AN INDEPENDENCE THEOREM FOR ORDERED SETS OF PRINCIPAL CONGRUENCES AND AUTOMORPHISM GROUPS OF BOUNDED LATTICES

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Dedicated to the memory of professor László Megyesi (1939–2015), former head of the Department of Algebra and Number Theory of the University of Szeged

ABSTRACT. For a bounded lattice L, the principal congruences of L form a bounded ordered set $\mathrm{Princ}(L)$. G. Grätzer proved in 2013 that every bounded ordered set can be represented in this way. Also, G. Birkhoff proved in 1946 that every group is isomorphic to the group of automorphisms of an appropriate lattice. Here, for an arbitrary bounded ordered set P with at least two elements and an arbitrary group G, we construct a selfdual lattice L of length sixteen such that $\mathrm{Princ}(L)$ is isomorphic to P and the automorphism group of L is isomorphic to G.

1. Introduction

1.1. **Our motivation and the result.** For a bounded lattice L, that is, a lattice with 0 and 1, let $\mathrm{Princ}(L) = \langle \mathrm{Princ}(L); \subseteq \rangle$ be the ordered set (also known as poset) of principal congruences of L. It is a bounded ordered set. G. Grätzer [16] proved that every bounded ordered set is isomorphic to $\mathrm{Princ}(L)$ for some lattice L of length 5. The ordered sets $\mathrm{Princ}(L)$ of countable but not necessarily bounded lattices L were characterized in Czédli [4]. Also, let $\mathrm{Aut}(L) = \langle \mathrm{Aut}(L); \circ \rangle$ stand for the group of automorphisms of L. We know from G. Birkhoff [2] that every group is isomorphic to $\mathrm{Aut}(L)$ for an appropriate lattice L. Our goal is to prove the following "simultaneous representation theorem" or, in another terminology, an "independence theorem".

Theorem 1.1. *If P is a bounded ordered set with at least two elements and G is an arbitrary group, then there exists a selfdual lattice L of length sixteen such that $\mathrm{Princ}(L)$ and $\mathrm{Aut}(L)$ are isomorphic to P and G, respectively. If P and G are finite, then we can construct a finite L with these properties.*

The theorem asserts that $\mathrm{Princ}(L)$ and $\mathrm{Aut}(L)$ are as independent as the trivial implication $|\mathrm{Princ}(L)| = 1 \Longrightarrow |\mathrm{Aut}(L)| = 1$ allows. Note that for *finite* lattices L, the lattice $\mathrm{Con}(L)$ of all congruences and $\mathrm{Aut}(L)$ are also independent in the same sense by a result V. A. Baranskiĭ [1] and A. Urquhart [26]; their result is generalized by G. Grätzer and E. T. Schmidt [21] and G. Grätzer and F. Wehrung [23].

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1.2. **Method and outline.** The rest of the paper is devoted to the proof of Theorem 1.1. Due to earlier results, the present paper is not particularly long. For those familiar with Czédli [7], the next paragraph and Examples 3.1–3.3, see later, are sufficient to understand our construction and the idea of the proof.

Let P and G be as in Theorem 1.1. Each of the papers G. Grätzer [16] and [17] and Czédli [4], [5], and [7] associates a lattice L with P such that $\mathrm{Princ}(L) \cong P$. In these papers, we start with a set $\{[a_p, b_p] : 0 \neq p \in P\}$ of "key" prime intervals, and add certain additional elements, which are organized into "gadgets", to obtain an appropriate L. Here, to get rid of the automorphisms inherited from P, we replace the key prime intervals with distinct simple bounded lattices that have no nontrivial automorphism. These lattices are constructed in Section 2. Next, the result from G. Sabidussi [25] allows us to represent G as the automorphism group of a graph $\langle V; E\rangle$. For each $v \in V$, we add a prime interval $[a_v, b_v]$ to our lattice together with appropriate gadgets to force that these new prime intervals generate the largest congruence. (Later, to make these intervals recognizable, we enlarge them to disjoint copies of an appropriate simple lattice.) Whenever $\langle u, v\rangle \in E$, we add a gadget between $[a_u, b_u]$ and $[a_v, b_v]$. The new gadgets encode the graph into the lattice without changing $\mathrm{Princ}(L)$. These details are discussed in Section 3, where both the quasi-coloring technique developed in Czédli [3]–[7] and the ideas of [4]–[7] and G. Grätzer [16] are intensively used. However, to follow the paper, it suffices only to keep [7] nearby.

2. Graphs and rigid simple lattices

By a graph we mean a pair $\langle V; E\rangle$ where V is a nonempty set and E is a subset of the set of two-element subsets of V. We refer to V and E as the *vertex set* and the *edge set* of the graph, respectively. The following statement is due to G. Sabidussi [25]; see also R. Frucht [11] and [12] for the finite case.

Lemma 2.1 ([25]). *For every group G, there exists a graph $\langle V; E\rangle$ such that G is isomorphic to* $\mathrm{Aut}(\langle V; E\rangle)$.

Next, we borrow some concepts from Czédli [7]. A *quasiorder* is a reflexive transitive relation. For a lattice or ordered set $L = \langle L; \leq\rangle$ and $x, y \in L$, $\langle x, y\rangle$ is called an *ordered pair* of L if $x \leq y$. If $x = y$, then $\langle x, y\rangle$ is a *trivial ordered pair.* The set of ordered pairs and that of nontrivial ordered pairs of L are denoted by $\mathrm{Pairs}^{\leq}(L)$ and $\mathrm{Pairs}^{<}(L)$, respectively. If $X \subseteq L$, then $\mathrm{Pairs}^{\leq}(X)$ will stand for $X^2 \cap \mathrm{Pairs}^{\leq}(L)$. We also need the notation $\mathrm{Pairs}^{\prec}(L) := \{\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(X) : x \prec y\}$ for the set of *covering pairs.* By a *quasi-colored lattice* we mean a structure

$$\mathcal{L} = \langle L, \leq; \gamma; H, \nu\rangle$$

where $\langle L; \leq\rangle$ is a lattice, $\langle H; \nu\rangle$ is a quasiordered set, $\gamma\colon \mathrm{Pairs}^{\leq}(L) \to H$ is a surjective map called *coloring*, and for all $\langle u_1, v_1\rangle, \langle u_2, v_2\rangle \in \mathrm{Pairs}^{\leq}(L)$,

(C1) if $\big\langle \gamma(\langle u_1, v_1\rangle)\,,\, \gamma(\langle u_2, v_2\rangle)\big\rangle \in \nu$, then $\mathrm{con}(u_1, v_1) \leq \mathrm{con}(u_2, v_2)$ and
(C2) if $\mathrm{con}(u_1, v_1) \leq \mathrm{con}(u_2, v_2)$, then $\big\langle \gamma(\langle u_1, v_1\rangle)\,,\, \gamma(\langle u_2, v_2\rangle)\big\rangle \in \nu$.

