A;\nu) \to \mathrm{Idl}(A/\theta;\nu/\theta) \quad \text{defined by} \quad X \to \{u/\theta : u \in X\} \tag{2.7}$$

is a lattice isomorphism. For a finite lattice L and $H \subseteq \mathrm{Edge}(L)$, we say that H is a *congruence-determining subset* of $\mathrm{Edge}(L)$ if

$$\{\mathrm{con}_L(x,y) : (x,y) \in H\} = \mathrm{Ji}(\mathrm{Con}(L)).$$

For example, $\{(a_*,a) : a \in \mathrm{Ji}(L)\}$, $\{(a,a^*) : a \in \mathrm{Mi}(L)\}$ (by (2.2) and its dual), and $\mathrm{Edge}(L)$ itself are such subsets. We define a quasiorder $\nu(L)$ of $\mathrm{Edge}(L)$ by letting

$$(x_1,y_1) \leq_{\nu(L)} (x_2,y_2) \overset{\text{def}}{\iff} \mathrm{con}_L(x_1,y_1) \leq \mathrm{con}_L(x_2,y_2).$$

For a congruence-determining subset H of $\mathrm{Edge}(L)$, we denote the quasiordered set $(H; H^2 \cap \nu(L))$ simply by $(H;\nu(L))$. (So, we make no notational distinction between $\nu(L)$ and its restriction to H.)

Clearly, the canonical poset associated with $(H; \nu(L))$ is isomorphic to the poset $\mathrm{Ji}(\mathrm{Con}(L))$. Thus, combining (2.1) with (2.7), we obtain that for every congruence-determining subset H of $\mathrm{Edge}(L)$,

$$\mathrm{Con}(L) \cong \mathrm{Idl}(H; \nu(L)). \tag{2.8}$$

Next, let Γ: $a^{(1)}, \dots, a^{(p)}$ be a repetition-free list of the elements of $\mathrm{Ji}(L)$. An $x \in L$ is a *join-deficit* (modulo Γ) if either $x \in \mathrm{Jr}(L)$, or x is of the form $a^{(i)} \in \mathrm{Ji}(L)$ such that there exists a $j \in \{1, 2, \dots, i-1\}$ with $\mathrm{con}({a^{(j)}}_*, a^{(j)}) = \mathrm{con}({a^{(i)}}_*, a^{(i)})$. *Meet-deficits* (modulo a repetition-free list Δ of $\mathrm{Mi}(L)$) are defined dually. Since our only purpose is to *count* the join-deficits and the meet-deficits of L, the choice of Γ and Δ is irrelevant. Hence, "(modulo Γ)" and "(modulo Δ)" will be dropped.

Lemma 1 (Three-Deficits Lemma). *If a finite lattice L has at least three join-deficits or at least three meet-deficits, then $\mathrm{cd}(L) \le 8/64$. In particular, if $|\mathrm{Jr}(L)| \ge 3$ or $|\mathrm{Mr}(L)| \ge 3$, then $\mathrm{cd}(L) \le 8/64$.*

P r o o f. Since L and its dual have the same congruence lattice, we can assume L has at least three join-deficits. Let $n := |L|$. As $0 \notin \mathrm{Ji}(L)$, (2.6) yields that

$$|\mathrm{Ji}(\mathrm{Con}(L))| \le |L| - 1 - 3 = n - 4.$$

Combining this with (2.1), we have that

$$|\mathrm{Con}(L)| \le 2^{n-4}.$$

Hence,

$$\mathrm{cd}(L) = |\mathrm{Con}(L)|/2^{n-1} \le 2^{-3} = 8/64,$$

as required. □

Lemma 2 (Glued Sum Lemma). *For $2 \le t \in \mathbb{N}^+$ and finite lattices $L_1, \dots, L_t$, $\mathrm{cd}(L_1 \dotplus \dots \dotplus L_t)$ is equal to the product $\mathrm{cd}(L_1) \dots \mathrm{cd}(L_t)$.*

P r o o f. Let $L := L_1 \dotplus L_2$. As Mureşan and Kulin [13] observed (and as it is easy to see), $\mathrm{Con}(L) \cong \mathrm{Con}(L_1) \times \mathrm{Con}(L_2)$ and $|L| = |L_1| + |L_2| - 1$. Thus

$$\mathrm{cd}(L) = \frac{|\mathrm{Con}(L)|}{2^{|L|-1}} = \frac{|\mathrm{Con}(L_1)|}{2^{|L_1|-1}} \cdot \frac{|\mathrm{Con}(L_2)|}{2^{|L_2|-1}} = \mathrm{cd}(L_1) \cdot \mathrm{cd}(L_2).$$

Thus, the lemma holds for $t = 2$, and a trivial induction completes the proof. □

Lemma 3 (Core Lemma). *For any finite lattice L, $\mathrm{cd}(L) = \mathrm{cd}(\mathrm{Cor}(L))$.*

P r o o f. The congruence density of any chain is 1. Hence (the Glued Sum) Lemma 2 implies Lemma 3. □

Lemma 4 (Glue-B_4 Lemma). *If L is a Hall–Dilworth gluing of B_4 and a finite lattice K, then $\mathrm{cd}(L) = \mathrm{cd}(K)/2$. In other words, if a finite lattice L is an edge gluing or a glued sum of a lattice K and B_4, then $\mathrm{cd}(L) = \mathrm{cd}(K)/2$.*

P r o o f. For $L \in \{K \dotplus \mathsf{B}_4, \mathsf{B}_4 \dotplus K\}$, (the Glued Sum) Lemma 2 and $\mathrm{cd}(\mathsf{B}_4) = 1/2$ imply that $\mathrm{cd}(L) = \mathrm{cd}(K)/2$. Let L be an edge gluing of K and B_4; by duality, we can assume that K is the lower lattice. Denote the atoms of B_4 by a and b so that $a \in K$ but $b \notin K$. Define

$$H := \mathrm{Edge}(K) \cup \{(0_{\mathsf{B}_4}, b)\}.$$

It follows from (2.3) and (2.5) that H is a congruence-determining subset of $\mathrm{Edge}(L)$, $\nu(K)$ is the restriction of $\nu(L)$ to $\mathrm{Edge}(K)$, and for every $(x, y) \in \mathrm{Edge}(K)$, neither $(0_{\mathsf{B}_4}, b) \leq_{\nu(L)} (x, y)$ nor $(x, y) \leq_{\nu(L)} (0_{\mathsf{B}_4}, b)$. Thus, each $X \in \mathrm{Idl}(H; \nu(L))$ can be written uniquely in the form $Y \cup Z$, where $Y \in \mathrm{Idl}(\mathrm{Edge}(K); \nu(K))$ and $Z \subseteq \{(0_{\mathsf{B}_4}, b)\}$ can be chosen independently. Hence,

$$|\mathrm{Idl}(H; \nu(L))| = 2 \cdot |\mathrm{Idl}(\mathrm{Edge}(K); \nu(K))|,$$

and (2.8) yields that $|\mathrm{Con}(L)| = 2 \cdot |\mathrm{Con}(K)|$. Dividing this equality by

$$2^{|L|-1} = 2^2 \cdot 2^{|K|-1},$$

we obtain the required equality $\mathrm{cd}(L) = \mathrm{cd}(K)/2$. □

Lemma 5 (Three-Covers Lemma). *If a finite lattice L has an element with at least three covers or at least three lower covers, then* $\mathrm{cd}(L) \leq 8/64 = 1/8$.

P r o o f. By duality, we can assume that an element $o \in L$ has $t \geq 3$ covers, $a_1, \ldots, a_t$. We will use only a_1, a_2, a_3. By (the Three-Deficits) Lemma 1, we can assume that $|\mathrm{Jr}(L)| \leq 2$. Hence, in particular, $a_1 \vee a_2$, $a_1 \vee a_3$, $a_2 \vee a_3$, which belong to $\mathrm{Jr}(L)$, are not pairwise distinct. After rearranging the subscripts if necessary, we have $a_1 \vee a_3 = a_2 \vee a_3 := i$. Depending on $a_1 \vee a_2$, there are two cases to consider.

