COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
29. UNIVERSAL ALGEBRA, ESZTERGOM (HUNGARY), 1977.

An application of Mal'cev type theorems to congruence varieties

G. Czédli

A lattice variety $\mathscr{U}$ is said to be a congruence variety if $\mathscr{U}=\mathrm{Con}(\mathscr{V})$ for some variety $\mathscr{V}$ of algebras where $\mathrm{Con}(\mathscr{V})$ denotes the lattice variety generated by the congruence lattices of all members of $\mathscr{V}$. Denote by $\mathscr{L}$ the variety of all lattices and by $\mathscr{M}$ the variety of modular lattices. Recently S. V. Polin – refuting the McKenzie–Nation conjecture – gave a variety $\mathscr{V}$ for which $\mathrm{Con}(\mathscr{V})\neq\mathscr{L}$ and $\mathrm{Con}(\mathscr{V})\subsetneqq\mathscr{M}$. However, there is a number of results stating that under certain conditions (on $\mathrm{Con}(\mathscr{V})$ or $\mathscr{V}$) $\mathrm{Con}(\mathscr{V})\neq\mathscr{L}$ implies $\mathrm{Con}(\mathscr{V})\subseteq\mathscr{M}$. For example, Freese and Nation ([2]) have proved this implication for any variety $\mathscr{V}$ of semigroups. For other results, when conditions are supposed on $\mathrm{Con}(\mathscr{V})$, the reader can see [3]. Here we intend to show for a certain kind of varieties $\mathscr{V}$ that $\mathrm{Con}(\mathscr{V})\neq\mathscr{L}$ implies $\mathrm{Con}(\mathscr{V})\subseteq\mathscr{M}$.

For an algebra Q, a subset of the set of all polynomial functions of Q is called a clone of Q if it is closed under superposition and contains all the projections. Let us recall some notations from Á. Szendrei's paper [4]. Let R be a ring with 1, S a subring of R ($1\in S$ is not supposed) and M a (unitary left) module over R. Denote by $\mathrm{Cl}_M(S)$ the clone of M consisting of all idempotent polynomials

$$f(x_1,x_2,\dots,x_n)=s_1x_1+s_2x_2+\dots+s_nx_n$$

where at most one of the coefficients $s_1,s_2,\dots,s_n$ is not in S.

Theorem (Á. Szendrei [4]). *Let R be a ring with 1. Then the following two conditions are equivalent.*

(i) *For an arbitrary module M over R, any idempotent clone of M (that is, any clone consisting of idempotent polynomials) is of the form $\bigcap_{\alpha<\beta}\mathrm{Cl}_M(S_\alpha)$ where each S_α $(\alpha<\beta)$ is a subring of R.*

(ii) *For any $r\in R$ there exist non-negative integers $n>0,a_0,a_1,\dots,a_n,b_0,b_1,\dots,b_n$*

such that

$$a_i+b_i \leqq \binom{n}{i} \quad (i = 0, 1, \ldots, n) \quad \textit{and}$$

$$\sum_{i=0}^{n} a_i r^i (1-r)^{n-i} = 1, \quad \sum_{i=0}^{n} b_i r^i (1-r)^{n-i} = r(1-r).$$

In what follows we shall denote the class of all rings with 1 satisfying the (equivalent) conditions of the above theorem by **K**. Á. Szendrei [4] has shown that the ring Z of integers, every ring with finite characteristic and $Z^n \times R$ ($n=1, 2, \ldots, R$ is any ring with finite characteristic) belong to **K**. Because of $Z \in \mathbf{K}$ the following theorem generalizes A. Huhn's similar result for Abelian group reducts (cf. [1]).

Theorem. *Let* $R \in \mathbf{K}$, M *be a module over* R, A *a reduct of* M *and* $\mathscr{V}$ *the variety generated by the algebra* A. *Then* $\mathrm{Con}(\mathscr{V}) \neq \mathscr{L}$ *implies* $\mathrm{Con}(\mathscr{V}) \subseteqq \mathscr{M}$.

Proof. We first describe the necessary Mal'cev conditions. Let $r \geqq 2$ and $s \geqq 2$ be integers, P a set of maps

$$P \subseteqq \{\pi | \pi \colon \{1, 2, \ldots, s\} \to \{1, 2, \ldots, s\}\},$$

and $H \subseteqq \{1, 2, \ldots, r\} \times P \times \{1, 2, \ldots, r\}$. The following condition "There exist s-ary *idempotent* polynomials $f_1, f_2, \ldots, f_r$ such that the identities $f_i(x_1, x_2, \ldots, x_s) = x_i$ (for $i=1, 2$) and $f_c(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(s)}) = f_d(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(s)})$ (for all $(c, \pi, d) \in H$) hold" will be called a *linear* Mal'cev condition (corresponding to r, s, H). By R. Wille's theorem [6], linear Mal'cev conditions U_{tv} ($t=2, 3, \ldots, v=2, 3, \ldots$) can be given for an arbitrary lattice identity λ such that the following two conditions are equivalent for any variety $\mathscr{V}$.