This concept is taken from Czédli [4] and [7]; for some earlier variants, the reader is referred to G. Grätzer, H. Lakser, and E.T. Schmidt [19], G. Grätzer [14, page 39], and Czédli [3]. For a quasiordered set $\langle H, \nu\rangle$, we let $\Theta_\nu = \nu \cap \nu^{-1}$. Then Θ_ν is an equivalence relation, and the definition

$$\langle x/\Theta_\nu, y/\Theta_\nu\rangle \in \nu/\Theta_\nu \overset{\mathrm{def}}{\iff} \langle x, y\rangle \in \nu \tag{2.1}$$

turns the quotient set H/Θ_ν into an ordered set $\langle H;\nu\rangle/\Theta_\nu := \langle H/\Theta_\nu;\nu/\Theta_\nu\rangle$. The importance of quasi-colored lattices is explained by the following lemma, which is a straightforward consequence of (C1) and (C2); see Czédli [4, Lemma 2.1] or [7, Lemma 4.7].

Lemma 2.2. *If $\mathcal{L} = \langle L,\le;\gamma;H,\nu\rangle$ is a quasi-colored lattice, then* Princ(L) *is isomorphic to $\langle H;\nu\rangle/\Theta_\nu$.*

To recall two concepts from G. Grätzer [14, pages 42–43], let L_1 be a sublattice of a lattice L_2. If each congruence of L_1 is the restriction of an appropriate congruence of L_2, then L_2 is a *congruence-reflecting extension* of L_1. If the "generation map" $\mathrm{Con}(L_1) \to \mathrm{Con}(L_2)$, defined by $\alpha \mapsto \mathrm{con}_{L_2}(\alpha)$, is bijective or, equivalently, the restriction map $\mathrm{Con}(L_2) \to \mathrm{Con}(L_1)$ is bijective, then L_2 is a *congruence-preserving extension* of L_1. To give an example for congruence-reflecting extensions, which will be important for us, we define an easy way to extend a prime interval into a bounded lattice. Let L be a lattice, $\langle a,b\rangle \in \mathrm{Pairs}^{\prec}(L)$, and let K be an arbitrary bounded lattice. The *extension $L(a,b,K)$ of L at $\langle a,b\rangle$ with K* is the union $L\cup K$ such that a and b are identified with 0_K and 1_K, respectively but otherwise the union is disjoint, that is, $L\cap K = \{a,b\} = \{0_K,1_K\}$, and the ordering is the extension of both the orderings of L and K in the unique way such that L and K are sublattices. In particular, for $x\in L\setminus K$ and $y \in K\setminus L$, $x \le_{a,b,K} y$ iff $x\le a$ and $y\le_{a,b,K} x$ iff $b\le x$.

An ordered pair in a subdirectly irreducible lattice is called *critical* if it generates the smallest nontrivial congruence; see, for example, R. Freese, J. Ježek, and J. B. Nation [10, page 55]. Motivated by this terminology, we say that a covering pair $\langle a,b\rangle \in \mathrm{Pairs}^{\prec}(L)$ is *locally critical* if the principal congruence $\mathrm{con}(x,a) = \mathrm{con}_L(x,a)$ collapses $\langle a,b\rangle$ for every $x<a$ and, dually, $\langle a,b\rangle \in \mathrm{con}(b,y)$ for every $y>b$. We emphasize that a locally critical pair is always a covering pair by definition.

Lemma 2.3. *If L is a lattice, $\langle a,b\rangle \in \mathrm{Pairs}^{\prec}(L)$ is a locally critical pair, and K is a bounded lattice, then $L(a,b,K)$ is a congruence-reflecting extension of L. If, in addition, L is simple, then this extension is congruence-preserving.*

We cannot omit the stipulation that $\langle a,b\rangle$ is a locally critical pair; this is witnessed by the four-element non-chain lattice playing the role of L and the five-element modular non-distributive lattice M_3 as K.

Proof of Lemma 2.3. For brevity, let $L' = L(a,b,K)$. For a lattice M and ordered pairs $\langle x_1,y_1\rangle, \langle x_2,y_2\rangle \in \mathrm{Pairs}^{\le}(M)$, we define

$$\begin{aligned}\langle x_1,y_1\rangle \twoheadrightarrow_{\mathrm{dn}} \langle x_2,y_2\rangle &\overset{\mathrm{def}}{\iff} x_2 = y_2\wedge x_1 \text{ and } y_2\le y_1,\\ \langle x_1,y_1\rangle \twoheadrightarrow_{\mathrm{up}} \langle x_2,y_2\rangle &\overset{\mathrm{def}}{\iff} y_2 = x_2\vee y_1 \text{ and } x_1\le x_2.\end{aligned} \tag{2.2}$$

We write $\langle x_1,y_1\rangle \twoheadrightarrow \langle x_2,y_2\rangle$ for the disjunction of

$$\langle x_1,y_1\rangle \twoheadrightarrow_{\mathrm{dn}} \langle x_2,y_2\rangle \text{ and } \langle x_1,y_1\rangle \twoheadrightarrow_{\mathrm{up}} \langle x_2,y_2\rangle.$$

If $x'\le x\le y\le y'$ in a lattice, then the ordered pair $\langle x',y'\rangle$ is said to be *superior* to the ordered pair $\langle x,y\rangle$. Let

$$\vec{s}: \langle x_0,y_0\rangle \twoheadrightarrow \langle x_1,y_1\rangle \twoheadrightarrow \cdots \twoheadrightarrow \langle x_n,y_n\rangle \tag{2.3}$$

be a *sequence of (congruence) perspectivities* (over M). We say that n, $\langle x_0, y_0\rangle$, and $\langle x_n, y_n\rangle$ are the *length*, the *initial ordered pair*, and the *terminal ordered pair* of $\vec s$, respectively. Let S be a finite set of sequences of perspectivities, $H \subseteq M^2$, and $\langle u, v\rangle \in \text{Pairs}^{\leq}(M)$. We say that S *witnesses* the containment $\langle u, v\rangle \in \text{con}_M(H)$ if there exist a nonnegative integer k and elements

$$w_0 < w_1 < \cdots < w_k \tag{2.4}$$

in M such that $w_0 = u$, $w_k = v$, and for each $i \in \{1, \dots, k\}$, there is an $\vec s \in S$ with initial ordered pair in H and terminal ordered pair superior to $\langle w_{i-1}, w_i\rangle$. Modifying a well-known fact on generation of lattice congruences, see G. Grätzer [15, Section III.1], in an obvious way, we obtain that

(2.5) $\langle u, v\rangle \in \text{con}_M(H)$ iff there exists a finite set S of sequences of perspectivities that witnesses the containment $\langle u, v\rangle \in \text{con}_M(H)$.

