A STRONGER ASSOCIATION RULE IN LATTICES, POSETS AND DATABASES

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Abstract. Galois closure operators associated with relations play an important role in many fields of pure and applied mathematics. Given a relation $\rho \subseteq A^{(0)} \times A^{(1)}$, in other terminology a context $(A^{(0)}, A^{(1)}, \rho)$, the pair of the induced Galois closure operators will be denoted by $\mathcal{G} = \mathcal{G}(A^{(0)}, A^{(1)}, \rho)$. The present paper studies a new pair $\mathcal{C} = \mathcal{C}(A^{(0)}, A^{(1)}, \rho)$ of closure operators which has been introduced and applied in [3]. For $\rho \in \{\le, <, \preceq, \prec\}$, the main theorem characterizes finite posets P with $\mathcal{C}(P, P, \rho) = \mathcal{G}(P, P, \rho)$. It is proved that $\mathcal{C}(J(L),\ M(L),\ \le) = \mathcal{G}(J(L),\ M(L),\ \le)$ when L is a finite modular lattice.

Since relations occur in many fields of pure mathematics, and also in some fields of applied mathematics including formal context analysis, decision making and knowledge discovery from databases, we have a general hope that $\mathcal{C}$ will be of some interest for at least some of these fields. The main theorem justifies this hope by asserting that $\mathcal{C}$ is often distinct from $\mathcal{G}$. Although we do not have applications with real existing contexts, a lot of possible contexts are mentioned to indicate that $\mathcal{C}$ does have a practical meaning in several cases.

1. Introduction and motivation

Although the terminology of formal context analysis and that of data bases (shortly: FCA) will be frequently used in this section, as long as we do not have real applications with concrete databases and contexts we cannot say that our purely mathematical investigations have a substantial overlapping with FCA. However, there is a hope and the present paper gives several indications that the stronger association rules we introduce are meaningful for other fields not just for pure algebra.

The history of science has several examples showing that a proper treatment, arrangement or visualization of information can be the source of new information. Many of these examples witness that the relevant mathematical tool was developed much before any application of this kind. For the classical periodic system of chemical elements Mendeleyev resorted to the ancient "mathematical" notion of binary tables. Formal concept analysis, cf. Wille [9] and Ganter and Wille [6], uses an old concept that goes back to Évariste Galois. In the rest of this section we recall Wille's theory, generalize its main tool, and pay a lot of attention to motivations.

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Following Wille's terminology, cf. [9] or [6], a triplet

$$(A^{(0)}, A^{(1)}, \rho)$$

is called a *context* if $A^{(0)}$ and $A^{(1)}$ are nonempty sets and $\rho \subseteq A^{(0)} \times A^{(1)}$ is a binary relation. From what follows, we fix a context $(A^{(0)}, A^{(1)}, \rho)$ and let

$$\rho_0 = \rho \text{ and } \rho_1 = \rho^{-1}.$$

From now on, unless otherwise stated, i will be an arbitrary element of $\{0, 1\}$. So whatever we say including i without specification, it will be understood as prefixed by $\forall i$. The set of all subsets of $A^{(i)}$ will be denoted by $P(A^{(i)})$.

It is often, especially in the finite case, convenient to depict our context in the usual form: a binary table with row labels from $A^{(0)}$, column labels from $A^{(1)}$, and a cross in the intersection of the x-th row and the y-th column iff $(x, y) \in \rho$. We will refer to this table as the *context table*. For example, a context (say, about juggling) is given by Table 1. (In an appropriate sense, cf. [4], this is the smallest interesting example.) The objects $a_1, \ldots, a_5$ are certain juggling tricks and the b_i are some relevant attributes; however, the reader need not know anything about juggling[1] to follow the rest of the paper.

	b_1	b_2	b_3	b_4
a_1	×	×		
a_2	×		×	
a_3	×	×		×
a_4	×			
a_5		×	×	×

Table 1

A mapping $\mathcal{D}^{(i)} : P(A^{(i)}) \to P(A^{(i)})$ is called a *closure operator* if it is *extensive* (i.e., $X \subseteq \mathcal{D}^{(i)}(X)$ for all $X \in P(A^{(i)})$), *monotone* (i.e., $X \subseteq Y$ implies $\mathcal{D}^{(i)}(X) \subseteq \mathcal{D}^{(i)}(Y)$), and *idempotent* (i.e., $\mathcal{D}^{(i)}(\mathcal{D}^{(i)}(X)) = \mathcal{D}^{(i)}(X)$ for all $X \in P(A^{(i)})$). By a *pair of extensive operators* we mean a pair $\mathcal{D} = (\mathcal{D}^{(0)}, \mathcal{D}^{(1)})$ where $\mathcal{D}^{(i)} : P(A^{(i)}) \to P(A^{(i)})$ is an extensive mapping for $i = 0, 1$. If these mappings are closure operators then $\mathcal{D}$ is called a *pair of closure operators*.

If $\mathcal{D} = (\mathcal{D}^{(0)}, \mathcal{D}^{(1)})$ and $\mathcal{E} = (\mathcal{E}^{(0}, \mathcal{E}^{(1)})$ are pairs of extensive operators then $\mathcal{D} \le \mathcal{E}$ means that $\mathcal{D}^{(i)}(X) \subseteq \mathcal{E}^{(i)}(X)$ for all $i \in \{0, 1\}$ and all $X \in P(A^{(i)})$.

Now, associated with $(A^{(0)}, A^{(1)}, \rho)$, we define some pairs of closure operators. The motivation will be given afterwards. For $X \in P(A^{(i)})$ let

$$X\rho_i = \{y \in A^{(1-i)} : \text{ for all } x \in X,\ (x, y) \in \rho_i\},$$

and, again for $X \in P(A^{(i)})$, define

$$\mathcal{G}^{(i)}(X) := (X\rho_i)\rho_{1-i} = \bigcap_{y \in X\rho_i} (\{y\}\rho_{1-i}) \ .$$

Then $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$ is the well-known *pair of Galois closure operators*, which plays the main role in formal concept analysis, cf. Wille [9] and Ganter and Wille [6]. The visual meaning of

$$\mathcal{G} = \mathcal{G}(A^{(0)}, A^{(1)}, \rho)$$

[1]The concrete meaning of the context is defined (partially) via video clips, cf. http://www.math.u-szeged.hu/∼czedli/jtable.html. For more about juggling cf. Polster [8].

is the following. The maximal subsets of ρ of the form $U^{(0)} \times U^{(1)}$ with $U^{(i)} \subseteq A^{(i)}$ are called the (formal) *concepts*, cf. [9] or [6]. Pictorially, they are the maximal full rectangles $U^{(0)} \times U^{(1)}$ of the context table. (Full means that each entry is a cross.) For $X_i \in P(A^{(i)})$ take all maximal full rectangles $U^{(0)} \times U^{(1)}$ such that $X \subseteq U^{(i)}$, then $\mathcal{G}^{(i)}(X)$ is the intersection of all the $U^{(i)}$'s.

