AN EASY WAY TO A THEOREM OF KIRA ADARICHEVA AND MADINA BOLAT ON CONVEXITY AND CIRCLES

GÁBOR CZÉDLI

Dedicated to the eighty-fifth birthday of Béla Csákány

Abstract. Kira Adaricheva and Madina Bolat have recently proved that if U_0 and U_1 are circles in a triangle with vertices A_0, A_1, A_2, then there exist $j \in \{0, 1, 2\}$ and $k \in \{0, 1\}$ such that U_{1-k} is included in the convex hull of $U_k \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})$. We give a short new proof for this result, and we point out that a straightforward generalization for spheres fails.

1. Aim and introduction

Our goal. The real n-dimensional space and the usual convex hull operator on it will be denoted by $\mathbb{R}^n$ and $\mathrm{Conv}_{\mathbb{R}^n}$. That is, for a set $X \subseteq \mathbb{R}^n$ of points, $\mathrm{Conv}_{\mathbb{R}^n}(X)$ is the smallest convex subset of $\mathbb{R}^n$ that includes X. In this paper, the Euclidean distance $(\sum_{i=1}^n (X_i - Y_i)^2)^{1/2}$ of $X, Y \in \mathbb{R}^n$ is denoted by $\mathrm{dist}(X, Y)$. For $P \in \mathbb{R}^2$ and $0 \le r \in \mathbb{R}$, the *circle* of center P and radius r will be denoted by

$$\mathrm{Circ}(P, r) := \{X \in \mathbb{R}^2 : \mathrm{dist}(P, X) = r\}.$$

Our aim is to give a new proof of the following theorem. Our approach is entirely different from and shorter than the original one given by Adaricheva and Bolat [3]. Roughly saying, the novelty is that instead of dealing with several cases, we prove that the "supremum of good cases" implies the result for all cases.

Theorem 1.1 (Adaricheva and Bolat [2, Theorem 3.1])**.** *Let A_0, A_1, A_2 be points in the plane. If U_0 and U_1 are circles such that $U_i \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\})$ for $i \in \{0, 1\}$, then there exist subscripts $j \in \{0, 1, 2\}$ and $k \in \{0, 1\}$ such that*

$$U_{1-k} \subseteq \mathrm{Conv}_{\mathbb{R}^2}\big(U_k \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big). \tag{1.1}$$

Notably enough, Adaricheva and Bolat [2, Theorem 5.1] states even more than [2, Theorem 3.1]; we formulate their more general result as follows.

Corollary 1.2 (Adaricheva and Bolat [2, Theorem 5.1])**.** *If C_0, C_1, C_2, U_0, and U_1 are circles in the plane such that $U_i \subseteq \mathrm{Conv}_{\mathbb{R}^2}(C_0 \cup C_1 \cup C_2)$ for $i \in \{0, 1\}$, then $U_{1-k} \subseteq \mathrm{Conv}_{\mathbb{R}^2}\big(U_k \cup \bigcup(\{C_0, C_1, C_2\} \setminus \{C_j\})\big)$ holds for some $j \in \{0, 1, 2\}$ and $k \in \{0, 1\}$.*

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Note that Adaricheva and Bolat [2] call the property stated in this corollary for circles the "Weak Carousel property". Note also that [2] gives a new justification to Czédli and Kincses [10], because Theorem 5.2 and Section 6 in [2] yield that the almost-circles in [10] cannot be replaced by circles. Also, [2] motivates Czédli [9] and Kincses [13]. This paper is self-contained. For more about the background of this topic, the reader may want, but need not, to see, for example, Adaricheva and Nation [5] and [6], Czédli [8], Edelman and Jamison [11], Kashiwabara, Nakamura, and Okamoto [12], Monjardet [14], and Richter and Rogers [16].

The results of Adaricheva and Bolat [2], that is, Theorem 1.1 and Corollary 1.2 above, and our easy approach raise the question whether the most straightforward generalizations hold for 3-dimensional spheres. In Section 4, which is a by-product of our method in some implicit sense, we give a negative answer.