We interrupt the proof to formulate the following auxiliary statement; remember that $\langle a, b\rangle$ is a locally critical pair of L and $L' = L(a, b, K)$.

Claim 2.4. If $\langle x, y\rangle \in \text{Pairs}^{\leq}(L)$ and $\langle x', y'\rangle \in \text{Pairs}^{\leq}(L') \setminus \text{Pairs}^{\leq}(L)$ such that $\langle x, y\rangle \twoheadrightarrow_{\text{dn}} \langle x', y'\rangle$, then $y' \in K \setminus L$ and

(i) either $x' = y' \in K \setminus L$,

(ii) or $x' \leq a$ and $\langle x', b\rangle \in \text{con}_L(x, y)$.

Proof of Claim 2.4. If we had $y' \in L$, then $x' = x \wedge y'$ would also be in L, contradicting $\langle x', y'\rangle \notin \text{Pairs}^{\leq}(L)$. Hence, $y' \notin L$. Thus, $a < y' < b \leq y$. Assume that $x' \in K \setminus L$. Since $x' \leq x \in L$, we have that $b \leq x$. But then $y' < x$ and $x' = x \wedge y' = y' \in K \setminus L$, which means the satisfaction of (i). So, we can assume that $x' \in L$. Since $x' \leq y' \in K \setminus L$, we have that $x' \leq a$. We distinguish two cases. First, assume that $x' < a$. Since $x' \leq x \wedge a \leq x \wedge y' = x'$, we have that $x' = x \wedge a$. Also, $a \leq y' \leq y$, that is, $\langle x, y\rangle \twoheadrightarrow_{\text{dn}} \langle x', a\rangle$. Trivially, or using (2.5) for the singleton set consisting of the sequence $\langle x, y\rangle \twoheadrightarrow_{\text{dn}} \langle x', a\rangle$ of length 1, $\langle x', a\rangle \in \text{con}_L(x', a) \leq \text{con}_L(x, y)$. Since $\langle a, b\rangle$ is a locally critical pair, $\langle a, b\rangle \in \text{con}_L(x', a) \leq \text{con}_L(x, y)$. By transitivity, $\langle x', b\rangle \in \text{con}_L(x, y)$, which means that (ii) holds. Second, assume that $x' = a$. Since $x \geq x' = a$, we have that $a \leq x \wedge b \leq b$. These elements are in L, where $a \prec_L b$, so either $x \wedge b = b$, or $x \wedge b = a$. The former of these two equalities would give $y' < b \leq x$, leading to $a = x' = x \wedge y' = y' \notin L$, which is impossible. Thus, $x \wedge b = a$. Since $b \leq y$, we have that $\langle x, y\rangle \twoheadrightarrow_{\text{dn}} \langle x', b\rangle = \langle a, b\rangle$. Hence, $\langle x', b\rangle = \langle a, b\rangle \in \text{con}_L(x, y)$ and (ii) holds again. This proves Claim 2.4. $\square$

We continue the proof of Lemma 2.3 by introducing the following concept. Besides $\vec s$ from (2.3), let

$$\vec s\,' : \langle x_0', y_0'\rangle \twoheadrightarrow \langle x_1', y_1'\rangle \twoheadrightarrow \cdots \twoheadrightarrow \langle x_n', y_n'\rangle \tag{2.6}$$

be another sequence of perspectivities. We say that $\vec s\,'$ is *superior* to $\vec s$ if

- the two sequences are of the same length (which will be denoted by n),
- $\langle x_{i-1}, y_{i-1}\rangle \twoheadrightarrow_{\text{up}} \langle x_i, y_i\rangle$ iff $\langle x_{i-1}', y_{i-1}'\rangle \twoheadrightarrow_{\text{up}} \langle x_i', y_i'\rangle$, for all $i \in \{1, \dots, n\}$ (that is, $\vec s$ and $\vec s\,'$ are of the same *pattern*),
- and $\langle x_i', y_i'\rangle$ is superior to $\langle x_i, y_i\rangle$ for $i \in \{0, \dots, n\}$.

Claim 2.5. Let $\vec s$ be a sequence of perspectivities written in the form (2.3). If $\langle x_0', y_0'\rangle$ is an ordered pair superior to $\langle x_0, y_0\rangle$, then $\langle x_0', y_0'\rangle$ can be continued to a sequence $\vec s\,'$ of perspectivities such that $\vec s\,'$ is superior to $\vec s$.

Proof of Claim 2.5. By duality, we can assume that $\langle x_0, y_0\rangle \twoheadrightarrow_{\mathrm{dn}} \langle x_1, y_1\rangle$. Let k be the largest subscript such that $\langle x_0, y_0\rangle \twoheadrightarrow_{\mathrm{dn}} \langle x_1, y_1\rangle \twoheadrightarrow_{\mathrm{dn}} \dots \twoheadrightarrow_{\mathrm{dn}} \langle x_k, y_k\rangle$, that is, k is the number of the $\twoheadrightarrow_{\mathrm{dn}}$ (down arrows) at the beginning of $\vec{s}$. For $i \in \{1, \dots, k\}$, let $y'_i := y_i$ and $x'_i := x'_0 \wedge x_i$. Since $x'_{i-1} \wedge y'_i = (x'_0 \wedge x_{i-1}) \wedge y'_i = x'_0 \wedge (x_{i-1} \wedge y_i) = x'_0 \wedge x_i = x'_i$, we obtain easily that $\langle x'_0, y'_0\rangle \twoheadrightarrow_{\mathrm{dn}} \langle x'_1, y'_1\rangle \twoheadrightarrow_{\mathrm{dn}} \dots \twoheadrightarrow_{\mathrm{dn}} \langle x'_k, y'_k\rangle$. For j in the (possibly empty) set $\{k + 1, \dots, n\}$, we let $\langle x'_j, y'_j\rangle := \langle x_j, y_j\rangle$. Since $x'_k \leq x_k$ and $y'_k = y_k$, it is clear that, in case $k < n$, $\langle x'_k, y'_k\rangle \twoheadrightarrow_{\mathrm{up}} \langle x'_{k+1}, y'_{k+1}\rangle$, while $\langle x'_t, y'_t\rangle \twoheadrightarrow \langle x'_{t+1}, y'_{t+1}\rangle$ for $t > k$ is obvious. Thus, $\vec{s}\,' : \langle x'_0, y'_0\rangle \twoheadrightarrow \dots \twoheadrightarrow \langle x'_n, y'_n\rangle$ is a sequence of perspectivities, and it is clearly superior to $\vec{s}$. This proves Claim 2.5 $\square$