Now we define a sequence $\mathcal{C}_i$, $i = 0, 1, 2, \dots$, of pairs of of closure operators. For $X \in P(A^{(i)})$ let

$$X\psi_i := \{Y \in P(A^{(1-i)}) : \text{ there is a surjection } \varphi : X \to Y \text{ with } \varphi \subseteq \rho_i\}.$$

Pictorially, the elements of $X\psi_i$ are easy to imagine. For example, let $i = 0$, i.e., let $X \subseteq A^{(0)}$ be a set of rows. Select a cross in each row of X, then the collection of the columns of the selected crosses is an element of $X\psi_0$, and each element of $X\psi_0$ is obtained this way. For example, if $X = \{a_1, a_2\}$ in Table 1 then $X\psi_0$ consists of $\{b_1\}$, $\{b_1, b_2\}$, $\{b_1, b_3\}$ and $\{b_2, b_3\}$.

Let $\mathcal{C}_0 = \mathcal{G}$. If $\mathcal{C}_n$ is already defined then let

$$\mathcal{C}^{(i)}_{n+1}(X) := \mathcal{C}^{(i)}_n(X) \cap \bigcap_{Y \in X\psi_i} \; \bigcup_{y \in \mathcal{C}^{(1-i)}_n(Y)} \{y\}\rho_{1-i} \,. \tag{1}$$

This defines the pair $\mathcal{C}_{n+1} = (\mathcal{C}^{(0)}_{n+1}, \mathcal{C}^{(1)}_{n+1})$.

The easiest way to digest formula (1) is to think of it pictorially. For example, let $i = 0$ and $X \subseteq A^{(0)}$, and suppose that $\mathcal{C}_n = (\mathcal{C}^{(0)}_n, \mathcal{C}^{(1)}_n)$ is already well-understood. Then a row z belongs to $\mathcal{C}^{(0)}_{n+1}(X)$ if and only if $z \in \mathcal{C}^{(0)}_n(X)$ and, in addition, for each set $Y \in X\psi_0$ of columns there is a column y in $\mathcal{C}^{(1)}_n(Y)$ such that y intersects the row z at a cross. (Notice that $X\psi_0$ has already been explained pictorially, $\mathcal{C}^{(0)}_n(X)$ and $\mathcal{C}^{(1)}_n(Y)$ are already well-known by assumption, and y need not be unique and it depends on Y.)

Finally, let

$$\mathcal{C} = (\mathcal{C}^{(0)}, \mathcal{C}^{(1)}) := (\bigwedge_{n=0}^{\infty} \mathcal{C}^{(0)}_n, \bigwedge_{n=0}^{\infty} \mathcal{C}^{(1)}_n),$$

which means that, for all $X \in P(A^{(i)})$,

$$\mathcal{C}^{(i)}(X) = \bigcap_{n=0}^{\infty} \mathcal{C}^{(i)}_n(X).$$

Although the above definitions look neither friendly nor natural at the first sight, they had a proper application in [3]; proper means that $\mathcal{C}$ was heavily used when proving a theorem which has nothing to do with the notion of $\mathcal{C}$. Notice also that it was routine to prove in [3] that we have indeed defined pairs of closure operators.

Lemma 1. *(cf. [3]) $\mathcal{C} = \mathcal{C}(A^{(0)}, A^{(1)}, \rho)$ and $\mathcal{C}_n = \mathcal{C}_n(A^{(0)}, A^{(1)}, \rho)$ $(n = 0, 1, \dots)$ are pairs of closure operators. Further, $\mathcal{G} = \mathcal{C}_0 \geq \mathcal{C}_1 \geq \mathcal{C}_2 \geq \cdots \geq \mathcal{C}$.*

It is well-known that, for each context $(A^{(0)}, A^{(1)}, \rho)$, the complete lattices $(\{X \in P(A^{(0)}) : \mathcal{G}^{(0)}(X) = X\}, \subseteq)$ and $(\{X \in P(A^{(1)}) : \mathcal{G}^{(1)}(X) = X\}, \subseteq)$ are dually isomorphic. The analogous statement is far from being true for $\mathcal{C}$; indeed, in case of the context given by Table 1, $|\{X \in P(A^{(0)}) : \mathcal{C}^{(0)}(X) = X\}| = 12$ while $|\{X \in P(A^{(1)}) : \mathcal{C}^{(1)}(X) = X\}| = 10$.

From now on *we always assume* that $(A^{(0)}, A^{(1)}, \rho)$ is *finite*. Then there are only finitely many pairs of operators, whence there is a smallest n with $\mathcal{C} = \mathcal{C}_n = \mathcal{C}_{n+1} = \mathcal{C}_{n+2} = \cdots$. The natural question if there is an upper bound on n will be answered at the very end of the paper. There is another question: why to deal with $\mathcal{C}$ rather then with $\mathcal{C}_1$. The first reason is that our goal is to study when $\mathcal{C} \neq \mathcal{G}$, which is clearly equivalent to $\mathcal{C}_1 \neq \mathcal{G}$. The other reason is that, according to our motivations that will be detailed later, $\mathcal{C}$ seems to be better than $\mathcal{C}_1$ from several aspects, so it is reasonable to study $\mathcal{C}$ first.

Now we explain why "association rule" occurs in the title of the paper, and this will be a part of our motivations. Closure operators have been playing an important role in the theory of relational databases and knowledge systems for a long time, cf. e.g., Caspard and Monjardet [2] for a survey. Nowadays most investigations of this kind belong to formal concept analysis, cf. Ganter and Wille [6] for an extensive survey. The theory of mining association rules goes back to Agrawal, Imielinski and Swami [1]; Lakhal and Stumme [7] gives a good account on the present status of this field.

For a data miner, the context is a *huge* binary database, and mining association rules is a popular knowledge discovery technique for warehouse basket analysis. In this case $A^{(0)}$ is the set of costumers' baskets, $A^{(1)}$ is the set of items sold in the warehouse, and the task is to figure out which items are frequently bought together. This information is expressed by so-called "association rules". For example,

$$\{\text{cereal, coffee}\} \to \{\text{milk}\}$$

is an association rule (in many real warehouses), and this association rule says that, with a given probability p, costumers buying cereal and coffee also buy milk. When milk $\in \mathcal{G}^{(1)}\{\text{cereal, coffee}\}$ then this probability is 1 and we speak about a *strong* association rule. The knowledge of association rules can help the warehouse in developing appropriate marketing strategies.