2. Homotheties and round-edged angles

2.1. **A single circle.** For $0 < r \in \mathbb{R}$ and $F, P \in \mathbb{R}^2$ with $\operatorname{dist}(F, P) > r$, let

(2.1) $\operatorname{Ang}(F, \operatorname{Circ}(P, r))$ be the grey-filled area in Figure 1;

it is called the *round-edged angle* determined by its *focus* F and *spanning circle* $\operatorname{Circ}(P, r)$. Note that $\operatorname{Ang}(F, \operatorname{Circ}(P, r))$ is not bounded from the right and F is outside both $\operatorname{Circ}(P, r)$ and $\operatorname{Ang}(F, \operatorname{Disk}(P, r))$. Note that $\operatorname{Ang}(F, \operatorname{Disk}(P, r))$ includes its boundary, which consists of a circular arc called the *front arc* and two half-lines.

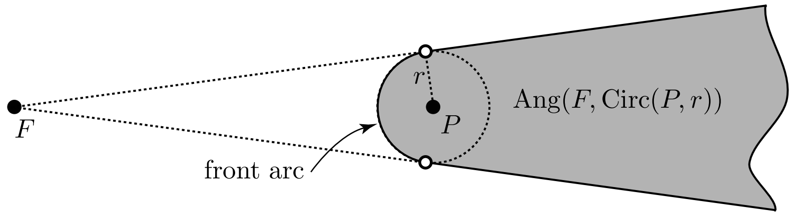

Figure 1. Round-edged angle

2.2. **Externally perspective circles.** First, recall or define some easy concepts and notations. For topologically closed convex sets $W_1, W_2 \subseteq \mathbb{R}^2$, we will say that

(2.2) W_1 is *loosely included* in W_2, in notation, $W_1 \overset{\text{loose}}{\subset} W_2$,

if every point of W_1 is an internal point of W_2. Given $P \in \mathbb{R}^2$ and $0 \neq \lambda \in \mathbb{R}$, the *homothety* with (homothetic) center P and ratio λ is defined by

(2.3) $\boldsymbol{\chi}_{P,\lambda} \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $X \mapsto PX\underline{\lambda} := (1 - \lambda)P + \lambda X$.

We will not need negative ratios λ and we use the Polish notation for the *barycentric operation* $\underline{\lambda}$. Homotheties are similarity transformations. In particular, they map the center of a circle to the center of its image. If C_1 and C_2 are circles and $C_2 = \boldsymbol{\chi}_{P,\lambda}(C_1)$ such that P is (strictly) outside both C_1 and C_2 (equivalently, if P is outside C_1 or C_2) and $0 < \lambda \in \mathbb{R}$, then C_1 and C_2 will be called *externally perspective circles.* Clearly, if C_1 and C_2 are of different radii and none of them is inside the other, then C_1 and C_2 are externally perspective, P is the intersection point of their external tangent lines, and λ is the ratio of their radii.

Lemma 2.1. *Let* $\mathrm{Circ}(P_1, r_1)$ *and* $\mathrm{Circ}(P_2, r_2)$ *be externally perspective circles in the plane with center* F *of perspectivity such that* $0 < r_2 < r_1$*; see Figure 2. If* G *is a point on the line segment* $[F, P_2]$ *such that* $r_2 < \mathrm{dist}(G, P_2) < \mathrm{dist}(F, P_2)$*, then* $\mathrm{Ang}(F, \mathrm{Circ}(P_1, r_1)) \overset{\text{loose}}{\subset} \mathrm{Ang}(G, \mathrm{Circ}(P_2, r_2))$*; see* (2.2) *and Figure 2.*