Now, to continue the proof of Lemma 2.3, consider a finite set S of sequences of perspectivities. The *total length* of S is $\max\{\mathrm{length}(\vec{s}) : \vec{s} \in S\}$. To prove that L' is a congruence-reflecting extension of L, it suffices to show that every $\alpha \in \mathrm{Con}(L)$ equals the restriction $(\mathrm{con}_{L'}(\alpha))\rceil_L$ of $\mathrm{con}_{L'}(\alpha)$ to L. The inclusion $\alpha \subseteq (\mathrm{con}_{L'}(\alpha))\rceil_L$ is obvious. To prove the converse inclusion, assume that $\langle u, v\rangle \in (\mathrm{con}_{L'}(\alpha))\rceil_L$, that is, $\langle u, v\rangle \in \mathrm{Pairs}^{\leq}(L)$ and $\langle u, v\rangle \in \mathrm{con}_{L'}(\alpha)$. In the sense of (2.5), take a finite set S of sequences of perspectivities over L' such that S witnesses the containment $\langle u, v\rangle \in \mathrm{con}_{L'}(\alpha)$ and S is of minimal total length. We assert that this total length is zero. Suppose, for a contradiction, that the total length of S is positive. To obtain a contradiction, it suffices to show that whenever $\vec{s} \in S$ is of positive length, then $\vec{s}$ can be abbreviated. Let $\vec{s} \in S$ be of the form (2.3) such that $\mathrm{length}(\vec{s}) > 0$. We know that $\langle x_0, y_0\rangle \in \alpha$. We can assume that $\langle x_1, y_1\rangle \notin \mathrm{Pairs}^{\leq}(L)$, because otherwise $\langle x_1, y_1\rangle$ would belong to α and we could abbreviate $\vec{s}$ by omitting its initial pair. By duality, we can assume that $\langle x_0, y_0\rangle \twoheadrightarrow_{\mathrm{dn}} \langle x_1, y_1\rangle$, If $x_1 = y_1$, then $x_i = y_i$ for all $i \in \{1, \dots, \mathrm{length}(\vec{s})\}$, but this contradicts (2.4). Therefore, Claim 2.4 yields that $y_1 \in K \setminus L$, $x_1 \leq a$ and $\langle x_1, b\rangle \in \mathrm{con}_L(x_0, y_0) \leq \alpha$. Observe that $\langle x_1, b\rangle$ is superior to $\langle x_1, y_1\rangle$. Omit $\langle x_0, y_0\rangle$ from $\vec{s}$, and apply Claim 2.5 to the remaining sequence, which we denote by $\vec{s}^{\,-}$. In this way, we obtain a sequence $\vec{s}\,'$ with initial pair $\langle x_1, b\rangle \in \alpha$ such that $\vec{s}\,'$ is superior to $\vec{s}^{\,-}$. Clearly, we can replace $\vec{s}$ in S with $\vec{s}\,'$; note that $\mathrm{length}(\vec{s}\,') = \mathrm{length}(\vec{s}^{\,-}) = \mathrm{length}(\vec{s}) - 1$. Thus, each sequence in S with positive length can be replaced with a shorter sequence, which contradicts the assumption that S was of minimal total length. Therefore, the total length of S is 0, and all sequences of perspectivities in S are of length 0. By the convexity of α-blocks, all the $\langle w_{i-1}, w_i\rangle$, see (2.4), belong to α. So does $\langle u, v\rangle = \langle w_0, w_k\rangle$ by transitivity. Consequently, $\alpha = (\mathrm{con}_{L'}(\alpha))\rceil_L$. This proves that L' is a congruence-reflecting extension of L.

Finally, assume that K is simple. To show that the restriction $\alpha\rceil_L$ of an arbitrary $\alpha \in \mathrm{Con}(L')$ determines α, it suffices only to consider the nontrivial ordered pairs. So let $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(L')$ be a nontrivial ordered pair. If $\langle x, y\rangle \in L$, then $\alpha\rceil_L$ clearly determines if $\langle x, y\rangle$ is in α or not. If $\langle x, y\rangle \in \mathrm{Pairs}^{\leq}(K)$, then $\langle x, y\rangle \in \alpha \iff \langle 0_K, 1_K\rangle = \langle a, b\rangle \in \alpha \iff \langle a, b\rangle \in \alpha\rceil_L$. We are left with the case where $|\{x, y\} \cap L| = 1$; by duality, we can assume that $x \in L$ and $y \notin L$. Then $\langle x, y\rangle \in \alpha$ iff $\langle x, a\rangle$ and $\langle a, y\rangle \in \alpha$, and both belong to the scope of the earlier cases. Hence, L' is a congruence-preserving extension of L. This completes the proof of Lemma 2.3 $\square$

In a simple lattice, every covering pair is a locally critical pair. Thus, we obtain the following statement.

Corollary 2.6. *If L and K are simple lattices, K is bounded and $\langle a,b\rangle \in \text{Pairs}^{\prec}(L)$, then $L(a,b,K)$ is also a simple lattice.*

Given a quasi-colored lattice $\mathcal{L} = \langle L, \leq; \gamma; H, \nu\rangle$, a locally critical pair $\langle a,b\rangle \in \text{Pairs}^{\prec}(L)$, and a simple bounded lattice K, we define a new quasi-colored lattice

$$\mathcal{L}(a,b,K) = \langle L(a,b,K), \leq_{a,b,K}; \gamma_{a,b,K}; H, \nu\rangle$$

as follows. The lattice $L(a,b,K)$ has already been defined; see Lemma 2.3. Since $\text{Pairs}^{\leq}(L) \subseteq \text{Pairs}^{\leq}(L(a,b,K))$, we can define $\gamma_{a,b,K}$ as the extension of γ such that, for $\langle x,y\rangle \in \text{Pairs}^{\leq}(L(a,b,K)) \setminus \text{Pairs}^{\leq}(L)$,

$$\gamma_{a,b,K}(\langle x,y\rangle) = \begin{cases} 0, & \text{if } x = y, \\ \gamma(\langle a,b\rangle), & \text{if } x,y \in K \text{ and } x \neq y, \\ \gamma(\langle a,y\rangle), & \text{if } x \in K \text{ and } y \notin K, \\ \gamma(\langle x,b\rangle), & \text{if } x \notin K \text{ and } y \in K. \end{cases} \tag{2.7}$$

Let us emphasize that, as opposed to Lemma 2.3, we only define $\mathcal{L}(a,b,K)$ if K is simple.