However, the importance of looking for the hidden regularities and rules is not restricted only to *huge* databases. Indeed, Mendeleyev's classical "database" and many real contexts from Ganter and Wille [6] are far from being huge. This means that exploring hidden rules in *small* databases may also lead to important results. From this aspect, the present paper offers $\mathcal{C}$, a mathematical tool, to formulate some regularities in abstract contexts. Since $\mathcal{C} \leq \mathcal{G}$, the "association rules" corresponding to $\mathcal{C}$ are *stronger* than the previously mentioned ones. This seems to be important, for finding associations is an integral part of any creative activity.

Now we use the context given by Table 1 to explain our motivations further. Let $X = \{a_1, a_2\} \subseteq A^{(0)} = \{a_1, \ldots, a_5\}$. Then $\{a_1, \ldots, a_4\} \times \{b_1\}$ is the only relevant maximal full rectangle to compute $\mathcal{G}^{(0)}(X) = \{a_1, \ldots, a_4\}$. Since $Y = \{b_2, b_3\} \in X\psi_0$ but there is no $y \in \mathcal{G}^{(1)}(Y) = \{b_2, b_3, b_4\}$ with $a_4 \in \{y\}\rho_1$, formula (1) gives $a_4 \notin \mathcal{C}_1^{(0)}(X)$. After the trivial and therefore omitted details we can easily see that $\mathcal{C} = \mathcal{C}_1$ and $\mathcal{C}^{(0)}(X) = \{a_1, a_2, a_3\}$.

Suppose our whole knowledge is decoded in the context and we have to associate an element with X. Usually we want an element outside X, and we look for something similar, i.e., we want an element which shares the common attributes of the elements of X. So the first answer is that we should associate some element of $\mathcal{G}^{(0)}(X) \setminus X = \{a_3, a_4\}$. This way we obtain more than one element, but we may want to chose only a single one. For example, which of a_3 and a_4 should a scientist choose if the context represents something in his research field and choosing both is not permitted? The unique element a_3 of $\mathcal{C}^{(0)}(X) \setminus X$? The other element, a_4? At this level of generality we cannot answer to this question of *decision making*. Fortunately, it is not our task to give an answer and specialists of concrete fields can interpret their contexts appropriately.

Well, if we choose the set of attributes in a random way, if we allow good and bad, important and unimportant attributes at the same time then probably we cannot tell which of a_3 and a_4 should be chosen. But the situation is much better if we assume that all the attributes (and all the objects) are *positive ones*. "Positive" here means something important that we want. It depends on the situation and the person (or the respective field of science) whether an attribute (or object) is positive or not.

To express this idea better, let us return the concrete juggling meaning of Table 1. Assume that a person M has already learnt the elements of $X = \{a_1, a_2\}$ but not the rest of the objects, and he has to decide which single one of the rest he should learn next. Assume also that M considers all the objects and attributes positive. In case of objects, positiveness may mean availability (i.e., the opposite of hopelessness). If M is ambitious then the positivity of attributes means that each attribute expresses some kind of difficulty (which is approximately the same as some kind of attractiveness)[2]. It is clear from definitions that the objects in $\mathcal{C}^{(0)}(X) \setminus X$ have "more" attributes[3] (somehow related with X) than the objects in $\mathcal{G}^{(0)}(X) \setminus \mathcal{C}^{(0)}(X)$. (Check Table 1 at this point, too.) Therefore it is clear that the ambitious M should choose from $\mathcal{C}^{(0)}(X) \setminus X$, i.e., he should choose a_3. If M is far from being ambitious and looks for the easiest way than he should choose from $\mathcal{G}^{(0)}(X) \setminus \mathcal{C}^{(0)}(X)$, i.e., he should choose a_4. (Or he can use another set of attributes expressing easiness.)

It is easy to imagine many similar examples where $X \subseteq A^{(0)}$ is accomplished in some sense and one has to choose the next object to accomplish. For example, the set of objects can consist of courses offered by a university (to take), musical compositions (to learn or listen), mountain peaks (to climb), foreign languages (to learn), books (to read), type of cars (to buy), dangers (to avoid), etc. For all of these sets of objects an involved person can easily define his own set of positive attributes. Natural or medical sciences may give rise to even more contexts with positive attributes, for example the objects can be pharmaceutical features while the attributes are some chemical compounds. However, only specialists can decide which objects and attributes are positive and which contexts are interesting. Another example is when $A^{(0)}$ is a certain set of persons and $X \subseteq A^{(0)}$ is public (political, scientific, etc.) body that intends to adopt some new members.

Notice that it is possible to consider contexts where the positive attributes form a proper subset of $A^{(1)}$, cf. [5], but this is not pursued in this paper.

We have seen that there is reasonable expectation that $\mathcal{C}$ gives some insights into various fields, and this raises the problem if, at least in certain cases, $\mathcal{C}$ can do more than $\mathcal{G}$. In the light of the above argument this problem seems to reduce to the question how often $\mathcal{C}$ is different from $\mathcal{G}$. This will be answered for certain

[2]Version A in the web site gives exactly this concrete meaning to the context.

[3]Of course, "more" here is not a numerical statement. Notice that, in case of numbers, "small" and "large" are relative notions and would not lead to a unique definition for all contexts.

contexts arising from lattices and posets in the next section, the main part of the paper, while there are some experimental results for other contexts in [4]. Notice that conditions guaranteeing $\mathcal{C} = \mathcal{G}$ can also be interesting, for they point out a specific property of $\mathcal{G}$ not studied before.

Last but not least we should not forget that the main motivation to investigate $\mathcal{C}$ is that [3], where $\mathcal{C}$ has a proper application, witnesses that $\mathcal{C}$ is useful in algebra.

2. Lattices and posets

There are many frequently used relations, and therefore contexts, when lattices and posets (=partially ordered sets) are studied. It is probably not always possible to describe those with $\mathcal{C} \neq \mathcal{G}$ in an elegant way. However, there are some particular relations where something interesting can be stated. As usual, for a finite lattice L the set of nonzero join irreducible elements is denoted by $J = J(L)$ while $M = M(L)$ is the set of meet irreducible elements distinct from 1. The context $(J, M, \leq)$ is famous since Wille [9] has pointed out that its concept lattice is isomorphic to L, so this context is a very economic way to describe L up to isomorphism.