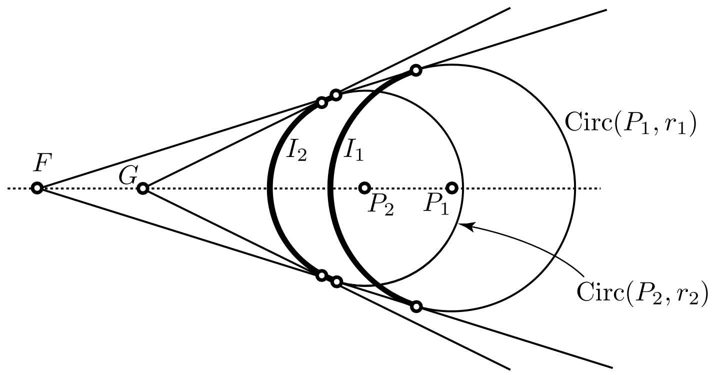

FIGURE 2. Illustration for Lemma 2.1

Proof. Clearly, $\mathrm{Circ}(P_1, r_1) = \boldsymbol{\chi}_{F,\lambda}(\mathrm{Circ}(P_2, r_2))$ with $\lambda = r_1/r_2 > 1$. The external tangent lines of our circles intersect at F. Since $\boldsymbol{\chi}_{F,\lambda}$ preserves tangency, it maps the circular arc I_2 of $\mathrm{Circ}(P_2, r_2)$ between the tangent points onto the circular arc I_1 of $\mathrm{Circ}(P_1, r_1)$ between the images of these tangent points; see the thick arcs in Figure 2. Hence, I_2 is strictly on the left of I_1 in the figure, implying the lemma. □

Lemma 2.2. *If* $\lambda, \mu \in \mathbb{R} \setminus \{0\}$*,* $F, Q \in \mathbb{R}^2$*, and* $R = \boldsymbol{\chi}_{F,\lambda}(Q)$*, then, composing maps from right to left,* $\boldsymbol{\chi}_{R,\mu} \circ \boldsymbol{\chi}_{F,\lambda} = \boldsymbol{\chi}_{F,\lambda} \circ \boldsymbol{\chi}_{Q,\mu}$.

Proof. $\boldsymbol{\chi}_{F,\lambda} \circ \boldsymbol{\chi}_{Q,\mu} \circ \boldsymbol{\chi}_{F,\lambda}^{-1}$ is clearly a homothety of ratio μ that fixes R. So this homothety is $\boldsymbol{\chi}_{R,\mu}$, which implies the lemma. □

Lemma 2.3. *If* $\lambda > 1$ *and* C_0 *and* C_1 *are internally tangent circles with center points* $C_0^\bullet$ *and* $C_1^\bullet$*, respectively, then either one of* $\boldsymbol{\chi}_{\lambda,C_0^\bullet}(C_0)$ *and* $\boldsymbol{\chi}_{\lambda,C_1^\bullet}(C_1)$ *is in the interior of the other, or* $C_0 = C_1$.

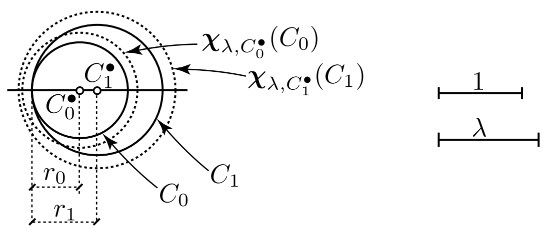

FIGURE 3. Illustration for Lemma 2.3

Proof. We can assume that the radii r_0 and r_1 are distinct, say, $r_0 < r_1$; see Figure 3. The distance $d := \mathrm{dist}(C_0^\bullet, C_1^\bullet)$ is $r_1 - r_0$. Since $\lambda r_1 = \lambda(r_0 + d) > \lambda r_0 + d$, $\boldsymbol{\chi}_{\lambda,C_0^\bullet}(C_0)$ is in the interior of $\boldsymbol{\chi}_{\lambda,C_1^\bullet}(C_1)$, as required. □

The following lemma resembles the 2-Carousel Rule in Adaricheva [1].

Lemma 2.4. *Let A_0, A_1, and A_2 be non-collinear points in the plane. If B_0 and B_1 are distinct* internal *points of* $\mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\})$*, then there exist $j \in \{0,1,2\}$ and $k \in \{0,1\}$ such that*

$$\{B_{1-k}\} \overset{\text{loose}}{\subset} \mathrm{Conv}_{\mathbb{R}^2}\big(\{B_k\} \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big).$$

Proof. Since the triangle $\mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\})$ is clearly of the form

$$\mathrm{Conv}_{\mathbb{R}^2}(\{B_0, A_1, A_2\}) \cup \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, B_0, A_2\}) \cup \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, B_0\}), \tag{2.4}$$

B_1 belongs to at least one of the triangles in (2.4). If one of these three triangles, say, $\mathrm{Conv}_{\mathbb{R}^2}(\{B_0, A_1, A_2\})$, contains B_1 as an internal point, then we let $k = 0$ and $j = 0$. Otherwise, there is a $j' \in \{0,1,2\}$ such that the line segment $[B_0, A_{j'}]$ contains B_1 in its interior, and we can clearly let $k = 1$ and $j = j'$. □