Lemma 2.7. *$\mathcal{L}(a,b,K)$ is a quasi-colored lattice.*

Proof. Let $\mathcal{L}' = \mathcal{L}(a,b,K)$ and $L' = L(a,b,K)$. We claim that, for $\langle x,y\rangle \in \text{Pairs}^{\leq}(L') \setminus \text{Pairs}^{\leq}(L)$,

(2.8) $\langle x,y\rangle$ and the corresponding pair on the right of (2.7) generate the same congruence of L'.

This is clear for the first two lines on the right of (2.7). Consider the third line, that is, assume that $x \in K \setminus L$ and $y \in L \setminus K$. Since K is simple, we have that $\text{con}_{L'}(x,y) = \text{con}_{L'}(x,b) \vee \text{con}_{L'}(b,y) = \text{con}_{L'}(a,b) \vee \text{con}_{L'}(b,y) = \text{con}_{L'}(a,y)$. A dual argument works for the fourth line, proving (2.8).

For $\langle u_i, v_i\rangle \in \text{Pairs}^{\leq}(L')$, let $\langle \hat{u}_i, \hat{v}_i\rangle$ be the "corresponding pair" in the sense of (2.8). We know from Lemma 2.3 that L' is a congruence-preserving extension of L. Hence, the map $\text{Con}(L) \to \text{Con}(L')$, defined by $\alpha \mapsto \text{con}_{L'}(\alpha)$, is a lattice isomorphism. Thus, using that $\text{con}_{L'}(\text{con}_L(u,v)) = \text{con}_{L'}(u,v)$,

$$\text{con}_L(\hat{u}_1,\hat{v}_1) \leq \text{con}_L(\hat{u}_2,\hat{v}_2) \iff \text{con}_{L'}(\hat{u}_1,\hat{v}_1) \leq \text{con}_{L'}(\hat{u}_2,\hat{v}_2). \tag{2.9}$$

Using the definition of $\gamma_{a,b,K}$, the validity of (C1) and (C2) in L, (2.9), and (2.8), we obtain that

$$\begin{aligned} &\big\langle \gamma_{a,b,K}(\langle u_1,v_1\rangle), \gamma_{a,b,K}(\langle u_2,v_2\rangle)\big\rangle \in \nu \\ &\iff \big\langle \gamma(\langle \hat{u}_1,\hat{v}_1\rangle), \gamma(\langle \hat{u}_2,\hat{v}_2\rangle)\big\rangle \in \nu \iff \text{con}_L(\hat{u}_1,\hat{v}_1) \leq \text{con}_L(\hat{u}_2,\hat{v}_2) \\ &\overset{(2.9)}{\iff} \text{con}_{L'}(\hat{u}_1,\hat{v}_1) \leq \text{con}_{L'}(\hat{u}_2,\hat{v}_2) \overset{(2.8)}{\iff} \text{con}_{L'}(u_1,v_1) \leq \text{con}_{L'}(u_2,v_2). \end{aligned}$$

Therefore, (C1) and (C2) hold in $\mathcal{L}'$, and $\mathcal{L}'$ is a quasi-colored lattice. □

A lattice or a graph is *automorphism-rigid* if its automorphism group is one-element. We are going to define a class $\{S(\alpha) : \alpha \text{ is an ordinal number}\}$ of pairwise non-isomorphic automorphism-rigid simple lattices of length twelve. Let α be an ordinal number, and let $\text{At}(\alpha) = \{g_\iota : \iota < 6+\alpha\}$ and $\text{CoAt}(\alpha) = \{g^{\iota\mu} : \iota < \mu < 6+\alpha\}$. We agree that these two sets are disjoint from each other and from $\{0,1\}$. On the set $T(\alpha) := \text{At}(\alpha) \cup \text{CoAt}(\alpha) \cup \{0,1\}$, we define an ordering as follows: 0

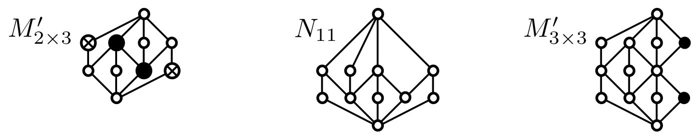

FIGURE 1. $M'_{2\times 3}$ and $M'_{3\times 3}$

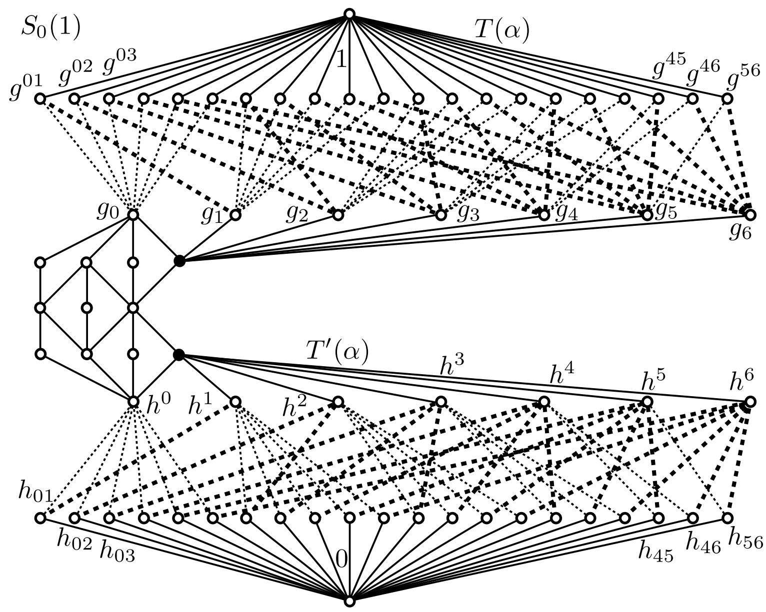

FIGURE 2. $S_0(\alpha)$ for $\alpha = 1$

and 1 are the bottom and top elements, $\mathrm{At}(\alpha)$ and $\mathrm{CoAt}(\alpha)$ are the sets of atoms and coatoms, respectively, and, for $\iota < 6+\alpha$ and $\kappa < \mu < 6+\alpha$,

$$g_\iota < g^{\kappa\mu} \overset{\text{def}}{\iff} \iota \in \{\kappa, \mu\}.$$

Similarly, let $\mathrm{At}'(\alpha) := \{h_{\iota\mu} : \iota < \mu < 6+\alpha\}$, $\mathrm{CoAt}'(\alpha) := \{h^\iota : \iota < 6+\alpha\}$, and $T'(\alpha) := \mathrm{At}'(\alpha) \cup \mathrm{CoAt}'(\alpha) \cup \{0,1\}$ such that $\mathrm{At}'(\alpha)$ and $\mathrm{CoAt}'(\alpha)$ are its sets of atoms and coatoms, respectively, and

$$h_{\kappa\mu} < h^\iota \overset{\text{def}}{\iff} \iota \in \{\kappa, \mu\}.$$

Therefore, $T'(\alpha)$ is the dual of $T(\alpha)$.