Theorem 1. *Let L be a finite lattice. If L is modular then*

$$\mathcal{C}(J(L), M(L), \leq) = \mathcal{G}(J(L), M(L), \leq).$$

Note that the converse is unfortunately not true. There are nonmodular lattices, like the five element ones: N_5 and M_3, for which $\mathcal{C} = \mathcal{G}$. But there are a plenty of nonmodular lattices for which $\mathcal{C} \neq \mathcal{G}$. The simplest such example is perhaps an n-crown with additional 0 and 1 for $n \geq 4$, i.e. the $(2n+2)$-element lattice $(\{0, a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}, 1\}, \leq)$ where the $\{a_0, \ldots, a_{n-1}\}$ is the set of atoms, $\{b_0, \ldots, b_{n-1}\}$ is the set of coatoms, and $a_j < b_k$ iff $k \in \{j, j+1\}$ where $j+1$ is understood modulo n. (Notice that this example follows as a particular case of Problem 1.)

Given a context $(A^{(0)}, A^{(1)}, \rho)$, by the *dual context* we mean

$$(A^{(1)}, A^{(0)}, \rho^{-1}).$$

Clearly, if $L_d = (L_d, \leq_d)$ denotes the dual of L, then $(J(L_d), M(L_d), \leq_d)$ is the dual of the context $(J(L), M(L), \leq)$. Similar observations are valid for posets occurring in the next theorem. Hence the lattice duality principle extends to our case and can be used in our proofs.

Proof. Let L be a finite modular lattice. Its ordering relation will also be denoted by $\rho = \rho_0$. Since modularity is a selfdual lattice property, by the duality principle it suffices to show that $\mathcal{C}^{(0)} = \mathcal{G}^{(0)}$. Let $X = \{a_1, \ldots, a_n\} \subseteq J$ with $|X| = n \geq 1$ and let

$$x \in \mathcal{G}^{(0)}(X) = (X\rho_0)\rho_1 = ([a_1 \vee \cdots \vee a_n) \cap M)\rho_1 = (a_1 \vee \cdots \vee a_n] \cap J$$

be an arbitrary element. Let $Y = \{b_1, \ldots, b_n\} \in X\psi_0$. This means that $a_j \leq b_j \in M$ for $j = 1, \ldots, n$ (but the b_j are not necessarily distinct). Then, dually to the above displayed formula, $\mathcal{G}^{(1)}(Y) = [b_1 \wedge \cdots \wedge b_n) \cap M$. Let $b = b_1 \wedge \cdots \wedge b_n$. According to formula (1), we have to show that

$$\text{there exists a } y \in [b) \cap M \text{ such that } x \leq y.$$

This is evident when $x \vee b \neq 1$, for $[x \vee b) \cap M$ is not empty in this case. So we assume that $x \vee b = 1$. Then

$$1 = x \vee b = (b_1 \wedge \cdots \wedge b_n) \vee x \leq b_1 \vee \cdots \vee b_n \vee x =$$
$$(a_1 \vee b_1) \vee \cdots (a_n \vee b_n) \vee x =$$
$$(b_1 \vee \cdots \vee b_n) \vee (a_1 \vee \cdots \vee a_n \vee x) =$$
$$= (b_1 \vee \cdots \vee b_n) \vee (a_1 \vee \cdots \vee a_n) =$$
$$(a_1 \vee b_1) \vee \cdots (a_n \vee b_n) = b_1 \vee \cdots \vee b_n.$$

Hence $b_1 \vee \cdots \vee b_n = 1$ and this happens in the interval $[b, 1] = [b, b \vee x]$. Since L is modular, this interval is isomorphic to the interval $[b \wedge x, x]$. But $x \in J$, so x is join irreducible also in the interval $[b \wedge x, x]$, whence 1 is join irreducible in $[b, 1]$, and we conclude that $b_j = 1$ for some j. But this is a contradiction, for $b_j \in M$ and $1 \notin M$. Thus we have shown that $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$ when $X \neq \emptyset$. When X is the empty set then $\emptyset\rho_0 = M$ and therefore $\mathcal{C}_1^{(0)}(\emptyset) = \mathcal{G}^{(0)}(\emptyset) = \emptyset$. This proves $\mathcal{C}_1^{(0)} = \mathcal{G}^{(0)}$, whence $\mathcal{C}^{(0)} = \mathcal{G}^{(0)}$. □

In order to formulate the main theorem, we need some definitions. Let $Q = (Q, \leq)$ be a finite poset. Let $\max(Q)$ resp. $\min(Q)$ denote the set of maximal resp. minimal elements of Q. Notice that Q is an antichain iff $\max(Q) = \min(Q) = Q$. If Q is a chain then the length of Q, denoted by $\operatorname{length}(Q)$, is $|Q| - 1$. In the general case, $\operatorname{length}(Q)$ is the maximum of the set $\{\operatorname{length}(C) : C \subseteq Q \text{ and } C \text{ is a chain}\}$. If X is a subset of Q then $L(X)$ denotes the set of lower bounds of X:

$$L(X) = \{y \in Q : y \leq x \text{ for all } x \in X\},$$

and, dually, $U(X)$ denotes the set of upper bounds of X. In particular, $U(\emptyset) = L(\emptyset) = Q$. For $X = \{x_1, \ldots, x_n\}$ we will write $U(x_1, \ldots, x_n)$ instead of $U(\{x_1, \ldots, x_n\})$, and the same convention applies for L. The disjoint union (or cardinal sum) of the posets $(Q_1, \leq_1)$ and $(Q_2, \leq_2)$ is $(Q_1 \cup Q_2, \leq_1 \cup \leq_2)$ where Q_1 is assumed to be distinct from Q_2. For example, an n-element antichain is the disjoint union of n chains of length 0.

Finally, we have to define three kinds of posets, cf. also Figure 1. For $1 \leq m, n$ we define an $(m + n + 1)$-element poset $T_{mn} = \{a_1, \ldots, a_m,\ b,\ d_1, \ldots, d_n\}$ such that $\min(T_{mn}) = \{a_1, \ldots, a_m\}$, $\max(T_{mn}) = \{d_1, \ldots, d_n\}$ and $a_j < b < d_k$ for all $(j, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$. In particular, T_{11} is the three element chain. For $2 \leq m, n$ we define two $(m + n)$-element posets, $G_{mn} = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ and $H_{mn} = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ such that $\min(G_{mn}) = \min(H_{mn}) = \{a_1, \ldots, a_m\}$, $\max(G_{mn}) = \max(H_{mn}) = \{b_1, \ldots, b_n\}$ and we have $a_j < b_k$ for all $(j, k) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ in G_{mn} while $a_j < b_k$ iff $1 \in \{k, j\}$ in H_{mn}.