3. Proving Theorem 1.1 with analytic tools

Proof of Theorem 1.1. If A_0, A_1, and A_2 are collinear points, then the circles are of radii 0 and (1.1) holds trivially (even without U_k on the right). Hence, in the rest of the proof, we assume that A_0, A_1, and A_2 are non-collinear points. We let

$$T := \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1, A_2\}).$$

Let P_i and r_i denote the center and the radius of U_i from the theorem. Note that

$$r_1 = 0 \text{ implies (1.1), by (2.4) applied for } B_0 \in U_0 \text{ and } B_1 = P_1; \tag{3.1}$$

and similarly for $r_0 = 0$. Therefore, we will assume that none of r_0 and r_1 is zero. From now on, we prove the theorem by way of contradiction. That is, we assume that U_0 and U_1 are circles satisfying the assumptions of Theorem 1.1, $r_0 r_1 > 0$, but (1.1) fails. For $0 \le \xi \le 1$ and $k \in \{0,1\}$, we denote $\mathrm{Circ}(P_k, \xi \cdot r_k)$ by $U_k(\xi)$. Let

$$\begin{aligned} &H := \{\eta \in [0,1] : (\forall \zeta \in [0,\eta])\, (\exists k \in \{0,1\})\, (\exists j \in \{0,1,2\}) \\ &\text{such that } U_{1-k}(\zeta) \subseteq \mathrm{Conv}_{\mathbb{R}^2}\big(U_k(\zeta) \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big)\}. \end{aligned} \tag{3.2}$$

In other words, H consists of those η for which $U_0(\zeta)$, $U_1(\zeta)$, A_0, A_1, and A_2 satisfy the theorem for all ζ in the closed interval $[0,\eta] \subseteq [0,1] \subseteq \mathbb{R}$. For brevity, we let

$$\begin{aligned} &W(j,k,\zeta) := \mathrm{Conv}_{\mathbb{R}^2}\big(U_k(\zeta) \cup (\{A_0, A_1, A_2\} \setminus \{A_j\})\big); \text{ then } H := \\ &\{\eta \in [0,1] : (\forall \zeta \in [0,\eta])\, (\exists k)\, (\exists j)\, (U_{1-k}(\zeta) \subseteq W(j,k,\zeta)\}. \end{aligned} \tag{3.3}$$

By (3.1), $0 \in H$. Since $U_k(1) = U_k$, for $k \in \{0,1\}$, our indirect assumption gives that $1 \notin H$. Clearly, if $0 \le \eta_1 \le \eta_2 \le 1$ and η_2 belongs to H, then so does η_1; in other words, H is an order ideal of the poset $\langle [0,1], \le \rangle$. From now on,

$$\text{let } \xi \text{ denote the supremum of } H. \tag{3.4}$$

We are going to show that

$$\xi \in H, \text{ whereby } \xi \text{ is actually the maximum of } H, \text{ and } \xi > 0. \tag{3.5}$$

Since $r_0, r_1 > 0$ and P_0 and P_1 are internal points of the triangle T, it follows from Lemma 2.4 that $\xi > 0$. In order to prove the rest of (3.5) by way of contradiction, suppose that $\xi \notin H$. However, for each i such that $\lceil 1/\xi \rceil < i \in \mathbb{N}$, in short, for each *sufficiently large* i, $\xi - 1/i \in H$. Hence, for each sufficiently large i, we can pick a $k_i \in \{0,1\}$ and a $j_i \in \{0,1,2\}$ such that $U_{1-k_i}(\xi - 1/i) \subseteq W(j_i, k_i, \xi - 1/i)$; see (3.3). Since $\{0,1\} \times \{0,1,2\}$ is finite, one of its pairs, $\langle k, j \rangle$, occurs infinitely many times in the sequence of pairs $\langle k_i, j_i\rangle$. Thus, there exist a $k \in \{0, 1\}$, a $j \in \{0, 1, 2\}$, and an infinite set $I \subseteq \mathbb{N}$ of sufficiently large integers i such that

$$\text{for all } i \in I, \text{ we have that } U_{1-k}(\xi - 1/i) \subseteq W(j, k, \xi - 1/i). \tag{3.6}$$