(i) λ holds in $\mathrm{Con}(\mathscr{V})$;

(ii) For any integer $t \geqq 2$ there exists an integer $v \geqq 2$ such that U_{tv} is satisfied in $\mathscr{V}$.

Returning to the proof, let C be the full idempotent clone of A (consisting of all idempotent A-polynomials). Let $B=(D, C)$ where D is the support of A (that is, the support of M). Denote by $\mathscr{W}$ the variety generated by the algebra B. The same linear Mal'cev conditions are satisfied in $A, B, \mathscr{V}$ and $\mathscr{W}$. So we obtain $\mathrm{Con}(\mathscr{V}) = \mathrm{Con}(\mathscr{W})$ from Wille's theorem mentioned above. We can assume that

($*$) the annihilator ideal I of M consists only of the zero element of R (otherwise M can be regarded as a module over $R/I \in \mathbf{K}$). Let E be the subring of R generated by 1.

If $\mathrm{Cl}_M(E) \subseteqq C$, then we obtain $\mathrm{Con}(\mathscr{W}) \subseteqq \mathrm{Con}(\mathbf{HSP}\{M^+\}) \subseteqq \mathscr{M}$ from Wille's theorem. ($M^+ = (D, \{+, -, 0\})$ means the additive group of M. So $\mathrm{Cl}_M(E)$ is the full idempotent clone of M^+.)

Now it is sufficient to show that $\mathrm{Cl}_M(E) \subsetneqq C$ implies $\mathrm{Con}(\mathscr{W}) = \mathscr{L}$. From Szendrei's theorem we obtain that $\mathrm{Cl}_M(E) \subsetneqq \mathrm{Cl}_M(S) \supseteqq C$ for a suitable subring S of R. Clearly $1 \notin S$. We show that

(∗ ∗) If a linear Mal'cev condition holds in $\mathscr{W}$ (and so in B), then it holds in each variety.

Let U be a linear Mal'cev condition corresponding to r, s, H which is satisfied by the polynomials $f_1, f_2, \ldots, f_r$ in B. It follows from (∗) that the f_i $(i=1, 2, \ldots, r)$ can be *uniquely* written in the form

$$f_i(x_1, x_2, \ldots, x_s) = r_{i1}x_1 + r_{i2}x_2 + \ldots + r_{is}x_s$$

where $r_{i1}, \ldots, r_{is} \in R$. Here $r_{i1} + r_{i2} + \ldots + r_{is} = 1 \notin S$ and $f_i \in C \subseteq \mathrm{Cl}_M(S)$. Thus all coefficients of f_i but exactly one belong to S. Let $r_{ij_i} \notin S$ for $i = 1, 2, \ldots, r$. Let $(c, \pi, d) \in H$ and denote by $\hat{\pi}$ the equivalence relation induced by π (that is, $n \equiv m\,(\hat{\pi})$ iff $\pi(n) = \pi(m)$). The identity

$$f_c(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(s)}) = f_d(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(s)})$$

holds in B, thus for each block ϱ of $\hat{\pi}$ $\sum_{i \in \varrho} r_{ci} = \sum_{i \in \varrho} r_{di}$.

Now $j_c \notin \varrho$ iff $\sum_{i \in \varrho} r_{ci} \in S$ iff $\sum_{i \in \varrho} r_{di} \in S$ iff $j_d \notin \varrho$. Therefore $j_c \equiv j_d\,(\hat{\pi})$. Consequently the linear Mal'cev condition U is satisfied in every variety by the projections $f_i(x_1, x_2, \ldots, x_s) = x_{ij_i}$ $(i = 1, 2, \ldots, r)$, which proves (∗ ∗).

Let $\mathscr{U}$ be the variety of sets, then $\mathrm{Con}\,(\mathscr{U}) = \mathscr{L}$. (By P. M. Whitman's theorem [5], every lattice is embeddable into an equivalence lattice.) Hence (∗ ∗) and Wille's theorem imply $\mathrm{Con}\,(\mathscr{W}) = \mathscr{L}$, which was to be proved.

The author is grateful to A.P. Huhn for a helpful suggestion.

References

[1] A. P. HUHN, Congruence varieties associated with reducts of Abelian group varieties, *Algebra Universalis* **9** (1979), 133–134.

[2] R. S. FREESE and J. B. NATION, Congruence lattices of semilattices, *Pacific J. Math.* **49** (1973), 51–59.

[3] B. JÓNSSON, Identities in congruence varieties, *Lattice theory (Proc. Conf. Szeged, 1974)*, pp. 195–205. Colloq. Math. Soc. J. Bolyai, 14, North-Holland, Amsterdam, 1976.

[4] Á. SZENDREI, On the idempotent reducts of modules, *this volume.*

[5] P. M. WHITMAN, Lattices, equivalence relations and subgroups, *Bull. Amer. Math. Soc.* **52** (1946), 507–522.

[6] R. WILLE, *Kongruenzklassengeometrien*, Springer, Lecture Notes in Math. 113, Berlin–Heidelberg–New York, 1970.

JATE Bolyai Institute
H-6720 Szeged,
Aradi vértanúk tere 1

Received August 11, 1977;
revised December 5, 1977

79-3364 — Szegedi Nyomda — Felelős vezető: Dobó József igazgató