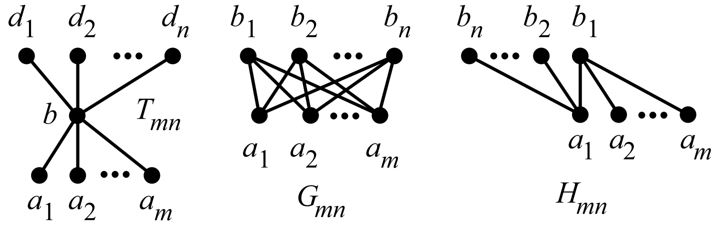

Figure 1

Theorem 2. *Let $Q = (Q, \le)$ be a finite poset, and let us consider the contexts $(Q, Q, <)$, $(Q, Q, \prec)$, $(Q, Q, \le)$ and $(Q, Q, \preceq)$. Then we have*

(A) $\mathcal{C}(Q, Q, <) = \mathcal{G}(Q, Q, <)$ if and only if $U(Q \setminus \max(Q)) \neq \emptyset$ and $L(Q \setminus \min(Q)) \neq \emptyset$.

(B) $\mathcal{C}(Q, Q, \prec) = \mathcal{G}(Q, Q, \prec)$ if and only if $\mathrm{length}(Q) \le 1$, $U(Q \setminus \max(Q)) \neq \emptyset$ and $L(Q \setminus \min(Q)) \neq \emptyset$.

(C) $\mathcal{C}(Q, Q, \le) = \mathcal{G}(Q, Q, \le)$ if and only if either $|\max(Q)| = |\min(Q)| = 1$, or $|\max(Q)| \ge 2$, $|\min(Q)| \ge 2$ and

$$(\forall x, y, z, t \in \max(Q))\ (x \neq y \text{ and } z \neq t \text{ imply } L(x, y) = L(z, t)) \text{ and}$$
$$(\forall x, y, z, t \in \min(Q))\ (x \neq y \text{ and } z \neq t \text{ imply } U(x, y) = U(z, t)).$$

(D) $(Q, Q, \preceq)$. Then $\mathcal{C}(Q, Q, \preceq) = \mathcal{G}(Q, Q, \preceq)$ if and only if either Q is (isomorphic to) T_{mn} for some $m, n \ge 1$, or Q is H_{mn} or G_{mn} for some $m, n \ge 2$, or $\mathrm{length}(Q) \le 1$ and Q is a disjoint union of chains.

Proof. Let $\rho = \rho_0$ denote the relation of the context in question, and remember that ρ_1 stands for ρ^{-1}. Since the conditions in the theorem are selfdual, by the duality principle it will suffice to deal with $\mathcal{C}^{(0)}$ and $\mathcal{G}^{(0)}$. Formula (1) will be used often without referring to it. Notice that $\mathcal{C} = \mathcal{G}$ iff $\mathcal{C}_1 = \mathcal{G}$, and this fact will be used implicitly either (so we usually drop the subscript of $\mathcal{C}_1$).

(A) Suppose that $\mathcal{C} = \mathcal{C}(Q, Q, <)$ coincides with $\mathcal{G} = \mathcal{G}(Q, Q, <)$. Let $A = Q \setminus \max(Q)$ and $B = Q \setminus \min(Q)$. By way of contradiction, suppose that $U(A)$ or $L(B)$ is empty. By the duality principle, it suffices to consider the case when $U(A)$ is empty. Then $A \neq \emptyset$, $A\rho_0 = \emptyset$ and $\mathcal{G}^{(0)}(A) = (A\rho_0)\rho_1 = Q$. Let $x \in \max(Q) \subseteq \mathcal{G}^{(0)}(A) = \mathcal{C}^{(0)}(A)$. Clearly, $A\psi_0$ is not empty, so we can choose a $Y \in A\psi_0$. However, since x is a maximal element, $x \in \{y\}\rho_1$, i.e. $x < y$, holds for no $y \in \mathcal{G}^{(1)}(Y)$. Hence $x \notin \mathcal{C}^{(0)}(A)$, a contradiction.

To prove the converse, suppose that A has an upper bound a and B has a lower bound b. We can assume that $a \in \max(Q)$ and $b \in \min(Q)$. If A or B is empty then Q is an antichain, and $\mathcal{C} = \mathcal{G}$ follows from the fact that $X\psi_i$ is empty when X is nonempty. Hence we assume that neither A nor B is empty. Clearly, $x < a$ for all $x \in A$ and $b < y$ for all $y \in b$. Moreover, $a \in B$ and $b \in A$, and therefore $b < a$. Notice that, for any $\emptyset \neq U \subseteq Q$, $U\rho_0 \subseteq B$ and $U\rho_1 \subseteq A$.

Let X be a subset of Q. If $X = \emptyset$ then $X\rho_0 = Q$ gives $\mathcal{G}^{(0)}(X) = Q\rho_1 = \emptyset$, so $\mathcal{C}^{(0)}(\emptyset) = \mathcal{G}^{(0)}(\emptyset)$. If $X \nsubseteq A$ then $X\psi_0 = \emptyset$ yields $\mathcal{C}^{(0)}(X) = \mathcal{G}^{(0)}(X)$.

Now let $X \subseteq A$. Then $X\rho_0 \supseteq \{a\}$ yields

$$\mathcal{G}^{(0)}(X) = (X\rho_0)\rho_1 \subseteq \{a\}\rho_1 = L(a) \setminus \{a\} \subseteq A.$$

Now let $x \in \mathcal{G}^{(0)}(X)$ and $Y \in X\psi_0$ be arbitrary. Then $Y\rho_1 \subseteq A$ gives $\mathcal{G}^{(1)}(Y) = (Y\rho_1)\rho_0 \supseteq A\rho_0 \ni a$. Since $x \in A$, $x \in \{a\}\rho_1$. Hence a can play the role of y in formula (1) and we obtain $x \in \mathcal{C}_1^{(0)}(X)$. This shows that $\mathcal{C}_1 = \mathcal{G}$, thus $\mathcal{C} = \mathcal{G}$, proving part (A) of the theorem.

(B) Suppose $\mathrm{length}(Q) \ge 2$. Then we can choose $a, b, c \in Q$ such that $a \prec b$, $b \prec c$ and $c \in \max(Q)$. Let $X = \{a, b\}$. Then $\mathcal{G}^{(0)}(X) = (\{a, b\})\rho_0)\rho_1 = \emptyset\rho_1 = Q$. If $\mathcal{C} = \mathcal{G}$ then $c \in Q = \mathcal{G}^{(0)}(X) = \mathcal{C}^{(0)}(X)$ and $Y = \{b, c\} \in X\psi_0$ implies that $c \in \{y\}\rho_1$, i.e. $c \prec y$, for some $y \in \mathcal{G}^{(1)}(Y)$, which contradicts $c \in \max(Q)$. Hence

$\mathcal{C} = \mathcal{G}$ implies length$(Q) \leq 1$. Then $(Q, Q, \prec)$ is exactly the same context as $(Q, Q, <)$, and the rest of part (B) follows from part (A).