Since, for all η and ζ, $0 \le \eta \le \zeta$ implies $W(j, k, \eta) \subseteq W(j, k, \zeta)$, (3.6) yields that

$$\text{for all } i \in I, \text{ we have that } U_{1-k}(\xi - 1/i) \subseteq W(j, k, \xi). \tag{3.7}$$

Next, let X be an arbitrary point of the circle $U_{1-k}(\xi)$. Denote by X_i the point $\boldsymbol{\chi}_{P_{1-k},(\xi-1/i)/\xi}(X)$; it belongs to $U_{1-k}(\xi - 1/i)$. Less formally, we obtain X_i as the intersection of $U_{1-k}(\xi - 1/i)$ with the line segment connecting X and P_{1-k}. As $i \in I$ tends to ∞, $X_i \to X$. Combining this with $X_i \in U_{1-k}(\xi - 1/i)$ and (3.7), we obtain that X is a *limit point* (AKA accumulation point or cluster point) of $W(j, k, \xi)$. The convex hull of a compact subset of $\mathbb{R}^n$ is compact; see, for example, Proposition 5.2.5 in Papadopoulos [15]. Hence, $W(j, k, \xi)$ from (3.3) is a compact set; whereby it contains its limit point, X. Thus, since X was an arbitrary point of $U_{1-k}(\xi)$, we conclude that $U_{1-k}(\xi) \subseteq W(j, k, \xi)$. By (3.3), this proves (3.5).

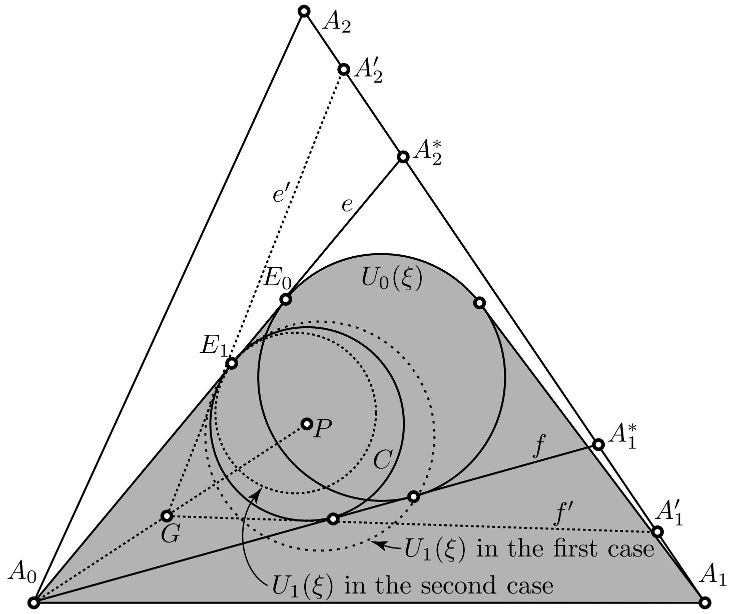

FIGURE 4. Illustration for (3.8)

Since $\xi \in H$, we can assume that the indices are chosen so that $U_1(\xi)$ is included in the grey-filled "round-backed trapezoid"

$$D(\xi) := \mathrm{Conv}_{\mathbb{R}^2}(\{A_0, A_1\} \cup U_0(\xi)); \text{ see Figure 4.} \tag{3.8}$$

If $U_1(\xi)$ was included in the interior of $D(\xi)$, then there would be a (small) positive $\boldsymbol{\varepsilon}$ such that $U_1(\xi+\delta) \subseteq D(\xi) \subseteq D(\xi+\delta)$ for all $\delta \in (0, \boldsymbol{\varepsilon}]$ and $\xi+\boldsymbol{\varepsilon}$ would belong to H, contradicting (3.4). Therefore, $U_1(\xi)$ is tangent to the boundary of $D(\xi)$. Since $\xi < 1$ and $U_1(1) = U_1$ is still included in the triangle T, $U_1(\xi)$ cannot be tangent to the side $[A_0, A_1]$ of T. If $U_1(\xi)$ was tangent to the back arc of the "round-backed trapezoid" $D(\xi)$ and so to $U_0(\xi)$, then one of $U_0 = U_0(1)$ and $U_1 = U_1(1)$ would be in the interior of the other by Lemma 2.3, and this would contradict the indirect assumption that (1.1) fails. Hence $U_1(\xi)$ is tangent to one of the "legs" of $D(\xi)$; this leg is an external tangent line e of the circles $U_1(\xi)$ and $U_0(\xi)$ through, say, A_0; see Figure 4. The corresponding touching points will be denoted by E_1 and E_0; see the figure. Let $\lambda := \mathrm{dist}(A_0, E_1)/\mathrm{dist}(A_0, E_0)$; note that $0 < \lambda < 1$. By well-known properties of homotheties, the auxiliary circle

