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A Note on Lattice Horn Sentences with Three Variables

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Lattice Horn sentences with three variables are shown to be trivial within the theory of modular lattices.

In the theory of modular lattices, all the interesting Horn sentences that have been investigated have at least four variables (cf., e. g., G. Grätzer, H. Lakser and B. Jónsson [4] and [2, 3]). The aim of the present note is to explain this phenomenon by the following

Theorem. *Let χ be a lattice Horn sentence on three variables. Then either χ is a consequence of the modular law or χ together with modularity imply distributivity.*

By a lattice Horn sentence χ with three variables we mean a universally quantified formula

$$p_1 \leqq q_1 \,\&\ldots\&\, p_k \leqq q_k \Rightarrow p \leqq q$$

where $p_1,\ldots,p_k, q_1,\ldots,q_k, p, q$ are lattice terms on the set $\{x, y, z\}$ of variables and $k \geqq 0$. (In case $k=0$ the premise is empty and χ is a lattice identity.) In what follows the reader is assumed to be familiar with some basic facts belonging to universal algebra and lattice theory and, first of all, with the diagram of $M = F_{\mathbf{M}}(x, y, z)$, the free modular lattice generated by $\{x, y, z\}$ (cf. any book on lattice theory, e. g., G. Grätzer [3, p. 39]).

Proof. Without loss of generality we may assume that $p, p_1,\ldots,p_k, q, q_1,\ldots,q_k$ are elements of M. If our statement is true for χ_i, $i=1, 2$, and χ is equivalent (modulo lattice theory) to the conjunction of χ_1 and χ_2 then the statement is also true for χ. On the other hand, in any lattice and for arbitrary lattice terms r, r_1, r_2, $r_1 + r_2 \leqq r$ holds iff $r_1 \leqq r$ and $r_2 \leqq r$ hold, and dually. This allows us to make the following assumption: $p, p_1,\ldots,p_k \in M$ are join-irreducible elements and $q, q_1,\ldots,q_k \in M$ are meet-irreducible elements. We may also suppose that $p_1 \nleqq q_1,\ldots,p_k \nleqq q_k$, $p \nleqq q$ in M and, in M, $p \leqq p_i$ simultaneously with $q_i \leqq q$ hold for no $i \in \{1,\ldots,k\}$ as otherwise χ would automatically hold in all modular lattices or some $p_i \leqq q_i$ could be omitted from the premise of χ. Finally, we assume that χ holds in $\mathbf{2}$, the two-element lattice, as otherwise χ would trivially imply distributivity. Therefore, by this assumption, χ holds in every distributive lattice as each distributive lattice is a subdirect power of $\mathbf{2}$.

Now let $u=xy+xz+yz$, $v=(x+y)(x+z)(y+z)$, $a=u+xv$, $b=u+yv$, $c=u+zv$. Then $M_3=[u, v]=\{u, a, b, c, v\}$ is the only diamond (i.e., five element non-distributive sublattice) in M. Let Θ denote the congruence of M corresponding to the partition $\{[x(y+z), x+yz], [y(x+z), y+xz], [z(x+y), z+xy], [0, u], [v, 1]\}$. We claim that, for any congruence Ψ of M, M/Ψ is distributive iff $\Psi\nsubseteq\Theta$. Indeed, if $\Psi\subseteq\Theta$ then $M_3\cong M/\Theta$ is a homomorphic image of M/Ψ, whence M/Ψ is not distributive. Conversely, if $\Psi\nsubseteq\Theta$ then, as the blocks of Ψ are intervals, there are $d, e\in M$ such that $(d, e)\in\Psi\setminus\Theta$ and e covers d. Further, $\{d, e\}\subseteq M_3$ can be assumed as the only other case is symmetric and/or dual to $d=a+x$, $e=v+x$, whence $(a, v)=(vd, ve)\in\Psi\setminus\Theta$. Since $\Psi\cap M_3^2\neq\emptyset$ and M_3 is a simple lattice, Ψ includes Θ_3, the congruence of M generated by M_3^2. Hence M/Ψ is a homomorphic image of M/Θ_3, which is a distributive (moreover, a free distributive) lattice. Thus M/Ψ is distributive.

Now let Θ_i denote the congruence of M generated by $(p_i, p_i q_i)$. Suppose $\Theta_i\nsubseteq\Theta$ holds for some $i\in\{1,\dots,k\}$. If L is an arbitrary modular lattice, $x_1, y_1, z_1\in L$ and $p_j(x_1, y_1, z_1)\leqq q_j(x_1, y_1, z_1)$ for $j=1,\dots,k$ then Θ_i is included in the kernel of the surjective homomorphism $M\to[x_1, y_1, z_1]$, $x\mapsto x_1$, $y\mapsto y_1$, $z\mapsto z_1$. Hence $[x_1, y_1, z_1]$, as a homomorphic image of M/Θ_i, is distributive, whereupon $p(x_1, y_1, z_1)\leqq q(x_1, y_1, z_1)$. This shows that χ holds in L.

From now on let us assume that $\Theta_i\subseteq\Theta$ holds for all $i\in\{1,\dots,k\}$. Then the premise of χ holds for $a, b, c\in M_3$. Really, considering the homomorphism $\tau: M\to M_3$, $x\mapsto a$, $y\mapsto b$, $z\mapsto c$, from $(p_i, p_i q_i)\in\Theta=\ker\tau$ we obtain $p_i(a, b, c)=p_i(x\tau, y\tau, z\tau)=p_i(x, y, z)\tau=p_i\tau=(p_i q_i)\tau=(p_i\tau)(q_i\tau)=p_i(x\tau, y\tau, z\tau)q_i(x\tau, y\tau, z\tau)=p_i(a, b, c)q_i(a, b, c)$ for $i\in\{1,\dots,k\}$. If $p(a, b, c)\nleqq q(a, b, c)$ then χ fails to hold in M_3. Hence χ and modularity imply distributivity. Therefore we assume that $p(a, b, c)\leqq q(a, b, c)$. In virtue of our former assumptions, a quick glance at M shows that, apart from symmetry and duality, $p=x$ and $q=y+z$. Since χ holds in $\{0, 1\}$, a distributive sublattice of M, but $1=p(1, 0, 0)\nleqq q(1, 0, 0)=0$, there is an $i\in\{1,\dots,k\}$ such that $p_i(1, 0, 0)\nleqq q_i(1, 0, 0)$, i.e., $p_i(1, 0, 0)=1$ and $q_i(1, 0, 0)=0$. Considering the homomorphism $\varphi: M\to\{0, 1\}$, $x\mapsto 1$, $y\mapsto 0$, $z\mapsto 0$, $\ker\varphi$ has only two blocks: $[x, 1]$ and $[0, y+z]$. Since $p_i\varphi=p_i(x, y, z)\varphi=p_i(x\varphi, y\varphi, z\varphi)=p_i(1, 0, 0)=1=1\varphi$ and $q_i\varphi=0=0\varphi$, we have $p_i\in[x, 1]$ and $q_i\in[0, y+z]$. Thus the contradiction $p\leqq p_i$ and $q_i\leqq q$ completes the proof.

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