Next, consider the lattice $M'_{3\times 3}$ given by Figure 1. The black-filled atom and the black-filled coatom determine a principal ideal I and a principal filter F, respectively. Form the Hall–Dilworth gluing of $T(\alpha)$ and $M'_{3\times 3}$ along F and the principal ideal $\downarrow g_0$. In the next step, form the Hall–Dilworth gluing of the lattice we have just obtained and $T'(\alpha)$ along I and the principal filter $\uparrow h^0$. The lattice we obtain in this way is $S_0(\alpha)$. For $S_0(1)$, see Figure 2.

For $\iota < \mu < 6+\alpha$, an edge of the form $\langle g_\iota, g^{\iota\mu}\rangle$, $\langle g_\mu, g^{\iota\mu}\rangle$, $\langle h_{\iota\mu}, h^\iota\rangle$, and $\langle h_{\iota\mu}, h^\mu\rangle$ is called an *upper left vertex edge*, an *upper right vertex edge*, a *lower left vertex edge*, and a *lower right vertex edge*, respectively. (This terminology is motivated by the connection between $S_0(\alpha)$ and Frucht's graphs; see in the proof later.) The (upper and lower) left vertex edges are indicated by densely dotted lines in Figure 2. The (upper and lower) right vertex edges are thick dotted lines, and there are also "ordinary" edges, the solid lines. We replace each upper left vertex edge and each lower right vertex edge of $S_0(\alpha)$ with a copy of the lattice N_{11} from Figure 1, using disjoint copies for distinct edges. Similarly, we replace each each lower left vertex edge and upper right vertex edge of $S_0(\alpha)$ with the dual $N_{11}^{(d)}$ of N_{11}, using disjoint copies for distinct edges again. The lattice we obtain is denoted by $S(\alpha)$.

Lemma 2.8. *For every ordinal α, $S(\alpha)$ is an automorphism-rigid simple selfdual lattice of length* 12. *Moreover, $S(\alpha) \cong S(\beta)$ iff $\alpha = \beta$.*

Note that there are also results on automorphism-rigid and even endomorphism-rigid *families* of lattices; see, for example, Czédli [6] and G. Grätzer and J. Sichler [22].

Proof. With $V := \{\iota : \iota < 6 + \alpha\}$ and $E := \{\{\iota, \mu\} : \iota < \mu < \alpha\}$, $\langle V; E\rangle$ is a graph. Notice that $T(\alpha)$ is the Frucht graph associated with $\langle V; E\rangle$; see R. Frucht [13] and G. Grätzer [14, Figure 15.1]. We know from G. Grätzer and H. Lakser [18] or G. Grätzer [14, Page 188] that $T(\alpha)$ is a simple lattice. (This is why we use $6 + \alpha$ rather than α in its definition.) Since $T'(\alpha)$, the dual of $T(\alpha)$, and $M'_{3\times 3}$ are also simple, it follows that $S_0(\alpha)$ is simple. Finally, since N_{11} and its dual are simple, Corollary 2.6 yields that $S(\alpha)$ is a simple lattice. Since $S_0(\alpha)$ is of length 8, $S(\alpha)$ is of length 12. Also, it is a ranked lattice, that is, any two maximal chains of $S(\alpha)$ have the same number of elements. While the graph $\langle V; E\rangle$ is encoded in $S_0(\alpha)$, the well-ordered set $\{\iota : \iota < 6 + \alpha\}$ is encoded in $S(\alpha)$ as follows: the elements h^ι and g_ι can be recognized as the elements of height 4 and the elements of dual height 4, respectively. Furthermore, $\iota < \mu$ iff the interval $[g_\iota, g_\iota \vee g_\mu]$ is isomorphic to N_{11} iff $[g_\mu, g_\iota \vee g_\mu] \cong N_{11}^{(d)}$ iff $[h^\iota \wedge h^\mu, h^\iota] \cong N_{11}^{(d)}$ iff $[h^\iota \wedge h^\mu, h^\mu] \cong N_{11}$. Hence, if $S(\alpha) \cong S(\beta)$, then $\{\iota : \iota < 6 + \alpha\}$ is order isomorphic to $\{\iota : \iota < 6 + \beta\}$, which yields that $6 + \alpha = 6 + \beta$, and we conclude that $\alpha = \beta$. This proves the second part of the lemma.

Clearly, $S(\alpha)$ is a selfdual lattice. Let f be an arbitrary automorphism of $S(\alpha)$. As we have noticed above, the elements g_ι are recognized by a first-order property. Hence, $f(\{g_\iota : \iota < 6+\alpha\} \subseteq \{g_\iota : \iota < 6+\alpha\}$. Actually, we have equality here, because the same kind of inclusion holds for f^{-1}. However, since the well-ordering of $\{\iota : \iota < 6 + \alpha\}$ is encoded in the lattice, we obtain that f induces an order automorphism on $\{\iota : \iota < 6 + \alpha\}$. It is well-known, and it follows by a straightforward transfinite induction, that $\langle\{\iota : \iota < 6 + \alpha\}; <\rangle$ is automorphism-rigid. Therefore, f acts as the identity map on $\{g_\iota : \iota < 6 + \alpha\}$. By duality, the same holds for the set $\{h^\iota : \iota < 6 + \alpha\}$. Since these two sets generate $T(\alpha)$ and $T'(\alpha)$, respectively, f acts identically on $T(\alpha) \cup T'(\alpha)$. In particular, the black-filled elements are fixed points of f, which implies that f acts identically on $M'_{3\times 3}$. Consequently, so does f on $S_0(\alpha)$. Finally, since N_{11} and $N_{11}^{(d)}$ are automorphism-rigid, we obtain that f is the identity map. Thus, $S(\alpha)$ is automorphism-rigid. ☐

3. A construction and the completion of the proof

Parallel to describing the construction in general, we also show how it works for the following example.

Example 3.1. Assume that we want to represent the ordered set $P = \langle P; \leq \rangle$ given in Figure 3 and the dihedral group $G := D_4$ of rank 4 simultaneously. First, we represent G as the automorphism group of the graph $\langle V; E \rangle$ given in Figure 3.