$$C := \boldsymbol{\chi}_{A_0,\lambda}(U_0(\xi)), \text{ with center } P := \boldsymbol{\chi}_{A_0,\lambda}(P_0), \tag{3.9}$$

touches e and, thus, $U_1(\xi)$ at E_1. Let f denote the other tangent of $U_0(\xi)$ through A_0. Let A_1^* and A_2^* be the intersection points of f and e with the line through A_1 and A_2, respectively. Since $U_0(1) = U_0$ is also included in T and $U_0(\xi)$ is a smaller circle concentric to U_0, both A_1^* and A_2^* are in the interior of the line segment $[A_1, A_2]$. By continuity, we can find a point G in the interior of the line segment $[A_0, P]$ such that G is outside C and G is so close to A_0 that the tangent lines e' and f' of C through G intersect the line segments $[A_2^*, A_2]$ and $[A_1, A_1^*]$ at some of their *internal* points, which we denote by A_2' and A_1', respectively. Since the "round-backed trapezoid" $\mathrm{Conv}_{\mathbb{R}^2}(\{A_1', A_2'\} \cup C)$ is clearly the intersection of the round-edged angle $\mathrm{Ang}(G, C)$ and one of the half-planes determined by the line through A_1' and A_2', we obtain from Lemma 2.1 that $U_0(\xi) \overset{\text{loose}}{\subset} \mathrm{Conv}_{\mathbb{R}^2}(\{A_1', A_2'\} \cup C)$. Combining this with the obvious $\mathrm{Conv}_{\mathbb{R}^2}(\{A_1', A_2'\} \cup C) \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup C)$, we obtain that $U_0(\xi) \overset{\text{loose}}{\subset} \mathrm{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup C)$. Thus, we conclude that there exists a (small) positive $\boldsymbol{\varepsilon}$ in the interval $(0, 1 - \xi)$ such that

$$U_0(\xi + \delta) \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup C) \text{ for all } \delta \in (0, \boldsymbol{\varepsilon}]. \tag{3.10}$$

Let r be the radius of C. Depending on r, there are two cases. First, if $r_1 > r$, then C is inside $U_1(\xi)$ and, consequently, also in $U_1(\xi + \delta)$, whereby (3.10) leads to

$$U_0(\xi + \delta) \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup C) \subseteq \mathrm{Conv}_{\mathbb{R}^2}(\{A_1, A_2\} \cup U_1(\xi + \delta))$$

for all $\delta \in (0, \boldsymbol{\varepsilon}]$. This gives that $\xi + \boldsymbol{\varepsilon} \in H$, contradicting (3.4).

Second, let $r_1 \le r$. Now $U_1(\xi)$ coincides with or is inside C. By Lemma 2.3,

$$\text{for all } \mu > 1,\ \boldsymbol{\chi}_{P,\mu}(U_1(\xi)) \text{ coincides with or is inside } \boldsymbol{\chi}_{P,\mu}(C). \tag{3.11}$$

Clearly, $C \overset{\text{loose}}{\subset} T$, since so is $U_0(\xi)$. Hence, we can choose a (small) positive δ such that $\boldsymbol{\chi}_{P_0,\mu}(U_0(\xi)) = U_0(\xi\mu)$ and $\boldsymbol{\chi}_{P,\mu}(C)$ are loosely included in T for every $\mu \in [1, 1 + \delta]$. Furthermore, for every $\mu \in [1, 1 + \delta]$,

$$\begin{aligned} \boldsymbol{\chi}_{P,\mu}(C) &\overset{(3.9)}{=} \boldsymbol{\chi}_{P,\mu}\big(\boldsymbol{\chi}_{A_0,\lambda}(U_0(\xi))\big) \\ &\overset{\text{Lemma 2.2}}{=} \boldsymbol{\chi}_{A_0,\lambda}(\boldsymbol{\chi}_{P_0,\mu}(U_0(\xi))) = \boldsymbol{\chi}_{A_0,\lambda}(U_0(\xi\mu)). \end{aligned} \tag{3.12}$$

