A note on congruence lattices of slim semimodular lattices

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Abstract. Recently, G. Grätzer has raised an interesting problem: Which distributive lattices are congruence lattices of slim semimodular lattices? We give an eight element slim distributive lattice that cannot be represented as the congruence lattice of a slim semimodular lattice. Our lattice demonstrates the difficulty of the problem.

1. Introduction

All lattices in the paper are assumed to be finite. A finite lattice L is *slim*, if $\mathrm{Ji}\,L$, the set of nonzero join-irreducible elements of L, is included in the union of two chains of L; see G. Czédli and E. T. Schmidt [3]. A slim lattice is finite by definition. In the planar semimodular case, this concept was first introduced by G. Grätzer and E. Knapp [8] in a different but equivalent way: a semimodular lattice is slim if it contains no M_3 sublattice; equivalently, if it contains no cover-preserving M_3 sublattice; see also G. Czédli and E. T. Schmidt [3, Lemma 2.3]. By [3, Lemma 2.2], slim lattices are planar. Our aim is to prove the following theorem.

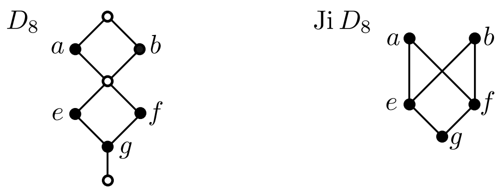

Figure 1. A non-representable slim distributive lattice

Theorem 1.1. *There is no slim semimodular lattice whose congruence lattice is isomorphic to the planar distributive lattice D_8 of Figure 1.*

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The historical background of this theorem is briefly the following. We know from J. Jakubík [13], see also G. Grätzer and E. T. Schmidt [12], that the congruence lattice $\operatorname{Con} M$ of a finite modular lattice M is boolean. (Interestingly, this result also follows from the deep result of G. Grätzer and E. Knapp [10, Theorem 7].) On the other hand, each finite distributive lattice is isomorphic to the congruence lattice of a planar semimodular lattice by G. Grätzer, H. Lakser, and E. T. Schmidt [11]. Let Con(SSL) denote the class of finite lattices isomorphic to the congruence lattices of slim semimodular lattices. In [7], G. Grätzer raised the problem to give an internal characterization for Con(SSL). There are many known facts about the lattices in Con(SSL). For example, G. Grätzer and E. T. Schmidt (personal communication) point out that if $D \in \mathrm{Con(SSL)}$ such that $\operatorname{Ji} D$ has exactly two maximal elements, then there exists a unique element $d \in D$ such that the filter $\uparrow d$ is a 4-element boolean lattice and $D = {\uparrow d} \cup {\downarrow d}$. Furthermore, in this case, G. Grätzer (personal communication) observed that each element of $\operatorname{Ji} D$ is covered by at most two elements of $\operatorname{Ji} D$. Although no specific property that hold for the members of Con(SSL) is targeted in the present paper, we note that the properties mentioned above follow from G. Czédli [2, Theorems 3.7 and 5.5] and G. Grätzer and E. Knapp [10, Theorem 7]. Since D_8 satisfies these properties, Theorem 1.1 indicates that the problem of characterizing Con(SSL) is more complex than one would expect.

2. Auxiliary statements and the proof of Theorem 1.1

For notation and concepts not defined in the paper, see G. Grätzer [6]. The *left boundary* (*chain*) and the *right boundary* (*chain*) of a planar lattice diagram D are denoted by $\mathrm{C}_\ell(D)$ and $\mathrm{C}_\mathrm{r}(D)$, respectively. Their union is the *boundary* of D. A double irreducible element on the boundary is called a *weak corner*. Note that 0 and 1 are not doubly irreducible elements. Following G. Grätzer and E. Knapp [8], a planar lattice diagram D is *rectangular* if it is semimodular, $\mathrm{C}_\ell(D)$ has exactly one weak corner, denoted by $\mathrm{lc}(D)$, $\mathrm{C}_\mathrm{r}(D)$ has exactly one weak corner, $\mathrm{rc}(D)$, and these two elements are complementary, that is, $\mathrm{lc}(D) \wedge \mathrm{rc}(D) = 0$ and $\mathrm{lc}(D) \vee \mathrm{rc}(D) = 1$. Note that a rectangular diagram consists of at least four elements. If a lattice L has a rectangular diagram, then L is a *rectangular lattice*. We know from G. Czédli and E. T. Schmidt [5, Lemma 4.9] that if one diagram of a planar semimodular lattice is rectangular, then so are all of its planar diagrams. Furthermore, if D is a planar diagram of a *slim* rectangular lattice R, then $\{\mathrm{C}_\ell(D), \mathrm{C}_\mathrm{r}(D)\}$ and $\{\mathrm{lc}(D), \mathrm{rc}(D)\}$ do not depend on D. Therefore, since our arguments are left-right symmetric, we can always think of a fixed diagram and we can use the notation $\mathrm{C}_\ell(R)$, $\mathrm{C}_\mathrm{r}(R)$, $\mathrm{lc}(R)$, and $\mathrm{rc}(R)$.