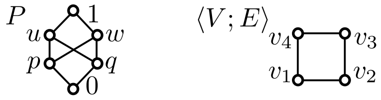

FIGURE 3. A small example

For brevity, we will often use the notation

$$P^{-01} := P \setminus \{0, 1\}.$$

In general, Lemma 2.1 always allows us to take a graph $\langle V; E \rangle$ whose automorphism group is isomorphic to G. We shall assume that V is disjoint from P. Let $H := P \cup V$, and consider the quasiordered set $\langle H; \nu \rangle$, where

$$\nu := \{\langle x, y \rangle \in P^2 : x \leq_P y\} \cup \big(H \times (V \cup \{1\})\big).$$

So each vertex $v \in V$ is added to P as an additional largest element and, consequently, $\langle H; \nu \rangle$ has $1 + |V|$ many largest elements. We let

$$I := J := E \cup \{\langle p, q \rangle \in P^{-01} \times P^{-01} : p < q\} \cup (\{1\} \times V).$$

Observe that $I \cup J \cup (\{0\} \times H) \cup (H \times \{1\})$ generates ν, that is, Czédli [7, (4.23)] holds. Let $\hat{\mathcal{L}}_0 = \langle \hat{L}_0, \leq_0; \gamma_0; H, \nu \rangle$ be the same quasi-colored lattice as $\mathcal{L}(H, I, J)$ from [7, (4.21) and Remark 6.6[1]], except that we use $M'_{2\times 3}$ rather than $M_{4\times 3}$ in its construction. (As opposed to $M'_{2\times 3}$, which is automorphism-rigid, $M_{4\times 3}$ has four automorphisms; this is why the latter is not appropriate here.) With Θ_ν defined in (2.1), $\langle H; \nu \rangle / \Theta_\nu \cong P$. We know from [7, Lemma 4.6] that

$$\hat{\mathcal{L}}_0 = \langle \hat{L}_0, \leq_0; \gamma_0; H, \nu \rangle \text{ is a quasi-colored lattice} \tag{3.1}$$

and it is selfdual. Thus, Lemma 2.2 yields that

$$\mathrm{Princ}(\hat{L}_0) \cong \langle H; \nu \rangle / \Theta_\nu \cong P. \tag{3.2}$$

Example 3.2. For the situation described in Example 3.1 and Figure 3, we visualize $\hat{L}_0$ and $I = J$ in Figure 4. We obtain the lattice in this figure by gluing $M'_{2\times 3}$ from Figure 1 and the chains $\{0 \prec a_x \prec b_x \prec 1\}$ for $x \in V \cup P^{-01}$ at their bottom and top elements. (Disregard the gray-filled ovals $S(0), \dots, S(0), S(1), \dots S(4)$ in the figure now.) The members of $I = J$ are indicated by arrows: if $\langle x, y \rangle \in I$, then there is an arrow from the prime interval $[a_x, b_x]$ to the prime interval $[a_y, b_y]$. We use two kinds of arrows: dotted arrows for $\langle x, y \rangle \in E$ and wavy arrows otherwise. Note that a dotted arc represents two arrows; one from left to right and another one from right to left. As it is explained in [7], we obtain $\hat{L}_0$ from Figure 4 by replacing, for every $\langle p, q \rangle \in I = J$, the corresponding arrow with the double gadget $\mathcal{G}^{\mathrm{db}}(p, q)$ given in [7, Figure 4].

[1]Remark 6.6 in [7]: Instead of $M_{4\times 3}$, we can use an arbitrary simple lattice having at least four elements; however, then we cannot guarantee that $L(H, I, J)$ is a lattice of length 5.

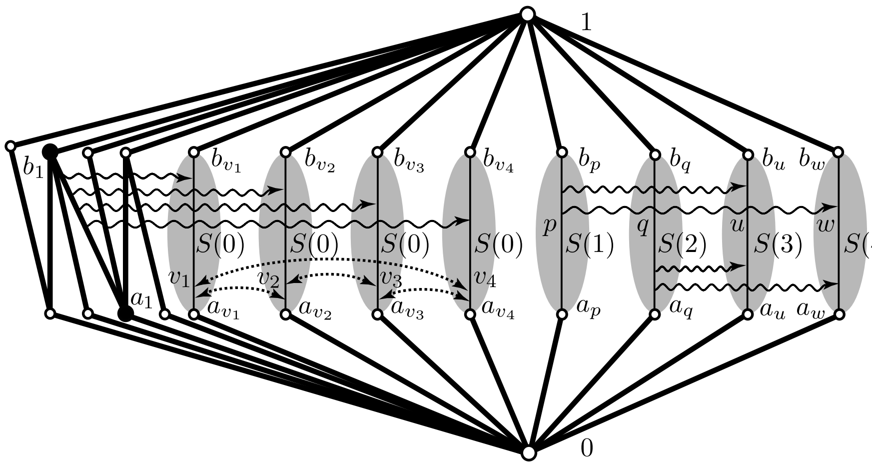

FIGURE 4. $\hat{L}_0$ (without the gray-filled ovals) and $\hat{L}$ (without the v_i-colored edges)

For each $p \in P^{-01}$, pick an ordinal number $\iota_p > 0$. We assume that $\iota_p \neq \iota_q$ if $p \neq q$. Also, if P is finite, then let all the ι_p be finite. To complete the construction, we replace the prime interval $[a_p, b_p]$ with $S(\iota_p)$ for $p \in P^{-01}$ and we replace $[a_v, b_v]$ with $S(0)$ for $v \in V$. The $S(\iota_p)$ and all copies of the $S(0)$ are pairwise disjoint, of course. By [7, the remark before Lemma 4.7[2]], the covering pairs $\langle a_x, b_x\rangle$, for $x \in P^{-01} \cup V$, are locally critical pairs. Using a trivial transfinite induction, it follows from Lemma 2.7 that

(3.3) we obtain a quasi-colored lattice $\hat{\mathcal{L}} = \langle \hat{L}, \leq; \gamma; H, \nu\rangle$

in this way.

Example 3.3. For P and G in Example 3.1, Figure 4 gives the lattice $\hat{L}$ with $\mathrm{Princ}(\hat{L}) \cong P$ and $\mathrm{Aut}(\hat{L}) \cong G$ as follows. Each arrow indicates a gadget, as explained in Example 3.2. An edge $\langle x, y\rangle \in \mathrm{Pairs}^{\prec}(L)$ is thick iff it generates the largest congruence iff $\langle 1, \gamma(\langle x, y\rangle)\rangle \in \nu$. The gray-filled ovals $S(0), \dots, S(4)$ stand for the lattices defined before Lemma 2.8; note that $S(1)$ is derived from $S_0(1)$ given in Figure 2.