Since $0 < \lambda < 1$, it follows that

$$\boldsymbol{\chi}_{P,\mu}(C) \overset{(3.12)}{=} \boldsymbol{\chi}_{A_0,\lambda}(U_0(\xi\mu)) \in \mathrm{Conv}_{\mathbb{R}^2}(\{A_0\} \cup U_0(\xi\mu)), \text{ whence} \tag{3.13}$$

$$U_1(\xi\mu) = \boldsymbol{\chi}_{P,\mu}(U_1(\xi)) \overset{(3.11)}{\subseteq} \mathrm{Conv}_{\mathbb{R}^2}(\boldsymbol{\chi}_{P,\mu}(C)) \overset{(3.13)}{\subseteq} \mathrm{Conv}_{\mathbb{R}^2}\big(\{A_0\} \cup U_0(\xi\mu)\big).$$

Since this holds for all $\mu \in [1, 1+\delta]$, we conclude that $\xi(1+\delta) \in H$. This contradicts (3.4), completing the proof of Theorem 1.1. □

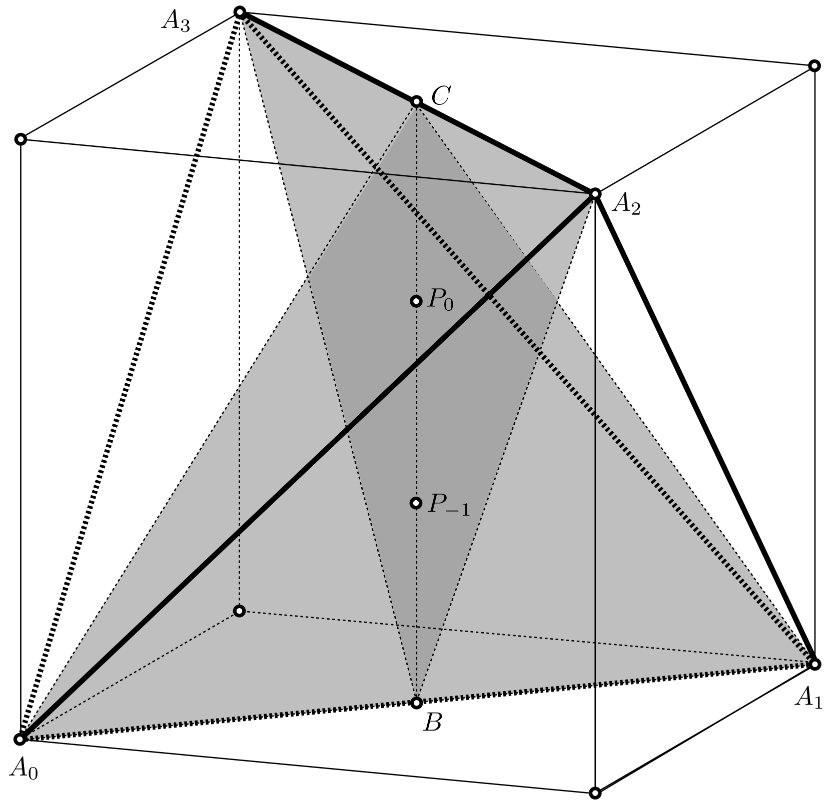

FIGURE 5. A regular tetrahedron in a cube

4. EXAMPLES

This section explains why we have been unable to generalize Theorem 1.1 for spheres so far. In our first example, one can change $\{-1, 0\}$ and $-1-k$ to $\{0, 1\}$ and $1-k$, respectively; we have chosen $\{-1, 0\}$ and $-1-k$ for a technical reason.