Suppose, for a contradiction, that Theorem 1.1 fails. In the rest of the paper, let L_8 denote a slim semimodular lattice such that $\operatorname{Con} L_8 \cong D_8$. We know from G. Grätzer and E. Knapp [10, Theorem 7] that each planar semimodular lattice L has a rectangular congruence-preserving extension. Analyzing the proof of this result, or referencing G. Czédli [1, Lemma 5.4] and G. Czédli and E. T. Schmidt [4, Lemma 21], we obtain that for a slim semimodular lattice L, there exists a slim rectangular lattice R such that $\mathrm{Con}\,L \cong \mathrm{Con}\,R$. Therefore, we can assume that L_8 is a slim rectangular lattice.

Next, let R be an arbitrary slim rectangular lattice. The *boundary* of R is $\mathrm{C}_\ell(R) \cup \mathrm{C}_\mathrm{r}(R)$, while $R \setminus \big(\mathrm{C}_\ell(R) \cup \mathrm{C}_\mathrm{r}(R)\big)$ is the *interior* of R. For a meet-irreducible element $x \in \mathrm{Mi}\,R$, x^* denotes the unique cover of x. We define an equivalence relation on $\mathrm{Mi}\,R$ as follows. For $x, y \in \mathrm{Mi}\,R$, let $\langle x, y\rangle \in \Theta$ mean that $x = y$, or both x and y are in the interior of R and $x^* = y^*$. The quotient set $(\mathrm{Mi}\,R)/\Theta$ is denoted by $\widehat{\mathrm{Mi}}\,R$. For $x \in \mathrm{Mi}\,R$, we denote the Θ-block of x by x/Θ. We define a relation $\widehat\sigma$ on $\widehat{\mathrm{Mi}}\,R$ by the rule

$$\begin{aligned}\langle x/\Theta, y/\Theta\rangle \in \widehat\sigma \text{ iff } & x/\Theta \neq y/\Theta,\ x \text{ is in the interior of } R,\ x^* \leq y^*,\\ & \text{but there are } x' \in x/\Theta \text{ and } y' \in y/\Theta \text{ such that } x' \not\leq y'.\end{aligned}$$

Finally, let $\widehat\tau$ be the reflexive transitive closure of $\widehat\sigma$ on $\widehat{\mathrm{Mi}}\,R$. We need the following result from G. Czédli [2, Theorem 7.3(ii)].

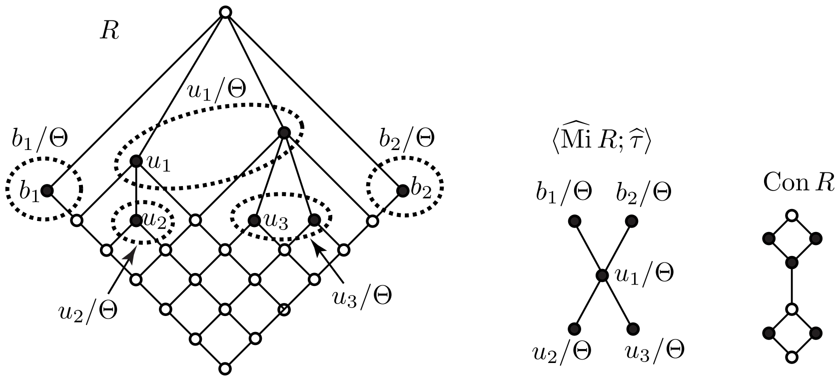

FIGURE 2. An illustration for Theorem 2.1

Theorem 2.1. *$\langle \widehat{\mathrm{Mi}}\,R; \widehat\tau\rangle$ is an ordered set, and it is isomorphic to the ordered set $\langle \mathrm{Ji}\,(\mathrm{Con}\,R); \leq\rangle$.*

The elements of $\widehat{\mathrm{Mi}}\,R$ belong to two categories. If $x \in \mathrm{Mi}\,R$ is on the boundary of R, then the singleton Θ-block $x/\Theta = \{x\}$ is a *boundary block*. For $x \in \mathrm{Mi}\,R$ such that x is in the interior of R, the block x/Θ is called an *umbrella*. For the slim rectangular lattice R of Figure 1, the elements of $\mathrm{Mi}\,R$ are black-filled, the Θ-blocks are indicated by dotted closed curves, b_1/Θ and b_2/Θ are boundary blocks, and u_1/Θ, u_2/Θ, and u_3/Θ are umbrellas. We need the following lemma.