(C) Consider the context $(Q, Q, \leq)$ and suppose that $\mathcal{C} = \mathcal{G}$. Suppose first that $|\min(Q)| = 1$, i.e., Q has a unique least element 0. Let $X = \emptyset$. Then $\mathcal{C}^{(0)}(X) = \mathcal{G}^{(0)}(X) = Q\rho_1 = \{0\}$ and $Y = \emptyset \in X\psi_0$ yields that there is a $y \in \mathcal{G}^{(1)}(Y)$ with $0 = x \in \{y\}\rho_1$. Thus $\mathcal{G}^{(1)}(Y) = \mathcal{G}^{(1)}(\emptyset) = (\emptyset\rho_1)\rho_0 = Q\rho_0 = \{z \in Q : t \leq z \text{ for all } t \in Q\}$ is nonempty. Therefore Q has a greatest element and $|\max(Q)| = 1$. Now the duality principle gives that $|\min(Q)| = 1$ iff $|\max(Q)| = 1$, and the condition of (C) holds.

Now suppose that $|\min(Q)| > 1$, then $|\max(Q)| > 1$ either. By way of contradiction let us assume that $L(u, v)$ (where $u \neq v$) is not a constant on $\max(Q)$. Then we can choose a three element subset $\{a, b, c\}$ of $\max(Q)$ such that $L(a, b) \not\subseteq L(a, c)$. Then there is an element $x \in L(a, b) \setminus L(a, c)$. Let $X = \{a, b\}$. We obtain $\mathcal{G}^{(0)}(X) = (X\rho_0)\rho_1 = L(U(a, b)) = L(\emptyset) = Q$, so $c \in \mathcal{G}^{(0)}(X)$. Let $Y = X$. Then $Y \in X\psi_0$ and $\mathcal{C} = \mathcal{G}$ imply that there is an element $y \in \mathcal{G}^{(1)}(Y)$ with $c \in \{y\}\rho_1$, i.e, $c \leq y$. Since $c \in \max(Q)$, $c = y \in \mathcal{G}^{(1)}(Y) = (Y\rho_1)\rho_0 = U(L(a, b))$. This and $x \in L(a, b)$ yield $x \leq c$, contradicting $x \in L(a, b) \setminus L(a, c)$. This shows that L is constant on $\{(u, v) : u, v \in \max(Q) \text{ and } u \neq v\}$. It follows from the duality principle that U is constant on $\{(u, v) : u, v \in \min(Q) \text{ and } u \neq v\}$.

Now, to prove the converse, suppose first that $0, 1 \in Q$, i.e., $|\max(Q)| = |\min(Q)| = 1$. Then $1 \in U(Q) = U(L(\emptyset)) = \mathcal{G}^{(1)}(\emptyset)$. Since $\mathcal{G}^{(1)}$ is monotone, $1 \in \mathcal{G}^{(1)}(Y)$ for any $Y \subseteq Q$. Moreover, $\{1\}\rho_1 = Q$. Hence 1 can always serve as y in formula (1), and we conclude that $\mathcal{C} = \mathcal{G}$.

From now on we suppose that $|\max(Q)| = |\min(Q)| \geq 2$, L is constant on $\{(u, v) : u, v \in \max(Q) \text{ and } u \neq v\}$ and U is constant on $\{(u, v) : u, v \in \min(Q) \text{ and } u \neq v\}$. Then $\mathcal{G}^{(0)}(\emptyset) = L(U(\emptyset)) = L(Q) = \emptyset$ gives $\mathcal{C}^{(0)}(\emptyset) = \mathcal{G}^{(0)}(\emptyset)$.

Now let us consider a nonempty subset X of Q, and an arbitrary $Y \in X\psi_0$. Then Y is nonempty either. We distinguish two cases according to $U(Y)$.

First suppose that $U(Y)$ is nonempty, and let us fix an element $z \in U(Y)$. Since $Y \in X\psi_0$, $U(X) \supseteq U(Y)$, so $U(X) \supseteq \{z\}$, whence $\mathcal{G}^{(0)}(X) = L(U(X)) \subseteq L(\{z\}) = \{z\}\rho_1$. On the other hand, the transitivity of the ordering gives $U(Y) \subseteq U(L(Y)) = \mathcal{G}^{(1)}(Y)$, whence $z \in \mathcal{G}^{(1)}(Y)$. Now it is clear from formula (1) that $\mathcal{C}^{(0)}(X) = \mathcal{G}^{(0)}(X)$.

Secondly, we suppose that $U(Y)$ is empty. Then there are $y_1, y_2 \in Y$ and $z_1, z_2 \in \max(Q)$ such that $y_1 \leq z_1$, $y_2 \leq z_2$ and $z_1 \neq z_2$. Since $\mathcal{G}^{(1)}(Y) = U(L(Y))$ is an order filter including Y, $\{z_1, z_2\} \subseteq \mathcal{G}^{(1)}(Y)$. Now let x be an arbitrary element of $\mathcal{G}^{(0)}(X)$, and choose an element $\tilde{x} \in \max(Q)$ such that $x \leq \tilde{x}$. If $\tilde{x} = z_j$ for some $j \in \{1, 2\}$ then we can chose $y = \tilde{x} = z_j$ in formula (1). Hence we can assume that $|\{\tilde{x}, z_1, z_2\}| = 3$. Using the assumption that L is constant for distinct maximal elements we obtain

$$\tilde{x} \in \mathcal{G}^{(1)}(\{\tilde{x}, z_1\}) = U(L(\tilde{x}, z_1)) = U(L(z_1, z_2)) = \\ \mathcal{G}^{(1)}(\{z_1, z_2\}) \subseteq \mathcal{G}^{(1)}(\mathcal{G}^{(1)}(Y)) = \mathcal{G}^{(1)}(Y),$$

and therefore the choice $y = \tilde{x}$ for formula (1) works again. This shows that $\mathcal{C}_1^{(0)}(X) = \mathcal{G}^{(0)}(X)$ for any $X \in P(A^{(0)})$. So $\mathcal{C} = \mathcal{G}$, proving part (C).