Case (i). We assume that $v := a_1 \vee a_2 < i$; see Fig. 2, where the bold edges denote coverings, the thin solid edges stand for "$<$", and the dotted edges indicate "$\leq$". As $|\mathrm{Jr}(L)| \leq 2$, $\mathrm{Jr}(L) = \{v, i\}$. In particular, $a_1, a_2, a_3 \in \mathrm{Ji}(L)$. Since $a_3 \leq v$ would lead to $i \leq v$, we have that $a_3 \not\leq v$. Let $j \in \{1, 2\}$. As $a_j \vee a_3 \in \mathrm{Jr}(L) = \{v, i\}$ but $a_j \vee a_3 = v$ would contradict that $a_3 \not\leq v$, we obtain that $a_j \vee a_3 = i$. Furthermore, $o \leq a_3 \wedge a_j < a_3$ and $o \prec a_3$ yield that $a_3 \wedge a_j = o$. Thus, $(o, a_j) \nearrow_{\mathrm{b}} (a_3, i)$, for $j \in \{1, 2\}$, and (2.5) yields that $\mathrm{con}(o, a_1) = \mathrm{con}(a_3, i) = \mathrm{con}(o, a_2)$. In particular,

$$\mathrm{con}(a_{1*}, a_1) = \mathrm{con}(a_{2*}, a_2).$$

Thus, $v, i \in \mathrm{Jr}(L)$ and a_2 are distinct join-deficits, and (the Three-Deficits) Lemma 1 implies that $\mathrm{cd}(L) \leq 8/64$, as required.

Case (ii). We assume that $a_1 \vee a_2 = i$; see Fig. 2 again. So

$$a_1 \vee a_2 = a_1 \vee a_3 = a_2 \vee a_3 = i.$$

Clearly,

$$a_1 \wedge a_2 = a_1 \wedge a_3 = a_2 \wedge a_3 = i.$$

Hence, for every $j, t \in \{1, 2, 3\}$ distinct, $(o, a_j) \nearrow_{\mathrm{b}} (a_t, i)$. Thus, (2.5) (or the fact that M_3 is a simple lattice) leads to

$$\mathrm{con}(o, a_1) = \mathrm{con}(a_3, i) = \mathrm{con}(o, a_2) = \mathrm{con}(a_1, i) = \mathrm{con}(o, a_3),$$

whereby

$$\mathrm{con}(o, a_1) = \mathrm{con}(o, a_2) = \mathrm{con}(o, a_3). \tag{2.9}$$

If

$$|\mathrm{Jr}(L) \cap \{a_1, a_2, a_3\}| \geq 2,$$

then $\mathrm{Jr}(L)$, which contains also i, has at least three elements, and the required $\mathrm{cd}(L) \leq 8/64$ follows from (the Three-Deficits) Lemma 1. If

$$|\mathrm{Jr}(L) \cap \{a_1, a_2, a_3\}| = 0,$$

then $\{a_1, a_2, a_3\} \subseteq \mathrm{Ji}(L)$, two members of $\{a_1, a_2, a_3\}$ are join-deficits by (2.9), and so is $i \in \mathrm{Jr}(L)$, whereby (the Three-Deficits) Lemma 1 applies again. Finally, assume that

$$|\mathrm{Jr}(L) \cap \{a_1, a_2, a_3\}| = 1.$$

Apart from indexing, $a_1 \in \mathrm{Jr}(L)$ and $\{a_2, a_3\} \in \mathrm{Ji}(L)$. By (2.9), a_3 is a join-deficit, and so are $a_1, i \in \mathrm{Jr}(L)$. So we can apply (the Three-Deficits) Lemma 1, completing the proof of Lemma 5. □

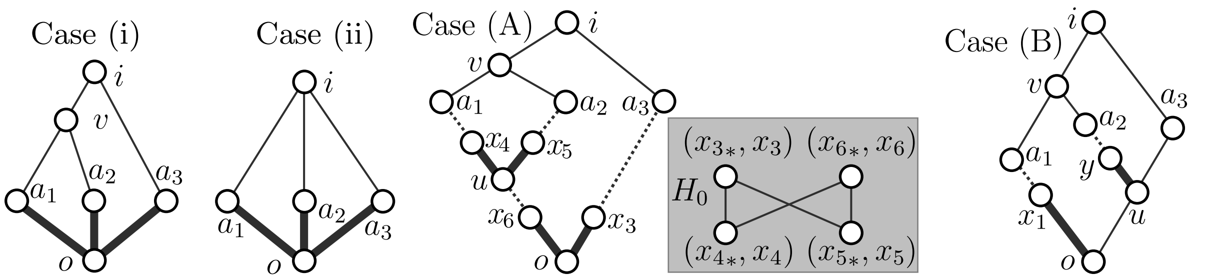

Figure 2. Illustrating the proofs of Lemmas 5 and 6.

Lemma 6 (Antichain Lemma). *If a finite lattice L has a three-element antichain, then* $\mathrm{cd}(L) \leq 8/64 = 1/8$.

P r o o f. Let $n := |L|$. By (the Three-Deficits) Lemma 1, we can assume that $|\mathrm{Mr}(L)| \leq 2$ and $|\mathrm{Jr}(L)| \leq 2$. Let $\{a_1, a_2, a_3\}$ be a three-element antichain in L. Let $o := a_1 \wedge a_2 \wedge a_3$ and $i := a_1 \vee a_2 \vee a_3$. Define the sets

$$S_\wedge := \{\{j, t\} : j \neq t \text{ and } a_j \wedge a_t > o\}$$

and

$$S_\vee := \{\{j, t\} : j \neq t \text{ and } a_j \vee a_t < i\}.$$

For $\{j, t\}, \{j', t'\} \in S_\wedge$, if $\{j, t\} \neq \{j', t'\}$, then $a_j \wedge a_t \neq a_{j'} \wedge a_{t'}$, as otherwise

$$a_j \wedge a_t = (a_j \wedge a_t) \wedge (a_{j'} \wedge a_{t'}) = a_1 \wedge a_2 \wedge a_3 = o$$

would contradict that $\{j, t\} \in S_\wedge$. Using this observation,

$$\{o\} \cup \{a_j \wedge a_t : \{j, t\} \in S_\vee\} \subseteq \mathrm{Mr}(L),$$

and $|\mathrm{Mr}(L)| \leq 2$, we obtain that $|S_\wedge| \leq 1$.

Assume that $|S_\wedge| = 0$ that is,

$$a_1 \wedge a_2 = a_1 \wedge a_3 = a_2 \wedge a_3 = o.$$

For $j \in \{1,2,3\}$, let $x_j \in [o,a_j]$ such that $o \prec x_j$. These covers of o are pairwise distinct; indeed, if we had that $x_j = x_t$ for $j \neq t$, then

$$o \prec x_j = x_j \wedge x_t \leq a_j \wedge a_t = o$$

would be a contradiction. Hence, the required inequality $\mathrm{cd}(L) \leq 8/64$ follows from (the Three Covers) Lemma 5.

Therefore, we can assume that $|S_\wedge| = 1$ and, by duality, $|S_\vee| = 1$. Apart from symmetry, there are two cases to deal with: either $S_\vee = \{\{1,2\}\}$ and $S_\wedge = \{\{1,2\}\}$ or $S_\vee = \{\{1,2\}\}$ and $S_\wedge = \{\{2,3\}\}$.

