Algebra Universalis, **15** (1982) 142–143 0001–5240/82/001142–02$01.50+0.20/0

Mailbox

A note on the compactness of the consequence relation for congruence varieties

G. Czédli

A congruence variety is a lattice variety of the form $\mathbf{Con}(U) = \mathbf{H}\,\mathbf{S}\,\mathbf{P}\{\mathbf{Con}(A) : A \in U\}$ where U is a variety of universal algebras. For a set Σ of lattice identities and a lattice identity ε let $\Sigma \models_c \varepsilon$ stand for "whenever all the identities of Σ are satisfied in a congruence variety V then ε is also satisfied in V". Let mod and dist stand for the modular law and the distributive law, respectively.

In [4, Problem 3.18] Jónsson asked whether the consequence relation $\models_c$ was compact. In Day and Freese [1] it is shown that if $\Sigma \models_c$ mod then $\Sigma' \models_c$ mod for some finite subset Σ' of Σ.

Our aim is to show the following

THEOREM. *If $\Sigma \models_c$ dist, then there exists a finite subset Σ' of Σ such that $\Sigma' \models_c$ dist.*

Proof. Suppose $\Sigma \models_c$ dist. By making use of the mentioned result of Day and Freese [1] we have $\Sigma_1 \models_c$ mod for some finite subset Σ_1 of Σ. Let P denote the set of prime numbers and $P_0 = P \cup \{0\}$. For $p \in P_0$ we denote the prime field of characteristic p by Q_p and let $M(Q_p)$ be the variety of all vector spaces over Q_p. Since $\mathbf{Con}(M(Q_p))$ is not distributive, for every $p \in P_0$ we can choose a λ_p in Σ such that λ_p is not satisfied in $\mathbf{Con}(M(Q_p))$. Let $D(m_p, n_p)$ denote the divisibility condition $(\exists x)(m_p \cdot x = n_p \cdot 1)$ corresponding to λ_p(see [3, Theorems 2 and 3]). Then we have

(1) For any $p, q \in P_0$, λ_p is satisfied in $\mathbf{Con}(M(Q_q))$ iff $D(m_p, n_p)$ holds in Q_q.

Therefore $D(m_p, n_p)$ is not satisfied in Q_p, whence $m_0 = 0$ and $n = n_0 > 0$. Let P_1 be the set of the prime divisors of n and set $\Sigma' = \Sigma_1 \cup \{\lambda_p : p \in P_1 \cup \{0\}\}$. We claim

Presented by B. Jónsson. Received April 2, 1979. Accepted for publication in final form October 29, 1979.

that $\Sigma' \models_c \text{dist}$. Suppose $\Sigma' \models_c \text{dist}$ is not true. Then Σ' is satisfied in some non-distributive modular congruence variety V. By Freese's result [2], $\mathbf{Con}(M(Q_q)) \subseteqq V$ and so Σ' holds in $\mathbf{Con}(M(Q_q))$ for some $q \in P_0$. Since λ_0 is satisfied in $\mathbf{Con}(M(Q_q))$, from (1) we have that q divides n. Hence $q \in P_1$ and λ_q holds in $\mathbf{Con}(M(Q_q))$, which is a contradiction. Q.E.D.

REFERENCES

[1] A. DAY and R. FREESE, A characterization of identities implying congruence modularity, Can. J. Math. (to appear).

[2] R. FREESE, Minimal modular congruence varieties, Abstract 76T–A14, Amer. Math. Soc. Notices 23 (1976), No. 1, A–4.

[3] G. HUTCHINSON and G. CZÉDLI, A test for identities satisfied in lattices of submodules, Algebra Univ. *8* (1978), 269–309.

[4] B. JÓNSSON, Congruence varieties, an appendix to G. Grätzer, Universal Algebra, 2nd edition.

József Attila University
Szeged
Hungary