(D) Consider the context $(Q, Q, \preceq)$ and suppose Q is one of the posets listed in (D). We need to show that $\mathcal{C} = \mathcal{G}$. If length$(Q) \leq 1$ then $(Q, Q, \preceq)$ coincides with

$(Q, Q, \leq)$ and part (C) easily implies that $\mathcal{C} = \mathcal{G}$. So we can assume that Q is T_{mn} for some $m, n \geq 1$. By the duality principle, it suffices to show that $\mathcal{C}^{(0)} = \mathcal{G}^{(0)}$. Let

$$K = \{X \in P(Q) : (\forall Z \in P(Q))\,(Z \subset X \Rightarrow \mathcal{G}^{(0)}(Z) \subset \mathcal{G}^{(0)}(X))\}.$$

If $\mathcal{C}^{(0)}$ and $\mathcal{G}^{(0)}$ would agree on K then for any $X \in P(Q)$ we could take a minimal element Z of $\{X' \in P(Q) : \mathcal{G}^{(0)}(X') = \mathcal{G}^{(0)}(X)\}$, and from $Z \in K$ we could deduce

$$\mathcal{C}^{(0)}(X) \supseteq \mathcal{C}^{(0)}(Z) = \mathcal{G}^{(0)}(Z) = \mathcal{G}^{(0)}(X),$$

implying $\mathcal{C}^{(0)} = \mathcal{G}^{(0)}$.

Hence it suffices to show that for all $X \in K$, $\mathcal{C}^{(0)}(X) = \mathcal{G}^{(0)}(X)$. Moreover, it suffices to consider a small subset K' of K such that for all X in K there is an automorphism of Q which maps X to an element of K'. Let $A = \{a_1, \ldots, a_m\}$, $A^+ = A \cup \{b\}$, $D = \{d_1, \ldots, d_n\}$, $D^+ = D \cup \{b\}$, and assume that $m \geq 2$ and $n \geq 2$. (The case $m = 1$ or $n = 1$ is simpler and will not be detailed.) Then an appropriate K' is given by the second row in Table 2, where, for brevity, we write x, y instead of $\{x, y\}$:

X	$\emptyset$	a_1	a_1, a_2	b	a_1, b	a_1, d_1	b, d_1	d_1	d_1, d_2
$X \in K'$?	yes	yes	yes	yes	yes	yes	no	yes	yes
$\mathcal{G}^{(0)}(X)$	$\emptyset$	a_1	A^+	b	A^+	Q	b, d_1	b, d_1	Q
$\mathcal{G}^{(1)}(X)$	$\emptyset$	a_1, b	Q	b	a_1, b	Q	D^+	d_1	D^+

Table 2

Now we can easily list all possible Y's from formula (1) (up to isomorphism, again) and check that $\mathcal{C}^{(0)}(X) = \mathcal{G}^{(0)}(X)$ for $X \in K'$; the tedious details will be omitted.

Now, in order to prove the converse direction, assume that $\mathcal{C} = \mathcal{G}$. If length$(Q) = 0$ then Q is an antichain, which is a disjoint union of chains, and there is nothing to prove.

Now assume that length$(Q) = 1$ and Q is not a disjoint union of chains. Then part (C) of the theorem applies, so $2 \leq |\max(Q)|$, $2 \leq |\min(Q)|$, L is constant on $\{(u, v) : u, v \in \max(Q),\ u \neq v\}$ and U is constant on $\{(u, v) : u, v \in \min(Q),\ u \neq v\}$. Since Q is not a disjoint union of chains, there are $a_1, b_1, b_2 \in Q$ such that $a_1 < b_1$ and $a_1 < b_2$, or dually. So we can assume that $a_1 < b_1$ and $a_1 < b_2$. If $c \in \max(Q) \cap \min(Q)$ then $\emptyset = L(b_1, c) \neq L(b_1, b_2) \supseteq \{a_1\}$ would lead to a contradiction. Therefore Q is the disjoint union of $\max(Q)$ and $\min(Q)$. Notice also that the Hasse diagram of Q is connected as a graph, for otherwise we could find an $x \in \max(Q)$ with $L(b_1, x) = \emptyset$. Let

$$B = \{x \in \max(Q) : a_1 \leq x\}, \text{ and remember that } b_1, b_2 \in B.$$

Since a_1 is connected with other elements of $\min(Q)$ in the graph, there is an $a_2 \in \min(Q) \setminus \{a_1\}$ which is less than some element of B. So we can assume that $a_2 < b_1$. Let

$$A := \{x \in \min(Q) : x < b_1\}, \text{ and notice that } a_1, a_2 \in A.$$

If $c \in \max(Q) \setminus B$ then $a_1 \notin L(b_1, c) = L(b_1, b_2) \supseteq \{a_1\}$ would be a contradiction. Hence $B = \max(Q)$, and we obtain $A = \min(Q)$ similarly.

Now Q is the disjoint union of A and B. Let $m = |A|$ and $n = |B|$. If, for $a \in A$ and $b \in B$, $a < b$ holds only when $\{a_1, b_1\} \cap \{a, b\} \neq \emptyset$ then Q is H_{mn}. Otherwise we may suppose that $a_2 < b_2$. Then, for any $b \in B \setminus \{b_1\}$, $a_2 \in L(b_1, b_2) = L(b_1, b)$

yields $a_2 < b$. Hence $U(a_1, a_2) = B$, and for any $a \in A \setminus \{a_1\}$ we have $U(a_1, a) = U(a_1, a_2) = B$. This means that $Q = G_{mn}$, and the case length$(Q) = 1$ is settled.

Now suppose that length$(Q) \geq 2$ and introduce the notation

$$\mathrm{mid}(Q) = Q \setminus (\max(Q) \cup \min(Q)).$$

Let us observe that for any $u, v \in Q$,

$$\begin{aligned} &\text{if } u \prec v \text{ then } \mathcal{G}^{(0)}(\{u,v\}) = \{x : x \preceq v\} \\ &\qquad \text{and } \mathcal{G}^{(1)}(\{u,v\}) = \{x : u \preceq x\}. \end{aligned} \tag{2}$$

Indeed, $\mathcal{G}^{(0)}(\{u,v\}) = (\{u,v\}\rho_0)\rho_1 = \{v\}\rho_1 = \{x : x \preceq v\}$, and the other equation follows by duality.

First we consider the case when length$(Q) \geq 3$. Then there are elements $a, b, c \in Q$ and $d \in \max(Q)$ such that $a \prec b \prec c \prec d$. Let $X = \{b, d\}$. Then $\mathcal{C}^{(0)}(X) = \mathcal{G}^{(0)}(X) = (X\rho_0)\rho_1 = \emptyset\rho_1 = Q$. Let $Y = \{c, d\} \in X\psi_0$. Then for any $y \in \mathcal{G}^{(1)}(Y)$ we have $c \leq y$ by (2), so $a \not\preceq y$, whence $a \notin \{y\}\rho_1$, and $a \notin \mathcal{C}^{(0)}(X) = Q$ by formula (1), a contradiction. Hence length$(Q) \geq 3$ is excluded, and from now on we assume that length$(Q) = 2$.