Case (A). We assume that $S_\vee = \{\{1,2\}\}$ and $S_\wedge = \{\{1,2\}\}$. This means that

$$v := a_1 \vee a_2 < i = a_1 \vee a_3 = a_2 \vee a_3$$

and

$$u := a_1 \wedge a_2 > o = a_1 \wedge a_3 = a_2 \wedge a_3.$$

Note that $\mathrm{Jr}(L) = \{v,i\}$ and $\mathrm{Mr}(L) = \{o,u\}$. Pick $x_4, x_5, x_6 \in L$ such that $u \prec x_4 \leq a_1$, $u \prec x_5 \leq a_2$, $o \prec x_6 \leq u$, and remember that $o \prec x_3 \leq a_3$; see Fig. 2. The elements $x_3, \dots, x_6$, being outside $\{v,i\}$, are in $\mathrm{Ji}(L)$. Since $x_3 \leq u$ would imply $x_3 \leq a_1 \wedge a_2 \wedge a_3 = o$ and $u \leq x_3$ would imply $u \leq o$, we obtain that $x_3 \parallel u$. Thus, as $\mathrm{Jr}(L) = \{v,i\}$, we obtain that $v \leq u \vee x_3$. Since

$$o \prec x_3 \quad \text{and} \quad o \leq u \wedge x_3 < x_3, \quad \text{we have} \quad u \wedge x_3 = o.$$

So, $(o,x_3) \nearrow_{\mathrm{b}} (u, u \vee x_3)$. Thus, by the convexity of the $\mathrm{con}(o,x_3)$-block of u, (2.4), and (2.5), we obtain that $\mathrm{con}(u,x_4) \leq \mathrm{con}(u,v) \leq \mathrm{con}(u, u \vee x_3) = \mathrm{con}(o,x_3)$ and $\mathrm{con}(u,x_5) \leq \mathrm{con}(o,x_3)$. Since $x_6 \leq a_3$ would imply $x_6 \leq o$ and $a_3 \leq x_6$ would lead to $a_3 \leq a_1$, we obtain that $x_6 \parallel a_3$. Combining this with $o \prec x_6$, $v \not\geq a_3$ (as otherwise $v \geq i$), and $\mathrm{Jr}(L) = \{v,i\}$, we conclude that $(o,x_6) \nearrow_{\mathrm{b}} (a_3,i)$. Hence, (2.5) yields that $\mathrm{con}(a_3,i) = \mathrm{con}(o,x_6)$. We have seen that $v \not\geq a_3$, while $v \leq a_3$ would lead to $a_1 \leq a_3$. So, $v \parallel a_3$. Trivially, $v \vee a_3 = i$. Using that $\mathrm{Mr}(L) = \{u,o\}$, we have that $\{u,o\} \ni v \wedge a_3 \leq u$. Hence, $(a_3,i) \searrow_{\mathrm{b}} (v \wedge a_3, v)$, and (2.5) gives that $\mathrm{con}(v \wedge a_3, v) = \mathrm{con}(a_3,i)$. By the convexity of the $\mathrm{con}(v \wedge a_3, v)$-block of v and $v \wedge a_3 \leq u$, $\mathrm{con}(u,x_4) \leq \mathrm{con}(v \wedge a_3, v)$ and $\mathrm{con}(u,x_5) \leq \mathrm{con}(v \wedge a_3, v)$. So, by transitivity, $\mathrm{con}(u,x_4) \leq \mathrm{con}(o,x_6)$ and $\mathrm{con}(u,x_5) \leq \mathrm{con}(o,x_6)$. We have seen that

$$\text{for } j \in \{4,5\} \text{ and } t \in \{3,6\}, \quad \mathrm{con}(x_{j_*}, x_j) \leq \mathrm{con}(x_{t*}, x_t). \tag{2.10}$$

On the set $H := \{(p_*,p) : p \in \mathrm{Ji}(L)\}$, define a quasiorder ρ by letting $(p_*,p) \leq_\rho (q_*,q)$ if and only if $p = q$ or there is a pair $(j,t) \in \{4,5\} \times \{3,6\}$ such that $p = x_j$ and $q = x_t$. Furthermore, as before, $(p_*,p) \leq_{\nu(L)} (q_*,q)$ means that $\mathrm{con}(p_*,p) \leq \mathrm{con}(q_*,q)$. Then we have two quasiordered sets, $(H;\rho)$ and $(H;\nu(L))$. By (2.10), $\nu(L)$ is coarser than ρ. Therefore, $\mathrm{Idl}(H;\nu(L)) \subseteq \mathrm{Idl}(H;\rho)$. Since H is a congruence-determining subset of $\mathrm{Edge}(L)$ by (2.6), we obtain from (2.8) that

$$|\mathrm{Con}(L)| = |\mathrm{Idl}(H;\nu(L))| \leq |\mathrm{Idl}(H;\rho)|. \tag{2.11}$$

The quasiordered set $(H;\rho)$ is actually a poset, and its subposet

$$H_0 := \{(p_*,p) : p \in \{x_3,x_4,x_5,x_6\}\}$$

is depicted in Fig. 2. Clearly, $|\mathrm{Idl}(H_0;\rho)| = 7$. The rule $X \mapsto X \cap H_0$ defines a map $g\colon \mathrm{Idl}(H;\rho) \to \mathrm{Idl}(H_0,\rho)$. Since

$$|H \setminus H_0| = |H| - |H_0| = (n - 1 - |\mathrm{Jr}(L)|) - 4 = n - 7,$$

each $Y \in \mathrm{Idl}(H_0;\rho)$ has at most 2^{n-7} preimages. Hence, $|\mathrm{Idl}(H,\rho)| \le 7\cdot 2^{n-7}$. Combining this with (2.11), we obtain the required

$$\mathrm{cd}(L) = |\mathrm{Con}(L)|/2^{n-1} \le (7\cdot 2^{n-7})/2^{n-1} = 7/2^6 \le 8/64.$$

Case (B). We assume that $S_\vee = \{\{1,2\}\}$ and $S_\wedge = \{\{2,3\}\}$. This means that

$$v := a_1 \vee a_2 < i = a_1 \vee a_3 = a_2 \vee a_3 \quad \text{and} \quad u := a_2 \wedge a_3 > o = a_1 \wedge a_2 = a_1 \wedge a_3.$$

Pick an element $y \in L$ such that $u \prec y \le a_2$, and remember that $o \prec x_1 \le a_1$; the situation is visualized in Fig. 2. As $\mathrm{Jr}(L) = \{v,i\}$, $\{x_1,y\} \subseteq \mathrm{Ji}(L)$. Since $x_1 \le a_3$ would lead to $x_1 \le a_1 \wedge a_3 = o$ and $x_1 \ge a_3$ would lead to $a_1 \ge a_3$, we have that $x_1 \parallel a_3$. So $o \le x_1 \wedge a_3 < x_1$ and $o \prec x_1$ give that $x_1 \wedge a_3 = o$. On the other hand, $x_1 \vee a_3 \in \mathrm{Jr}(L) = \{v,i\}$. As $x_1 \vee a_3 = v$ would lead to $a_3 \le v$ and so $i = v \vee a_3 = v$, we have that $x_1 \vee a_3 = i$. Hence, $(o,x_1) \nearrow_{\mathrm{b}} (a_3,i)$, whereby (2.5) yields that $\mathrm{con}(a_3,i) = \mathrm{con}(o,x_1) = \mathrm{con}(x_{1*},x_1)$.

Next, $y \le a_3$ would lead to $y \le a_2 \wedge a_3 = u$ while $y \ge a_3$ to $a_2 \ge a_3$, whence $a_3 \parallel y$. Combining $u \le a_3 \wedge y < y$ with $u \prec y$, we obtain that $a_3 \wedge y = u$. Since $a_3 \vee y = \mathrm{Jr}(L) = \{v,i\}$ and $a_3 \le a_3 \vee y = v$ would lead to $i \le v$, we have that $a_3 \vee y = i$. Thus, $(u,y) \nearrow_{\mathrm{b}} (a_3,i)$, whence (2.5) gives that $\mathrm{con}(a_3,i) = \mathrm{con}(u,y) = \mathrm{con}(y_*,y)$. Hence, $\mathrm{con}(y_*,y) = \mathrm{con}(x_{1*},x_1)$, whereby y is a join-deficit. So y, v, and i are three join-deficits, and (the Three-Deficits) Lemma 1 gives the required inequality $\mathrm{cd}(L) \le 8/64$), completing the proof of Lemma 6. □

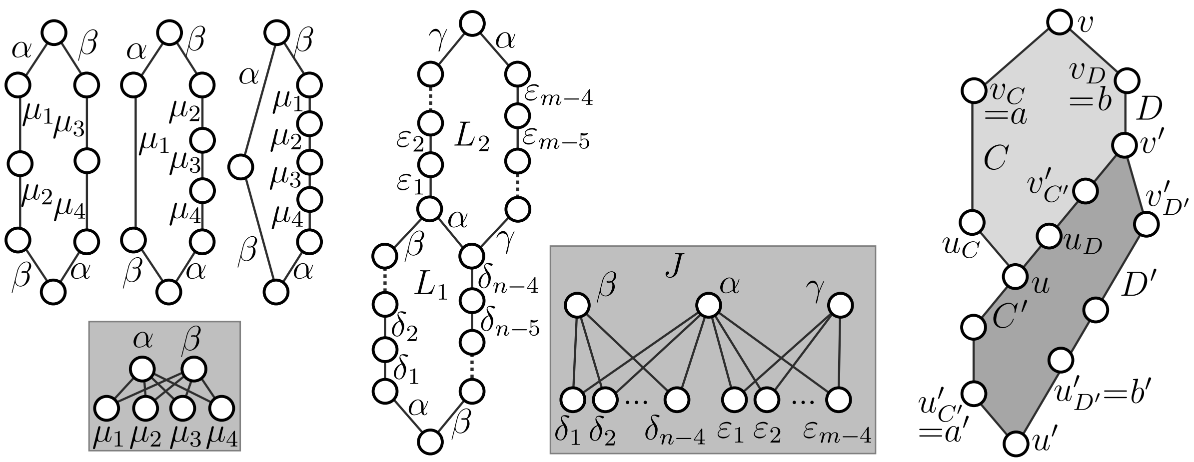

Figure 3. Illustrating the proofs of Lemmas 7, 8, and 9.