Example 4.1. Let $A_0, \dots, A_3$ be the vertices of a regular tetrahedron as well as some vertices of a cube; see Figure 5. Let B and C be the middle points of the line segments $[A_0, A_1]$ and $[A_2, A_3]$, respectively, and let P_{-1} and P_0 divide $[B, C]$ into three equal parts as the figure shows. Finally, let S_{-1} and S_0 be spheres in the interior of the tetrahedron $\mathrm{Conv}_{\mathbb{R}^3}(\{A_0, \dots, A_3\})$ with centers P_{-1} and P_0 and of the same positive radius. Then, for all $j \in \{0, 1, 2, 3\}$ and $k \in \{-1, 0\}$,

$$S_{-1-k} \nsubseteq \mathrm{Conv}_{\mathbb{R}^3}\big(S_k \cup \bigcup(\{A_0, A_1, A_2, A_3\} \setminus \{A_j\})\big). \tag{4.1}$$

Proof. By symmetry, it suffices to show (4.1) only for $j = 3$. First, let $k = 0$. We denote by π the orthogonal projection of $\mathbb{R}^3$ to the plane containing A_2, A_3 and B. Suppose for a contradiction that $S_{-1} \subseteq \mathrm{Conv}_{\mathbb{R}^3}(S_0 \cup A_0 \cup A_1 \cup A_2)$; this inclusion is preserved by π. Since π commutes with the formation of convex hulls and the disk $\pi(S_{-1})$ is not included in $\mathrm{Conv}_{\mathbb{R}^2}(\pi(S_0) \cup \pi(\{A_0, A_1, A_2\}))$, the grey-filled area in Figure 6, which is a contradiction. Second, if $k = -1$, then the argument is essentially the same but the grey-filled area in Figure 6 has to be changed. $\square$

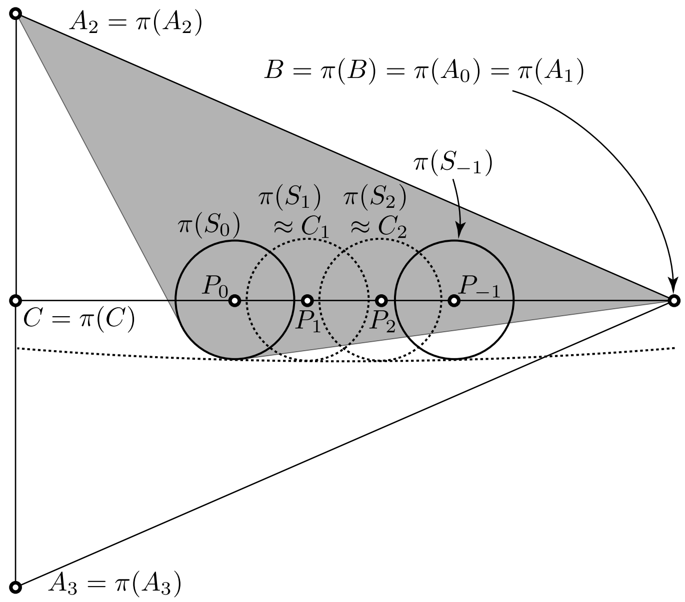

FIGURE 6. The π-images of our spheres

Example 4.2. For $t \in \{3, 4, 5, \dots\}$, add $t-2$ additional spheres to the previous example in the following way. Let $P_1, \dots, P_{t-2}$ divide the line segment $[P_0, P_{-1}]$ equidistantly; see Figure 6 for $t = 4$. This figure contains also a circular dotted arc with a sufficiently large radius; its center is far above the triangle. Besides the boundary circles of the disks $\pi(S_0)$ and $\pi(S_{-1})$ from the previous example, let $C_1, \dots, C_{t-2}$ be additional circles with centers $P_1, \dots, P_{t-2}$ such that all the (little) circles are tangent to the dotted arc; this idea is taken from Czédli [8, Figure 5]. For $i \in \{1, \dots, t-2\}$, let S_i be the sphere obtained from C_i by rotating it around the line through B and C. Note that $\pi(S_i) \approx C_i$ in Figure 6 means that the circle C_i is the boundary of the *disk* $\pi(S_i)$. Now, for all $j \in \{0, 1, 2, 3\}$ and $k \in \{-1, 0, \dots, t-2\}$, S_k is *not* a subset of

$$\mathrm{Conv}_{\mathbb{R}^2}\bigl(\bigcup(\{S_{-1}, S_0, \dots, S_{t-2}\} \setminus \{S_k\}) \cup \bigcup(\{A_0, A_1, A_2, A_3\} \setminus \{A_j\})\bigr),$$

while all the S_k are still included in the tetrahedron $\mathrm{Conv}_{\mathbb{R}^3}(\{A_0, \dots, A_3\})$.

Proof. Combine the previous proof and Czédli [8, Example 4.3]. □

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E-mail address: czedli@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/~czedli/

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, Hungary 6720