Lemma 2.2. *If R is a slim rectangular lattice, then the maximal elements of $\langle \widehat{\mathrm{Mi}}\,R; \widehat\tau\rangle$ are exactly the boundary blocks.*

Proof. It follows from the definition of $\widehat{\sigma}$ and $\widehat{\tau}$ that the boundary blocks are maximal elements.

Conversely, assume that $x \in \operatorname{Mi} R$ is an interior element of R. We have to show that the umbrella x/Θ is not a maximal element in $\langle \widehat{\operatorname{Mi} R}; \widehat{\tau}\rangle$. Denote $\operatorname{lc}(R)$ and $\operatorname{rc}(R)$ by w_ℓ and w_r, respectively. Let $a = w_\ell \vee x$ and $b = w_r \vee x$. If $a = w_\ell$ and $b = w_r$, then $x \le w_\ell$ and $x \le w_r$ imply that $x \le w_\ell \wedge w_r = 0$, which contradicts $0 = w_\ell \wedge w_r \notin \operatorname{Mi} R$. Therefore, we can assume that $a \ne w_\ell$, that is, $w_\ell < a$. By G. Grätzer and E. Knapp [10, Lemma 4], $\uparrow w_\ell$ is a chain and a subset of the left boundary chain of R. Hence, there is a unique $c \in \uparrow w_\ell$ such that $c \prec a$. We know from G. Grätzer and E. Knapp [10, Lemma 3] that $c \in \operatorname{Mi} L$. Hence, c/Θ is a boundary block. Since x is in the interior of R but a is not, $x \le a$ yields that $x < a$. This, together with $\{x, c\} \subseteq \operatorname{Mi} R$, implies that $x^* \le a = c^*$. If $x \le c$, then $a = x \vee w_\ell \le c$, which contradicts that $c \prec a$. Thus, $x \not\le c$, and we conclude that $\langle x/\Theta, c/\Theta\rangle \in \widehat{\sigma}$. Consequently, $\langle x/\Theta, c/\Theta\rangle \in \widehat{\tau}$. Since c/Θ is a boundary block and x/Θ is an umbrella, we have that $x/\Theta \ne c/\Theta$. Therefore, x/Θ is not a maximal element. □

Now, we are ready to prove our result.

Proof of Theorem 1.1. Applying Theorem 2.1 and Lemma 2.2 to $R = L_8$, we obtain that $\langle \widehat{\operatorname{Mi} L_8}; \widehat{\tau}\rangle$ consists of two boundary blocks, A and B, and three umbrellas, E, F and G, such that, with the notation of Figure 1, A, B, E, F, and G correspond to a, b, e, f, and g, according to the order isomorphism $\langle \widehat{\operatorname{Mi} L_8}; \widehat{\tau}\rangle \to \langle \operatorname{Ji} D_8; \le\rangle$ provided by Theorem 2.1.

For a Θ-block U, in particular, for an umbrella U, the element x^* is the same for all $x \in U$; by the *top* U^* of U we mean x^*, where $x \in U$. On the other hand, the *bottom* U_* of U is $\bigwedge U$, the meet of all elements of U.

It follows from the main result of G. Grätzer and E. Knapp [9] that L_8 contains an element with at least three distinct lower covers. Consequently, it is clear, and it follows rigorously from D. Kelly and I. Rival [14], that the interior of L_8 is non-empty. Let u be a maximal element of the interior of L_8. Suppose, for a contradiction, that $u \notin \operatorname{Mi} L_8$. Then u is the meet of meet-irreducible elements that belong to $\mathrm{C}_\ell(L_8) \cup \mathrm{C}_\mathrm{r}(L_8)$ of L_8. Since $\mathrm{C}_\ell(L_8)$ and $\mathrm{C}_\mathrm{r}(L_8)$ are chains, we can assume that $u = v_1 \wedge v_2$, where $v_1 \in \mathrm{C}_\ell(L_8)$, $v_2 \in \mathrm{C}_\mathrm{r}(L_8)$, $\{v_1, v_2\} \subseteq \operatorname{Mi} L_8$, and $v_1 \parallel v_2$. By G. Grätzer and E. Knapp [10, Lemma 4],

$$\mathrm{C}_\ell(L_8) = \downarrow\operatorname{lc}(L_8) \cup \uparrow\operatorname{lc}(L_8). \tag{2.1}$$

Since every element in $\downarrow\operatorname{lc}(L_8) \setminus \{\operatorname{lc}(L_8)\}$ has at least two covers by G. Czédli and E. T. Schmidt [5, (2.14)], we obtain that $v_1 \ge \operatorname{lc}(L_8)$. Similarly, $v_2 \ge \operatorname{rc}(L_8)$. We cannot have equality in both cases, because this would imply $u = v_1 \wedge v_2 = \operatorname{lc}(L_8) \wedge \operatorname{rc}(L_8) = 0$, which is not in the interior of L_8. Let, say, $v_1 > \operatorname{lc}(L_8)$. Then v_1/Θ, $\operatorname{lc}(L_8)/\Theta$, and $\operatorname{rc}(L_8)/\Theta$ are three distinct boundary blocks, which is a contradiction. This proves that $u \in \operatorname{Mi} L_8$.