Lemma 7 (Circle Lemma). *If* $4 \le n \in \mathbb{N}^+$ *and* $L \in \mathrm{Circ}(n)$, *then*

$$\mathrm{cd}(L) = \frac{8 + 3/2^{n-7}}{64} = \frac{1}{8} + \frac{3}{2^{n-1}}.$$

P r o o f. The set of *minimal elements* and that of *maximal elements* of a poset P will be denoted by $\mathrm{Min}(P)$ and $\mathrm{Max}(P)$, respectively. Let $L \in \mathrm{Circ}(n)$. Label its two edges departing from 0_L with α and β, and label the remaining of edges that are disjoint from 1_L with $\mu_1,\dots,\mu_{n-4}$. These labels stand for the congruences generated by the corresponding edges. As an illustration, all $L \in \mathrm{Circ}(8)$ are drawn on the left of Fig. 3. The two α-labeled edges of each of these diagrams are perspective (i.e., $\to_{\mathrm{b}}$), and the same holds for the β-labeled edges.

Let $J := \mathrm{Ji}(\mathrm{Con}(L))$, (2.2) and (2.3) give that $\mathrm{Max}(J) = \{\alpha, \beta\}$, $\mathrm{Min}(J) = \{\mu_1, \dots, \mu_{n-4}\}$, J is the disjoint union of $\mathrm{Max}(J)$ and $\mathrm{Min}(J)$, and $\xi < \psi$ holds for every $\xi \in \mathrm{Min}(J)$ and $\psi \in \mathrm{Max}(J)$. There are 2^{n-4} members of $\mathrm{Idl}(J)$ that are disjoint from $\mathrm{Max}(J)$. If $X \in \mathrm{Idl}(J)$ is not disjoint from $\mathrm{Max}(J)$, then $\mathrm{Min}(J) \subseteq X$; so there are 3 ways to pick such an X. Therefore, (2.1) leads to

$$\mathrm{cd}(L) = |\mathrm{Con}(L)|/2^{n-1} = |\mathrm{Idl}(J)|/2^{n-1} = (2^{n-4} + 3)/2^{n-1} = 1/8 + 3/2^{n-1},$$

completing the proof of Lemma 7. □

The concept of edge gluing, defined in the paragraph right after Theorem 1, is meaningful for any two non-singleton finite lattices: if we form a Hall–Dilworth gluing of L_1 and L_2 so that we identify a two-element filter of L_1 with a two-element ideal of L_2, then we obtain an *edge-gluing* of L_1 and L_2.

Lemma 8 (Two-Circles Lemma). *Assume that $m \geq 4$ and $n \geq 4$ are integers, $L_1 \in \mathrm{Circ}(n)$, $L_2 \in \mathrm{Circ}(m)$, and L is an edge-gluing of L_1 and L_2. Then the following two assertions hold.*

(1) *If $m \geq 5$ and $n \geq 5$, then* $\mathrm{cd}(L) \leq 6.5/64 = 13/128$.
(2) *If either $m = 4$ and $n \geq 6$ or $m \geq 6$ and $n = 4$, then* $\mathrm{cd}(L) \leq 7/64$.

P r o o f. For part (1), the situation is visualized in the middle of Fig. 3. The distribution of the $\delta_1, \dots, \delta_{n-4}$ between the left boundary and the right boundary of L_1 is unimportant, regardless of how many are on each side. The same holds for $\varepsilon_1, \dots, \varepsilon_{m-4}$ in case of L_2. By (2.2) and (2.3), we obtain $J := \mathrm{Ji}(\mathrm{Con}(L))$, see the figure again. In the computation below, N_α, $N_{\neg\alpha,\beta}$, etc. will denote $|\{X \in \mathrm{Idl}(J) : \alpha \in X\}|$, $|\{X \in \mathrm{Idl}(J) : \alpha \notin X,\ \beta \in X\}|$, etc., respectively. By (1.1) and (2.1), we can compute as follows

$$\begin{aligned}\mathrm{cd}(L) = |\mathrm{Idl}(J)| \,/\, 2^{|L|-1} &= (N_{\neg\alpha,\neg\beta,\neg\gamma} + N_{\neg\alpha,\beta,\neg\gamma} + N_{\neg\alpha,\neg\beta,\gamma} + N_{\neg\alpha,\beta,\gamma} + N_\alpha) \,/\, 2^{n+m-3} \\ &= (2^{n+m-8} + 2^{m-4} + 2^{n-4} + 1 + 4) \,/\, 2^{n+m-3} \\ &= (2^1 + 2^{-(n-5)} + 2^{-(m-5)} + 5 \cdot 2^{-(n-5)-(m-5)-1}) \,/\, 2^6 \\ &\leq (2 + 1 + 1 + 5/2) \,/\, 2^6 = 6.5/64,\end{aligned}$$

as required by part (1). Part (2) is a trivial consequence of (the Glue-B_4) Lemma 4 and (the Circle) Lemma 7, completing the proof of Lemma 8. □

3. The final steps of the proof of the theorems

In addition to proving the new result, Theorem 2, no extra effort is required to provide an entirely new proof of Theorem 1 as well. Thus, we prove both theorems. We proceed with a lemma.

Lemma 9. *For a finite lattice L, the following two conditions are equivalent*:

(1) $|\mathrm{Jr}(L)| = 2$, $|\mathrm{Mr}(L)| = 2$, *and L contains no three-element antichain.*
(2) *There are $4 \leq m, n \in \mathbb{N}^+$ and lattices $L_1 \in \mathrm{Circ}(m)$ and $L_2 \in \mathrm{Circ}(n)$ such that $\mathrm{Cor}(L)$ is the glued sum or an edge-gluing of L_1 and L_2.*

P r o o f. As the second condition implies the first trivially, we assume that the first condition holds. Pick a pair (a, b) of incomparable elements of L. Let $u := a \wedge b$, $v := a \vee b$, and $n := |[u, v]|$. Furthermore, let

$$C := [u, v] \cap (\mathrm{idl}(a) \cup \mathrm{fil}(a)) \quad \text{and} \quad D := [u, v] \cap (\mathrm{idl}(b) \cup \mathrm{fil}(b)).$$

In the rightmost diagram of Fig. 3, $[u, v]$ is indicated by the light-grey color. Note that the diagram is loosely connected to the proof. While certain features of the diagram align with specific parts of the proof, other features, such as the presence of three-element antichains, illustrate (indirect) assumptions that must be excluded. Additionally, the diagram primarily supports the "$u < v'$" portion of the proof (i.e., the second half). We claim that

$$C \quad \text{and} \quad D \quad \text{are chains,} \quad C \cup D = [u, v], \quad \text{and} \quad C \cap D = \{u, v\}. \tag{3.1}$$

For the sake of contradiction, assume that there are $x, y \in C$ such that $x \parallel y$. By duality, we can assume that $x, y \in \mathrm{fil}(a) \cap [u, v]$. Since $x \parallel y$, we have that $x < v$ and $y < v$. If we had that $b \le x$, then $v = a \vee b \le x$ would be a contradiction, while $x \le b$ would contradict that $a \parallel b$. Hence, $x \parallel b$. Similarly, $y \parallel b$. So, $\{x, y, b\}$ is a three-element antichain, which is a contradiction. This proves that C is a chain. By symmetry, so is D. Next, assume that $x \in C \cap D$. As $a \parallel b$, either $x \in \mathrm{idl}(a) \cap \mathrm{idl}(b)$ or $x \in \mathrm{fil}(a) \cap \mathrm{fil}(b)$. In the first case, $u = a \wedge b \ge x \in [u, v]$ gives that $x = u$. Dually, the second case leads to $x = v$, and we have shown that $C \cap D = \{u, v\}$. Finally, if we had an element y in $[u, v] \setminus (C \cup D)$, then $\{a, b, y\}$ would be a three-element antichain. Thus, $[u, v] = C \cup D$, and we obtain the validity of (3.1).