Now, we are in the position to complete the paper as follows.

Proof of Theorem 1.1. We are going to show that $\mathrm{Princ}(\hat{L}) \cong P$ and $\mathrm{Aut}(\hat{L}) \cong G$. Comparing $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}$, see (3.1) and (3.3), we obtain from Lemma 2.2 that $\mathrm{Princ}(\hat{L}) \cong \mathrm{Princ}(\hat{L}_0)$. This equality and (3.2) give that $\mathrm{Princ}(\hat{L}) \cong P$. Hence, it suffices to deal with $\mathrm{Aut}(\hat{L})$. We say that a subset X of $\hat{L}$ is *rigid*, if the restriction of every member of $\mathrm{Aut}(\hat{L})$ to X is the identity map of X. If $f(X) \subseteq X$ for all $f \in \mathrm{Aut}(\hat{L})$, then X is an *invariant* subset. For such a subset X, $X = f(f^{-1}(X)) \subseteq f(X)$. That is, if X is an invariant subset, then $f(X) = X$ holds for all $f \in \mathrm{Aut}(\hat{L})$.

Since $M'_{2\times 3}$ is automorphism-rigid and it is isomorphic to no other cover-preserving $\{0, 1\}$-sublattice of $\hat{L}$, it follows that $M'_{2\times 3}$ and, in particular, $\{a_1, b_1\}$ are rigid

[2]This remark in [7] notes that for $p \in H$, $x < a_p$, and $y > b_p$, both $\langle x, a_p\rangle$ and $\langle b_p, y\rangle$ are 1-colored, so each of them generates the largest congruence of $L(H, I, J)$.

subsets. The elements a_x, for $x \in V \cup P^{-01}$, are characterized by the properties that $[0, a_x]$ is of length at most 2 and a_x is covered by at least $\binom{6}{2} = 15$ elements. (Remark 3.5, which we do not need in the moment, will shed more light on this part of the proof.) Therefore, taking duality also into account,

$$\{a_x : x \in V \cup P^{-01}\} \text{ and } \{b_x : x \in V \cup P^{-01}\} \text{ are invariant subsets.} \tag{3.4}$$

For distinct $p, q \in P^{-01}$ and $v \in V$, observe that a_p, a_q, and a_v are the bottoms of $S(\iota_p)$, $S(\iota_q)$, and $S(0)$. Since $S(\iota_p)$, $S(\iota_q)$, and $S(0)$ are pairwise non-isomorphic by Lemma 2.8, no automorphism maps a_p to a_q or a_v. Hence,

$$\{a_p : p \in P^{-01}\} \text{ is a rigid subset, and so is } \{b_p : p \in P^{-01}\} \tag{3.5}$$

by duality. For $x \neq y \in H$, there is at most one gadget (that is, at most one arrow in Figure 4) from $[a_x, b_x]$ to $[a_y, b_y]$. If there is a gadget from $[a_x, b_x]$ to $[a_y, b_y]$ and $f \in \mathrm{Aut}(\hat{L})$, then the restriction of f to $\{a_x, b_x, a_y, b_y\}$ determines its restriction to the whole the gadget. Since $a_x \leq b_y$ iff $x = y$, it follows that if $f(a_x) = a_y$, then $f(b_x) = b_y$. Also, $[a_x, b_x] \cong S(\iota_x)$ for $x \in P^{-01}$ and $[a_v, b_v] \cong S(0)$ for $v \in V$ are automorphism-rigid by Lemma 2.8. Putting all the above facts, including (3.4), and (3.5), together, we obtain that

$$\{a_v : v \in V\} \text{ is an invariant subset and } f \in \mathrm{Aut}(\hat{L}) \text{ is determined by its restriction to this subset.} \tag{3.6}$$

For distinct $x, y \in V \cup \{1\}$, f and $f^{(-1)}$ clearly preserve the property "there is a gadget from $[a_x, b_x]$ to $[a_y, b_y]$". But $[a_1, b_1] = \{a_1, b_1\} \subseteq M'_{2\times 3}$ is a rigid subset, so the case $x, y \in V$ is only interesting from this point of view. In the spirit of Figure 2, f preserves the dotted arrows, and also the absence of these arrows. Therefore, f induces an automorphism of the graph $\langle V; E\rangle$. Conversely, since the intervals $[a_v, b_v]$, $v \in V$, of $\hat{L}$ are isomorphic and they are only in connection with themselves (and, all in the same way, with $[a_1, b_1]$), we conclude that each automorphism of the graph induces a unique automorphism of the sublattice $\bigcup\{[a_v, b_v] : v \in V\}$ and, consequently, of $\hat{L}$. This proves that $\mathrm{Aut}(\hat{L}) \cong \langle V; E\rangle$, as required. □

Remark 3.4. Besides the lattices $S(\alpha)$, see Lemma 2.8, there are other ways to construct automorphism-rigid simple selfdual lattices. For example, the CM_n lattices, the Kirby Baker lattices KB_n, and the Ralph Freese lattices RF_n from G. Grätzer and R. W. Quackenbush [20] are simple lattices; see also R. Freese [9] and the L'_n from Czédli and M. Maróti [8] for the original sources when available. From those that are not selfdual we can easily obtain selfdual lattices; either by gluing such a lattice K to their dual, or taking $M_3 = \{0, a, b, c\}$ and forming $M_3\big(M_3(0, a, K)(a, 1, K^{\mathrm{dual}})\big)$, see Corollary 2.6. If necessary, we can get rid of nontrivial automorphisms by adding some extra elements like we added the two $\otimes$-shaped elements to $M_{2\times 3}$ in order to obtain $M'_{2\times 3}$ in Figure 1. However, we prefer the $S(\alpha)$ in the paper, because the proof benefits from the fact that they are of the same finite length, twelve.

Remark 3.5. In the proof, we used that the elements a_x can be recognized as elements of length at most 2 with at least 15 covers. This gives the second reason why we used $6 + \alpha$ rather than α in the definition of $S(\alpha)$, since otherwise an element of height 2 with many covers need not be of the form a_x. Namely, if there is no arrow from or to the edge $\langle a_p, b_p\rangle$ in $\hat{L}_0$ for some $p \in P^{-01}$, then h_{01} of $S(\iota_p)$ has nine covers and it is of height 2 in $\hat{L}$.

Remark 3.6. Since our proof does not use G. Birkhoff's result from [2], the present paper gives a new proof of the fact that every group can be represented as the automorphism group of an appropriate lattice L. Note, however, that [2] proves more by constructing a distributive lattice in a shorter way.

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