By the maximality of u, the element u^* is on the boundary of L_8. Let, say $u^* \in \mathrm{C}_\ell(L_8)$. Since $\mathrm{C}_\ell(L_8) \supseteq {\downarrow}\mathrm{lc}(L_8)$ by G. Grätzer and E. Knapp [10, Lemma 4], if u^* is in ${\downarrow}\mathrm{lc}(L_8)$, then $u \in \mathrm{C}_\ell(L_8)$, which is a contradiction. Hence, $u^* \notin {\downarrow}\mathrm{lc}(L_8)$, and (2.1) implies that $\mathrm{lc}(L_8) < u^*$. If $u^* < 1$, then $u^* \in \mathrm{Mi}\, L_8$ by [10, Lemma 3], whence u^*/Θ, $\mathrm{lc}(L_8)/\Theta$, and $\mathrm{rc}(L_8)/\Theta$ are three distinct boundary blocks, which is a contradiction. Consequently, $u^* = 1$. This proves that L_8 has an umbrella whose top is 1.

Next, we show that this umbrella is not G. Suppose, for a contradiction, that $G^* = 1$. Since $g < e$, we have that $\langle G, E\rangle \in \widehat{\tau}$. Hence, there exists a sequence

$$G = X_0, \dots, X_s = E \text{ in } \widehat{\mathrm{Mi}}\, L_8 \text{ such that } \langle X_{i-1}, X_i\rangle \in \widehat{\sigma} \tag{2.2}$$

for $i \in \{1, \dots, s\}$. Thus $X^*_{i-1} \le X^*_i$ for all i, and we obtain that $G^* \le E^*$. Therefore, $G^* = 1 = E^*$, which is a contradiction, because distinct umbrellas clearly have distinct tops by the definition of Θ. This proves that $G^* \ne 1$. Since the role of E and F is symmetric, we can assume that $E^* = 1$.

Next, we assert that

$$F^* \le E_*. \tag{2.3}$$

To prove this, observe that $f \parallel e$ gives that $\langle F, E\rangle \notin \widehat{\tau}$ and, consequently, $\langle F, E\rangle \notin \widehat{\sigma}$. This, together with $F^* \le 1 = E^*$, yields that, for all $x' \in F$ and $y' \in E$, $x' \le y'$. Hence, for all $x' \in F$, we have that $x' \le E_*$. There are two cases. First, assume that $|E| \ge 2$, and let $x' \in F$. Since $E_* \notin \mathrm{Mi}\, L_8$ and $x' \in \mathrm{Mi}\, L_8$, we have that $x' \ne E_*$. Thus the inequality $x' \le E_*$ implies that $x' < E_*$. Therefore, $F^* = x'^* \le E_*$. Second, assume that $E = \{y'\}$ is a singleton. Let $x' \in F$. We know that $x' \le E_* = y'$. If we have equality here, then $F^* = x'^* = y'^* = E^*$ contradicts the fact that distinct umbrellas have distinct tops. Thus $x' < E_*$, which implies that $F^* = x'^* \le E_*$ again. This proves (2.3).

Next, we turn our attention to the inequality $g < f$. Since $\langle G, F\rangle \in \widehat{\tau}$, we obtain from a $\widehat{\sigma}$-sequence similar to (2.2) that $G^* \le F^*$. (In fact, $G^* < F^*$, but we do not need the sharp inequality here.) Combining this with (2.3), we obtain that $G^* \le E_*$. Since $g < e$, we have that $\langle G, E\rangle \in \widehat{\tau}$, which is witnessed by a *shortest* sequence described in (2.2). If X_1 in the sequence equals F, then (2.2) gives $\langle F, E\rangle \in \mathrm{quor}(\widehat{\sigma}) = \widehat{\tau}$, contradicting $f \not\le e$. This excludes that $X_1 = F$. Similarly, since $a \not\le e$ and $b \not\le e$, we exclude $X_1 \in \{A, B\}$. Since (2.2) is the shortest sequence, $X_1 \ne G$. Now, after that all but one element of $\widehat{\mathrm{Mi}}\, L_8 = \{A, B, E, F, G\}$ have been excluded, we conclude that $X_1 = E$. Therefore, $\langle G, E\rangle \in \widehat{\sigma}$. This implies the existence of an element $z' \in G$ and an element $y' \in E$ such that $z' \not\le y'$. But this is a contradiction, because $z' \le G^* \le E_* \le y'$. □

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