Next, let $u_C \in C$ and $v_C \in C$ be the (unique) elements of C such that u_C covers u and v_C is covered by v. Similarly, $u_D \in D$ and $v_D \in D$ denote the (unique) elements of D such that $u \prec u_D$ and $v_D \prec v$. We claim that for each $x, y \in L$,

$$\text{if} \quad x > u \quad \text{then} \quad x \ge u_C \quad \text{or} \quad x \ge u_D, \quad \text{and if} \quad y < v \quad \text{then} \quad y \le v_C \quad \text{or} \quad y \le v_D. \tag{3.2}$$

To see this, note that $x > u$ implies the existence of an x_0 such that $u \prec x_0 \le x$. As there is no three-element antichain, $x_0 \in \{u_C, u_D\}$. Thus, (3.2) follows by duality.

We continue by showing that

$$\text{every} \quad x \in \mathrm{Jr}(L) \cup \mathrm{Mr}(L) \quad \text{is comparable with both} \quad u \quad \text{and} \quad v. \tag{3.3}$$

To see this, we assume by duality that $x \in \mathrm{Jr}(L)$. Then x is comparable with v, as otherwise v, x, and $v \vee x$ would be three distinct elements of $\mathrm{Jr}(L)$. Assume, aiming for a contradiction, that $x \parallel u$. Then $u \vee x = v$, since $\mathrm{Jr}(L) = \{v, x\}$. But then $x < v$ and (3.2) give that $x \in \mathrm{idl}(v_C) \cup \mathrm{idl}(v_D)$, whence $v = u \vee x \le v_C$ or $v = u \vee x \le v_D$, which is a contradiction proving (3.3).

Next, we show that if there exist $a', b' \in L$ such that

$$a', b' \in \mathrm{idl}(u) \cup \mathrm{fil}(v) \quad \text{and} \quad a' \parallel b', \quad \text{then } \mathrm{Cor}(L) \text{ is of the required form.} \tag{3.4}$$

By duality, it suffices to deal with $a', b' \in \mathrm{idl}(u)$. Let $u' := a' \wedge b'$, $v' := a' \vee b'$, and $m := |[u', v']|$. Since $u' < v' \le u$, $v' \le u < v$, and $|\mathrm{Mr}(L)| = |\mathrm{Jr}(L)| = 2$, we have that $\mathrm{Mr}(L) = \{u', u\}$ and $\mathrm{Jr}(L) = \{v', v\}$. We know that $u' < v' \le u < v$, and we claim that $\{u', v', u, v\} \subseteq \mathrm{Nar}\,(L)$. If we had an element x such that $x \parallel u'$, then $x \wedge u'$ would belong to $\mathrm{Mr}(L)$, contradicting that $|\mathrm{Mr}(L)| = 2$. Hence, there is no such x, and so $u' \in \mathrm{Nar}\,(L)$. Dually, $v \in \mathrm{Nar}\,(L)$. By duality, it suffices to deal with u alone, out of v' and u. Seeking a contradiction, suppose that there is an $x \in L$ such that $x \parallel u$. As we already know that $v \in \mathrm{Nar}\,(L)$, we have that $x < v$, as $v \le x$ would lead to $u < x$. By (3.2), $u \vee x \le v_C$ or $u \vee x \le v_D$, and so $u \vee x \ne v$. As $v' \le u < u \vee x$, $u \vee x \ne v'$. Thus,

$u \vee x$, which is clearly join-reducible, cannot belong to $\{v', v\} = \mathrm{Jr}(L)$. This contradiction rules out the existence of $x \in L$ with $x \parallel u$, and we have shown that $\{u', v', u, v\} \subseteq \mathrm{Nar}\,(L)$. Therefore,

$$L = \mathrm{idl}(u') \dotplus [u', v'] \dotplus [v', u] \dotplus [u, v] \dotplus \mathrm{fil}(v). \tag{3.5}$$

Each of $\mathrm{idl}(u')$, $[v', u]$, and $\mathrm{fil}(v)$ is a chain, since otherwise we would obtain a third join-reducible element. Since $C \setminus \{u, v\}$ is disjoint from both $\mathrm{Jr}(L) = \{v', v\}$ and $\mathrm{Mr}(L) = \{u', u\}$, the elements of $C \setminus \{u, v\}$ are doubly irreducible. As the same holds for $D \setminus \{u, v\}$, (3.1) yields that $[u, v] \in \mathrm{Circ}(n)$. Similarly, $[u', v'] \in \mathrm{Circ}(m)$. These facts and (3.5) imply(3.4).

If necessary, we can select new a and b with the same $v = a \vee b$ as before and allow u to change accordingly. Therefore, in the remainder of the proof, we assume that $a \prec v = a \vee b$ and $b \prec v$. (These covering relations mean that $a = v_C$ and $b = v_D$.) Obviously, (3.1)–(3.4) remain valid. However, the extra assumption allows us to strengthen (3.1) to

$$[u, v] \in \mathrm{Circ}(n) \tag{3.6}$$

To verify this, it suffices to augment (3.1) with the following property: for any $x \in C \setminus \{u, v\}$ and $y \in D \setminus \{u, v\}$, x and y are incomparable. This is straightforward. Indeed, $x \leq y$ leads to $x = x \wedge y \leq v_C \wedge v_D = a \wedge b = u$, a contradiction, while swapping C an D shows that $y \leq x$ is impossible, either. We have verified (3.6).

We obtain from (3.3) that (the two-element) $\mathrm{Jr}(L)$ is a chain. Therefore, we can assume that, in addition to $a \prec v$ and $b \prec v$, v is the largest element of $\mathrm{Jr}(L)$. Let u' be the unique element of $\mathrm{Mr}(L) \setminus \{u\}$, and pick $a', b' \in L$ such that $u' = a' \wedge b'$, $u' \prec a'$, and $u' \prec b'$. Let $v' := a' \vee b' \in \mathrm{Jr}(L)$ and, as earlier, $m := |[u', v']|$. In the interval $[u', v']$, we define C', D', $v'_{C'}$, and $v'_{D'}$ in the same way as their non-primed counterparts are defined in the interval $[u, v]$. Furthermore, $u'_{C'} = a'$ and $u'_{D'} = b'$.

By (3.3), u and v' are comparable. If $v' \leq u$, then $a', b' < u$ and (3.4) applies. Therefore, we can assume that $u < v'$. As v is the largest element of $\mathrm{Jr}(L)$, we know that $v' \leq v$. By (3.3), u' and u (which are distinct) are comparable. Assume, aiming for a contradiction, that $u < u'$. Then $a', b' \in [u', v'] \subseteq [u, v]$ and $u < u' = a' \wedge b' \in \mathrm{Mr}([u, v])$ contradicts (3.6). Hence, $u' < u$.