The first step in the case length$(Q) = 2$ is to show that for any $b \neq c$

$$\text{if } b, c \in \mathrm{mid}(Q) \text{ and } b \parallel c \text{ then } |L(b,c)| \leq 1 \text{ and } |U(b,c)| \leq 1. \tag{3}$$

Suppose, by way of contradiction, that $d_1, d_2 \in U(b,c)$ and $d_1 \neq d_2$. Let $X = \{d_1, d_2\}$, and choose an element a such that $a \prec b$. Since $X \subseteq \max(Q)$, we obtain that $a \in Q = \emptyset\rho_1 = \mathcal{G}^{(0)}(X) = \mathcal{C}^{(0)}(X)$. Let $Y = X \in X\psi_0$. By formula (1) there is a $y \in \mathcal{G}^{(1)}(Y)$ with $a \preceq y$. But $Y\rho_1 \supseteq \{b, c\}$ implies $y \in \mathcal{G}^{(1)}(Y) = (Y\rho_1)\rho_0 \subseteq \{b,c\}\rho_0$, i.e., $b \preceq y$ and $c \preceq y$. Since $b \parallel c$, we obtain $a \prec b \prec y$, which contradicts $a \preceq y$. This and the duality principle prove (3).

Now, to sharpen the previous assertion, we prove that for any $b \neq c$

$$\text{if } b, c \in \mathrm{mid}(Q) \text{ and } b \parallel c \text{ then } L(b,c) = U(b,c) = \emptyset. \tag{4}$$

Suppose the contrary. By the duality principle, we may assume that $L(b,c)$ is nonempty. Let $L(b,c) = \{a\}$. We can choose $d_1, d_2 \in \max(Q)$ such that $b \prec d_1$ and $c \prec d_2$. If possible, then we choose them equal: $d_1 = d_2$. Let $X = \{b, c\}$. If $U(b,c)$ is nonempty then $d_1 = d_2$, $X\rho_0 = \{d_1\}$ and we have $d_1 \in \mathcal{G}^{(0)}(X) = \mathcal{C}^{(0)}(X)$. If $U(b,c)$ is empty then so is $X\rho_0$ and we have $d_1 \in \mathcal{G}^{(0)}(X) = \mathcal{C}^{(0)}(X)$ again. Let $Y = X = X\psi_0$. Then, by formula (1), $d_1 \preceq y$ for some $y \in \mathcal{G}^{(1)}(Y)$. Since $d_1 \in \max(Q)$, $d_1 = y \in \mathcal{G}^{(1)}(Y) = (Y\rho_1)\rho_0 = \{a\}\rho_0$. This gives $a \preceq d_1$, which contradicts $a \prec b \prec d_1$. This shows (4).

Based on (4) we can prove even more: for any elements of Q we have

$$\text{if } c \in Q,\ b \in \mathrm{mid}(Q) \text{ and } b \parallel c \text{ then } L(b,c) = U(b,c) = \emptyset. \tag{5}$$

Suppose the contrary. By (4), $c \notin \mathrm{mid}(Q)$. By the duality principle we can assume that $c \in \max(Q)$. Then $U(b,c) = \emptyset$. Let $a \in L(b,c)$ and choose an element $d \in \max(Q)$ with $b \prec d$. For $X = \{a, d\}$ from $X\rho_0 = \emptyset$ we obtain $c \in Q = \mathcal{G}^{(0)}(X) = \mathcal{C}^{(0)}(X)$. Let $Y = \{b, d\} \in X\psi_0$. Then $c \preceq y$ for some $y = \mathcal{G}^{(1)}(Y)$ by (1) and $b \preceq y$ by (2). This together with $b \parallel c$ imply $c \prec y$, which contradicts $c \in \max(Q)$. This proves (5)

Now we are in the position to show that

$$\text{if } b \in \mathrm{mid}(Q) \text{ then there is no } c \in Q \text{ with } b \parallel c. \tag{6}$$

Suppose the contrary, and choose $a, d \in Q$ with $a \prec b \prec d$. If $X = \{a, d\}$ and $Y = \{b, d\} \in X\psi_0$ then, exactly the same way as in the previous step, we obtain an element y with $c \preceq y$ and $b \preceq y$, and we conclude that $y \in U(b, c)$, which contradicts (5). This proves (6)

Now, since length$(Q) = 2$, we can choose elements $a_1 \prec b \prec d_1$ in Q. It follows from (6) that for any further element x either $x < b$ or $b < x$. Let $m = |\{x \in Q : x < b\}|$ and $n = |\{x \in Q : b < x\}|$, then clearly Q is T_{mn}. □

Now we mention an open problem about $\mathcal{C}$. For motivation and a possible application cf. [3]. Let us say that $(A^{(0)}, A^{(1)}, \rho)$ is a decomposable context if there are nonempty sets $B^{(i)}$ and $C^{(i)}$ with $B^{(i)} \cup C^{(i)} = A^{(i)}$ and $B^{(i)} \cap C^{(i)} = \emptyset$ such that

$$\rho = (\rho \cap (B^{(0)} \times B^{(1)})) \cup (\rho \cap (C^{(0)} \times C^{(1)})).$$

Otherwise $(A^{(0)}, A^{(1)}, \rho)$ is called an *indecomposable context*. We say that it is a *uniform context* if $|\{x\}\rho_i| = |\{y\}\rho_i|$ for all $x, y \in A^{(i)}$. In the terminology of context tables, if any two columns contain the same number of crosses and any two rows contain the same number of crosses. For example, each finite block design (P, B, I) and, in particular, each finite projective space (P, L, I) is a uniform context.

Problem 1. *Is it true that for each indecomposable uniform context* $(A^{(0)}, A^{(1)}, \rho)$ *with* $|A^{(0)}| \geq 3$ *and* $|A^{(1)}| \geq 3$ *there exists an* $i \in \{0, 1\}$ *and there are* $x, y, z \in A^{(i)}$ *such that*

$$\mathcal{C}^{(i)}(\{x, y\}) \cap \mathcal{C}^{(i)}(\{y, z\}) \cap \mathcal{C}^{(i)}(\{z, x\}) = \emptyset?$$

Interestingly enough, this problem is connected with a much easier one, suggested and solved by an anonymous referee. Namely, the definition of $\mathcal{C}$ raises the question how long the sequence $\mathcal{G} = \mathcal{C}_0 > \mathcal{C}_1 > \mathcal{C}_2 > \mathcal{C}_3 \cdots$ can be. (Of course, the context is assumed to be finite.) The answer is that it can be arbitrarily long, and this is exemplified by indecomposable uniform contexts with the property $|\{x\}\rho_0| = 2$ for all $x \in A^{(0)}$. These contexts play the key role in [3], and it is straightforward to extract the proof of this statement from [3].

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