After a series of smaller observations, we will strengthen $v' \leq v$ to $v' < v$. If we had that $u < a'$, then $u' < u < a'$ would contradict the covering $u' \prec a'$. Thus, $u \not< a'$. Similarly, $u \not< b'$. Thus, since $\{u, a', b'\}$ cannot be a three-element antichain and $a' \parallel b'$, at least one of $a' \leq u$ and $b' \leq u$ holds. On the other hand, at least one of them fails, as $u < v' = a' \vee b'$. Hence, after swapping the roles of a' and b' if necessary, we know that $a' \leq u$ but $b' \not\leq u$, so $a' \leq u$ and $b' \parallel u$. Since $b' < v' \leq v$, (3.2) implies that $b' \leq v_C = a$ or $b' \leq v_D = b$. So, after swapping the roles of a and b if necessary, we have that $b' \leq b$. Using the inequalities established in this paragraph, we obtain that $v' = a' \vee b' \leq u \vee b = b < v$. Thus, we have shown that $v' < v$, and we have also obtained that $v' \leq b$. Let us summarize; $a \parallel v'$ below comes from $a = v_C$, $u < v' \leq b = v_D$, and (3.6), while "$[u', v'] \in \mathrm{Circ}(m)$" follows from (3.6) by duality:

$$u' \prec a' \leq u < v' \leq b \prec v, \quad u \parallel b', \quad a \parallel v', \quad \text{and} \quad [u', v'] \in \mathrm{Circ}(m). \tag{3.7}$$

Next, we claim that

$$[u', v] = [u', v'] \cup [u, v]. \tag{3.8}$$

To derive a contradiction, assume that $[u', v] \neq [u', v'] \cup [u, v]$. As "$\supseteq$" in the place of "$\neq$" follows from the inequalities summarized in (3.7), we have an element $x \in [u', v]$ such that $x \notin [u', v'] \cup [u, v]$. We know that $x \parallel u$, since otherwise either $u \leq x$ and x would belong to $[u, v]$, or $x \leq u \leq v'$ and x would be in $[u', v']$. Similarly, $x \parallel v'$, as otherwise we would have that $x \in [v', v] \subseteq [u, v]$ or

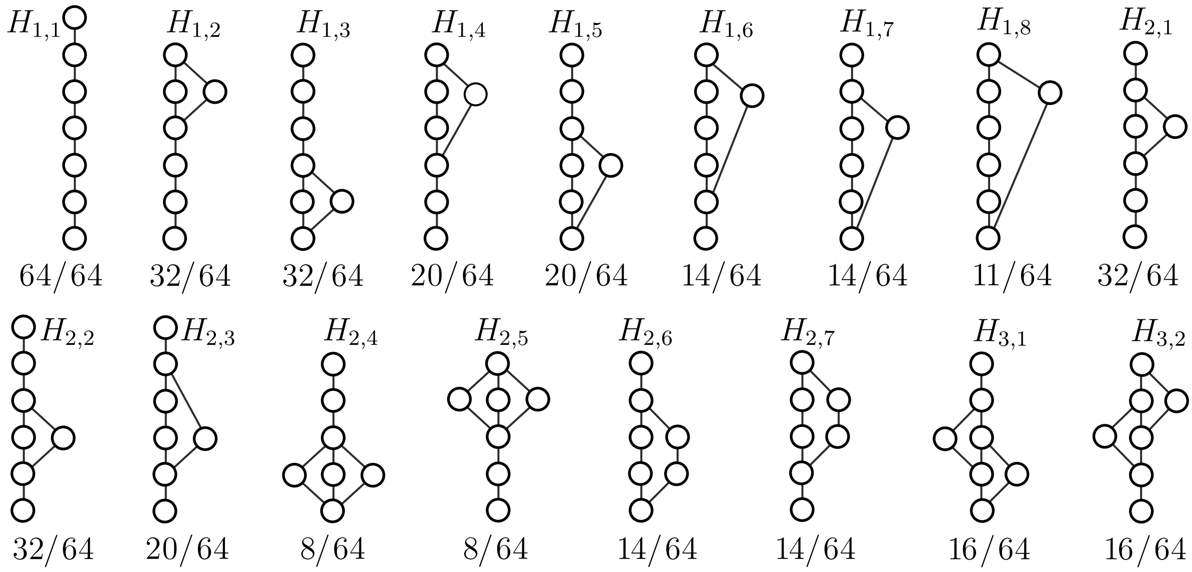

Figure 4. The seven-element lattices $H_{1,1}$–$H_{3,2}$.

$x \in [u', v']$. As $x \parallel v'$, $x \vee u \neq v'$. But $x \vee u \in \mathrm{Jr}(L) = \{v, v'\}$ since $x \parallel u$. Hence, $x \vee u = v$. On the other hand, $x < v$ and (3.2) yield that $x \le v_C = a$ or $x \le v_D = b$. Hence $v = u \vee x \le a$ or $v = u \vee x \le b$, which is a contradiction implying (3.8).

By (3.8), every element of $[u', v]$ is in $[u, v]$ or $[u', v']$, but this will not be repeated in the remainder of the proof. Moving forward, we will show that

$$u \prec v'. \tag{3.9}$$

Suppose the contrary, that is, $u \not\prec v'$; see Fig. 3. We know from $b = v_D$ and (3.7) that $v' \in D$ and $u < v'$. These facts and $u \not\prec v'$ imply that $u_D < v'$. Using (3.7) and $a' = u'_{C'} \in C'$, we have that $u \in C' \setminus \{u'\}$, whereby $u < u_D < v'$ yields that $u_D \in C'$, and so $u_D \le v'_{C'}$. In particular, as $v' \in D \setminus \{v\}$, $u_D \in D$, and $u_D \le v'_{C'} < v'$, we obtain that $v'_{C'} \in D \setminus \{u, v\}$. Since $u \in C' \setminus \{u', v'\}$, the $[u', v'] \in \mathrm{Circ}(m)$ part of (3.7) implies that $u \parallel v'_{D'}$ and $u \vee v'_{D'} = v'$. Hence, if we had that $u_C \ge v'_{D'}$, then $u < u \vee v'_{D'} \le u_C$ and $u \prec u_C$ would lead to $u \vee v'_{D'} = u_C$, and combining this equality with the previously established $u \vee v'_{D'} = v'$, we would obtain that $u_C = v' \in D$, contradicting (3.6). Therefore, $u_C \not\ge v'_{D'}$. On the other hand, $u_C \le v'_{D'}$ would lead to $u_C \le v' \le b$ by (3.7), whereby (3.6) would imply that $v = u_C \vee b = b = v_D$, a contradiction. Thus, $u_C \not\le v'_{D'}$, and we have obtained that $u_C \parallel v'_{D'}$; this was the first step to show that $Y := \{u_C, v'_{C'}, v'_{D'}\}$ is a three-element antichain. By the $[u', v'] \in \mathrm{Circ}(m)$ part of (3.7), $v'_{C'} \parallel v'_{D'}$. We have already seen that $v'_{C'} \in D \setminus \{u, v\}$, whence (3.6) gives that $u_C \parallel v'_{C'}$. So Y is a three-element antichain, which is a contradiction that proves (3.9).

Finally, (3.6), the $[u', v'] \in \mathrm{Circ}(m)$ part of (3.7), (3.8), and (3.9) imply that $[u', v]$ is an edge-gluing of $L_1 := [u', v'] \in \mathrm{Circ}(m)$ and $L_2 := [u, v] \in \mathrm{Circ}(n)$. Hence,

$$\{v, v'\} \subseteq \mathrm{Jr}(L), \quad \{u, u'\} \subseteq \mathrm{Mr}(L), \quad |\mathrm{Jr}(L)| = 2, \quad \text{and} \quad |\mathrm{Mr}(L)| = 2$$

yield that $\mathrm{Cor}(L) = [u', v]$. Thus, $\mathrm{Cor}(L)$ is of the required form, completing the proof of Lemma 9. □

Most of the following lemma summarizes what we have already proved. By a *core lattice* we mean a finite lattice L that is its own core, that is, $L = \mathrm{Cor}(L)$.

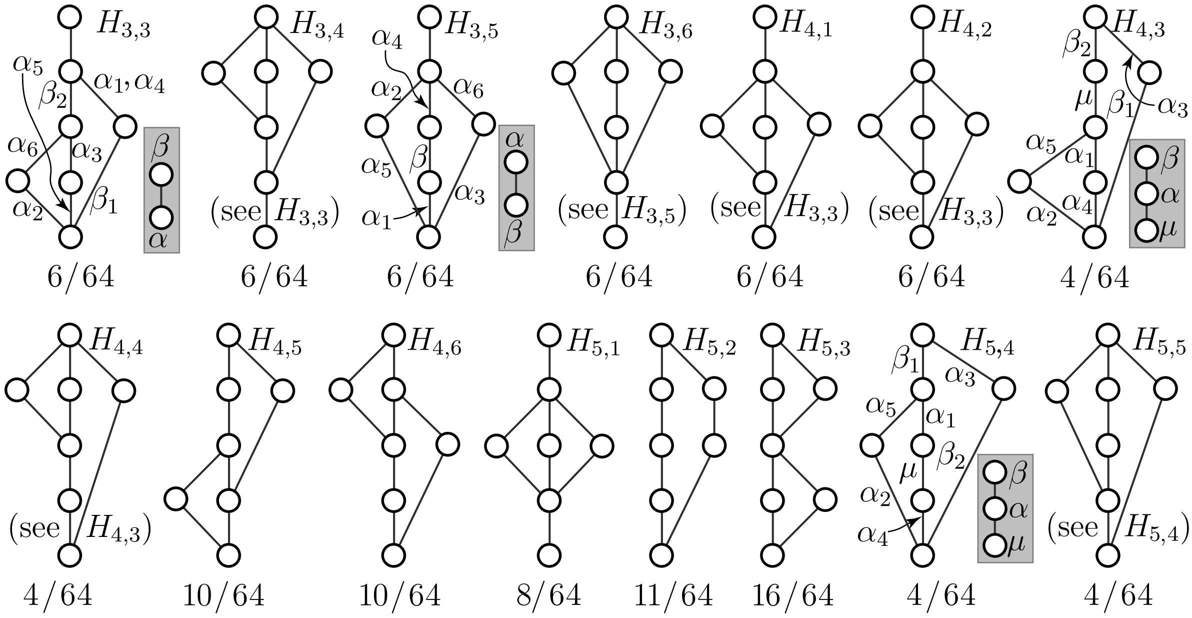

Figure 5. The seven-element lattices $H_{3,3}$–$H_{5,5}$.

Lemma 10. *The complete list of core lattices L with $\mathrm{cd}(L) > 8/64$, along with their sizes and congruence densities, is the following.*

(1) *The singleton lattice; its congruence density is* $64/64$, *its size is* 1.
(2) B_4, *the only element of* $\mathrm{Circ}(4)$; $\mathrm{cd}(\mathsf{B}_4) = 32/64$ *and* $|\mathsf{B}_4| = 4$.
(3) N_5, *the only element of* $\mathrm{Circ}(5)$; $\mathrm{cd}(\mathsf{N}_5) = 20/64$ *and* $|\mathsf{N}_5| = 5$.
(4) $\mathsf{C}_2 \times \mathsf{C}_3$, *that is, the edge gluing of two copies of* B_4; $\mathrm{cd}(\mathsf{C}_2 \times \mathsf{C}_3) = 16/64$ *and* $|\mathsf{C}_2 \times \mathsf{C}_3| = 6$.
(5) $\mathsf{B}_4 \dotplus \mathsf{B}_4$; $\mathrm{cd}(\mathsf{B}_4 \dotplus \mathsf{B}_4) = 16/64$ *and* $|\mathsf{B}_4 \dotplus \mathsf{B}_4| = 7$.
(6) *All* $L \in \mathrm{Circ}(6)$, *where* $\mathrm{cd}(L) = 14/64$ *and* $|L| = 6$.
(7) *All* $L \in \mathrm{Circ}(7)$, *where* $\mathrm{cd}(L) = 11/64$ *and* $|L| = 7$.
(8) *All edge gluings L of* B_4 *and* N_5; $\mathrm{cd}(L) = 10/64$ *and* $|L| = 7$.
(9) $L = \mathsf{B}_4 \dotplus \mathsf{N}_5$ *or* $L = \mathsf{N}_5 \dotplus \mathsf{B}_4$; $\mathrm{cd}(L) = 10/64$ *and* $|L| = 8$.
(10) *All* $L \in \mathrm{Circ}(n)$ *for* $8 \le n \in \mathbb{N}^+$, *where* $\mathrm{cd}(L) = (8 + 3/2^{n-7})/64$ *and* $|L| = n$.

P r o o f. It follows from (the Glued Sum) Lemma 2, (the Glue-B_4) Lemma 4, and (the Circle) Lemma 7 that the congruence densities are correctly given in the lemma. Clearly, so are the sizes of the lattices that occur. We need to show only that every core lattice L with $\mathrm{cd}(L) > 8/64$ is in the list supplied by the lemma. So, assume that L is a core lattice such that $\mathrm{cd}(L) > 8/64$.

By (the Three-Deficits) Lemma 1, we can assume that $|\mathrm{Jr}(L)|$, $|\mathrm{Mr}(L)| \in \{0,1,2\}$. Clearly, $|\mathrm{Jr}(L)| = 0$ if and only if $|\mathrm{Mr}(L)| = 0$ if and only if L is a chain. As the only core chain, the singleton chain, is in the list, we can assume that $|\mathrm{Jr}(L)|$, $|\mathrm{Mr}(L)| \in \{1,2\}$.

Assume that $|\mathrm{Jr}(L)| = 1$. Pick two incomparable elements a and b (in notation, $a \parallel b$), and denote $a \vee b$ and $a \wedge b$ by v and u, respectively. So v is the only nontrivial join. If $L = \{u, a, b, v\}$, then $L = \mathsf{B}_4$ is in the list. Assume that x is an additional element of L. Consider the following six intervals of L:

$$[0, u], \quad [v, 1], \quad [u, a], \quad [u, b], \quad [a, v], \quad [b, v]. \tag{3.10}$$

If we had that $x \parallel v$, then $x \vee v$ would be a second nontrivial join, which is excluded. If x was incomparable with u, then we would have $x \vee u = v$, since v is the only nontrivial join. But then $a \le x$ or $b \le x$ would violate $x \parallel u$, while $x \le a$ or $x \le b$ would contradict $x \vee u = v$. Therefore,

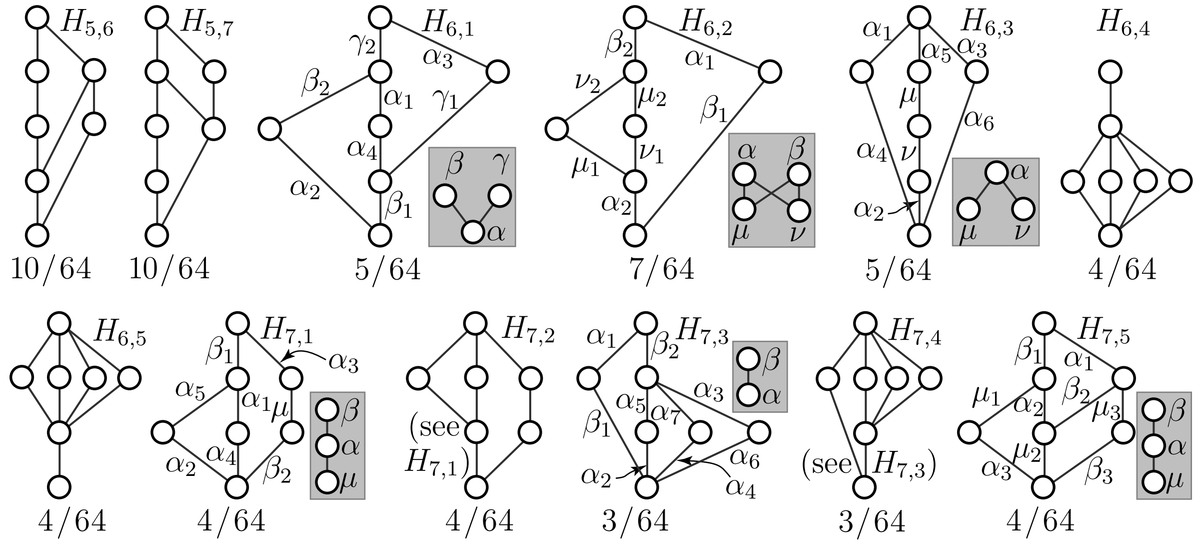

Figure 6. The seven-element lattices $H_{5,6}$–$H_{7,5}$.

$\{a, b, x\}$ would form a three-element antichain, contradicting (the Antichain) Lemma 6. Hence, x is comparable with both u and v. So x is either in one of the first two intervals listed in (3.10), or $x \in [u, v]$. Assume that $x \in [u, v]$. As we know from (the Antichain) Lemma 6 that $\{a, b, x\}$ is not an antichain, x belongs to one of the last four intervals given in (3.10). We have seen that

$$\text{each } x \in L \setminus \{u, a, b, v\} \quad \text{is in one of the six intervals given in (3.10).} \tag{3.11}$$

We cannot have two incomparable elements in $[a, v]$, as these two elements and b would form a three-element antichain, contradicting (the Antichain) Lemma 6. By symmetry, $[b, v]$ cannot contain two incomparable elements either. Two incomparable elements in the remaining four intervals in (3.10) would give a nontrivial join distinct from v, which would violate $|\mathrm{Jr}(L)| = 1$. We have obtained that

$$\text{each of the six intervals given in (3.10) is a chain.} \tag{3.12}$$

Therefore, as $L = \mathrm{Cor}(L)$, we obtain that $[0, u]$ and $[v, 1]$ are singletons. This fact, (3.11), (3.12), and $|\mathrm{Ji}(L)| = 1$ imply that $L \in \mathrm{Circ}(n)$, and so L is in the list.

We have seen that L is in the list when $|\mathrm{Jr}(L)| = 1$. By duality, L is in the list when $|\mathrm{Mr}(L)| = 1$. We are left with the case where $|\mathrm{Jr}(L)| = |\mathrm{Mr}(L)| = 2$. Then, by (the Antichain) Lemma 6 and Lemma 9, there are $4 \le n_1, n_2 \in \mathbb{N}^+$, $L_1 \in \mathrm{Circ}(n_1)$, and $L_2 \in \mathrm{Circ}(n_2)$ such that L is the glued sum or an edge-gluing of L_1 and L_2.

First, let $L = L_1 \dotplus L_2$. We obtain from (the Circle) Lemma 7 that, for $n_i = 4, 5, 6, \ldots$, $\mathrm{cd}(L_i)$ equals $1/2$, $5/16$, $7/32$, $\ldots$, respectively. By (the Glued Sum) Lemma 2,

$$8/64 < \mathrm{cd}(L) = \mathrm{cd}(L_1) \cdot \mathrm{cd}(L_2).$$

But the product of two (not necessarily distinct) entries of the sequence $(1/2, 5/16, 7/32, \ldots)$ is greater than $8/64$ if and only if two copies of $1/2$ are multiplied or we form the product $(1/2)\cdot(5/16)$. Thus, L is $\mathsf{B}_4 \dotplus \mathsf{B}_4$, $\mathsf{B}_4 \dotplus \mathsf{N}_5$, or $\mathsf{N}_5 \dotplus \mathsf{B}_4$, whereby L is in the list, as required.

Second, assume that L is an edge-gluing of L_1 and L_2. Since $\mathrm{cd}(L) > 8/64$, (the Two-Circles) Lemma 8 implies that $(n_1, n_2) \in \{(4, 4), (4, 5), (5, 4)\}$. Hence, as (4) and (8) in Lemma 10 show, L is in the list again. Thus, the proof of Lemma 10 is complete. □

Based on the work carried out in this paper so far, the following proof is short.

P r o o f. Lemma 10 implies Theorems 1 and 2 in a trivial way. □

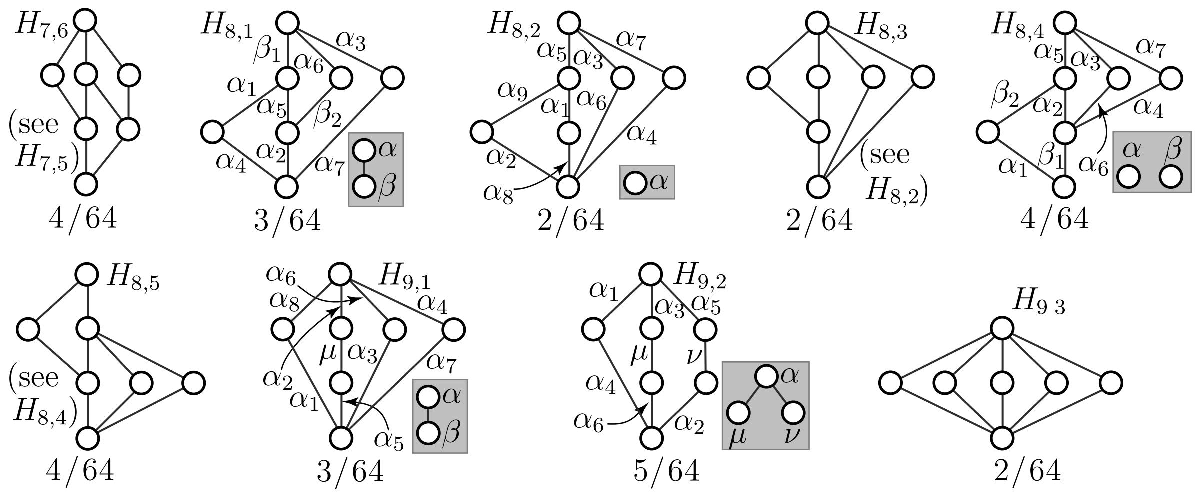

Figure 7. The seven-element lattices $H_{7,6}$–$H_{9,3}$

4. The congruence densities of seven-element lattices

By Kyuno [12], there are exactly 53 seven-element lattices (up to isomorphism). Here, we list all of them in Fig. 4–7, and we compute their congruence densities. Out of these 53 lattices, we have already mentioned $H_{5,3}$ and $H_{5,7}$ as examples of glued sums and edge-gluing constructs, respectively. With these two exceptions, this section is not needed in the earlier parts of the paper.

To make the completeness of our list easy to verify, the jth lattice in the ith line of Kyuno's drawings is denoted by $H_{i,j}$. In most cases, $\mathrm{cd}(H_{i,j})$ is trivially obtained by (the Glued Sum) Lemma 2, (the Glue-B_4) Lemma 4, (the Core) Lemma 3, (the Circle) Lemma 7, or from an earlier case by duality. In all other cases, we diagram $\mathrm{Ji}(\mathrm{Con}(H_{i,j}))$ and adopt the following convention: For each Greek letter ξ in Fig. 4–7, the edges labeled by ξ_1, ξ_2, $\xi_3, \dots$ generate the same congruence. Moreover, the equality of the congruence generated by a ξ_{j-1}-labeled edge and that generated by a ξ_j-labeled edge follows directly from applying (2.5).

Based on the list, Fig. 4–7, we obtain $\mathrm{lcd}_7(k)$ for $k \in \{1, \dots, 14\}$, and we also obtain that $\mathrm{lcd}_7(k)$ does not exist for $k > 14$. With respect to the common denominator $64 = 2^{7-1}$, the numerators of the $\mathrm{lcd}_7(k)$ for $k \in \{1, \dots, 14\}$ are given in the Table 1. Note that these numerators are the sizes $|\mathrm{Con}(H_{i,j})|$. Each "i, j" entry gives one of the witnesses $H_{i,j}$ of the corresponding congruence density. Based on (the Core) Lemma 3, one could easily obtain the counterparts of the Table 1 for $|L| = 5$ and $|L| = 6$ from Fig. 4–7. Obtaining analogous tables for $|L| = 8, 9, \dots$ would be excessively tedious, as the computational complexity rapidly increases with $|L|$.

5. Conclusion

For $n \geq 7$, let $\mathrm{SCL}(n)$ denote the set of the numbers of congruence relations of n-element lattices $L = (L; \vee, \wedge)$. For all positive integers $k \geq 6$ and $n \geq \max\{7, k\}$, we determined the k-th

Table 1. The numerators of the $\mathrm{lcd}_7(k)$ for $k \in \{1, \ldots, 14\}$.

k	1	2	3	4	5	6	7	8
$64 \cdot \mathrm{lcd}_7(k)$	64	32	20	16	14	11	10	8
$H_{\mathrm{a\ witness}}$	1,1	1,2	1,4	3,1	1,6	1,8	4,5	2,4
k (cont'd)	9	10	11	12	13	14	15	16
$64 \cdot \mathrm{lcd}_7(k)$	7	6	5	4	3	2	—	—
$H_{\mathrm{a\ witness}}$	6,2	3,3	6,1	4,3	7,3	8,2	—	—

largest element $\mathrm{lnc}(n, k)$ of $\mathrm{SCL}(n)$. Furthermore, we gave an explicit structural description of the lattices witnessing $\mathrm{lnc}(n, k)$. (Analogous results for $k \leq 5$ were previously known.)

By examining the kth largest number of the *congruences* within the class of *all* n-element *lattices*, this paper could inspire further research where one or more of the concepts —"kth", "congruences","all", and "lattices"— are replaced with alternative, yet meaningful and fruitful notions. (The first paragraph of the Introduction already highlighted a few existing examples that align with this direction.)

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