A CHARACTERIZATION FOR CONGRUENCE SEMI-DISTRIBUTIVITY

Gábor Czédli *

1. INTRODUCTION. A variety of algebras is said to be congruence-meet-semi-distributive if in the congruence lattices of its algebras the <u>semi-distributive law</u>,

$(SD_\wedge)$ $\quad (\forall\alpha)(\forall\beta)(\forall\gamma)\ (\alpha\wedge\beta = \alpha\wedge\gamma \Rightarrow \alpha\wedge\beta = \alpha\wedge(\beta\vee\gamma)),$

holds. From the general description of properties that can be characterized by Mal'cev conditions (Taylor [10], Neumann [7]) it follows that there exists a weak Mal'cev condition characterizing congruence meet semi-distributivity of varieties (Jónsson [4, Theorem 2.16]). However, $SD_\wedge$ has seemed the simplest (characterizable) property of congruence lattices for which no concrete weak Mal'cev condition has been known. The aim of this note is to present such a condition and some corollaries to it. (Note that the dual law, $SD_\vee$, has been characterized in [1].)

2. A WEAK MAL'CEV CONDITION. Our Mal'cev conditions will be given by means of certain graphs. First for any lattice term $p = p(\alpha,\beta,\gamma)$ we define a set $\underset{\sim}{G}(p)$ of graphs associated with p. The edges of any $G \in \underset{\sim}{G}(p)$ will be coloured by the variables α, β, and γ, and two distinguished vertices, the so-called <u>left</u> and <u>right</u> <u>endpoints</u>, will have special roles. In figures these endpoints will be always placed on the left-hand side and on the right-hand side, respectively. For all $k \geq 2$ $G_k(p)$ will be a distinguished member of $\underset{\sim}{G}(p)$, but $\underset{\sim}{G}(p)$ will be different from $\{G_k(p) : k \geq 2\}$ in general. Before defining $\underset{\sim}{G}(p)$ we introduce two kinds of operations for graphs. We obtain the parallel connection of graphs G_1 and G_2 by taking disjoint copies of G_1 and G_2 and identifying their left (right, resp.) endpoints (Figure 1). By taking disjoint graphs $H_1, H_2, \dots, H_\ell$ $(\ell \geq 1)$ such that $H_j \cong G_i$ for $i \equiv j \mod (2)$ and

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identifying the right endpoint of H_i and left endpoint of H_{i+1} for $i = 1, 2, \ldots, \ell - 1$ we obtain the serial connection of length ℓ of the graphs G_1 and G_2. (The left endpoint of H_1 and the right one of H_ℓ are the endpoints of the serial connection, cf. Figure 2.)

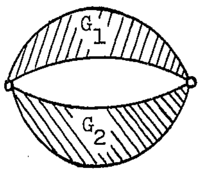

Figure 1

Figure 2

Now, if p is a variable then, for all $k \geq 2$, let $G_k(p)$ be the following graph

$$\overset{p}{\circ\!\!-\!\!-\!\!-\!\!-\!\!\circ}$$

which consists of a single edge coloured by p, and let $\underset{\sim}{G}(p)$ be the singleton $\{G_k(p)\}$. Let $\underset{\sim}{G}(p_1 \wedge p_2)$ ($\underset{\sim}{G}(p_1 \vee p_2)$, respectively) be the set of all parallel (serial, resp.) connections of G_1 and G_2 with G_i belonging to $\underset{\sim}{G}(p_i)$. Furthermore let $G_k(p_1 \wedge p_2)$ and $G_k(p_1 \vee p_2)$ be the parallel connection and the serial connection of length k of the graphs $G_k(p_1)$ and $G_k(p_2)$, respectively.

For $m \geq 2$ the smallest equivalence relation of $\{0, 1, \ldots, m\}$ collapsing 0 and m will be denoted by $\alpha(m)$. Similarly, $\beta(m)$ ($\gamma(m)$, respectively) is the smallest equivalence of $\{0, 1, \ldots, m\}$ that collapses $(i, i+1)$ for $0 \leq i < m$, i even (odd, respectively). If $\pi \in \{\alpha, \beta, \gamma\}$ and $j \leq m$ then the smallest member of $\{0, 1, \ldots, m\}$ that is congruent to j modulo $\pi(m)$ will be denoted by $j\pi(m)$ or $j\pi$.

Given a lattice term $p = p(\alpha, \beta, \gamma)$, an integer $m \geq 2$ and a graph $G \in \underset{\sim}{G}(p)$ we associate the following (strong, i.e. finite) Mal'cev condition $U(m, G)$ with G and m:

"For any vertex f_i of G there exists an $(m+1)$-ary term $f_i(x_0, x_1, \ldots, x_n)$ such that for each $\pi \in \{\alpha, \beta, \gamma\}$ and any π-coloured edge connecting, say, f_i and f_j the identity $f_i(x_{0\pi}, x_{1\pi}, \ldots, x_{m\pi}) = f_j(x_{0\pi}, x_{1\pi}, \ldots, x_{m\pi})$ holds (here π abbreviates $\pi(m)$), and for the left and right endpoints f_0 and f_1 the endpoint identities $f_0(x_0, x_1, \ldots, x_m) = x_0$, $f_1(x_0, x_1, \ldots, x_m) = x_m$ are satisfied."

We shall consider the ternary lattice terms $\beta_n = \beta_n(\alpha, \beta, \gamma)$ and $\gamma_n = \gamma_n(\alpha, \beta, \gamma)$, $n = 0, 1, 2, \ldots$, defined by the following induction: $\beta_0 = \beta$, $\gamma_0 = \gamma$, $\beta_{n+1} = \beta \vee (\alpha \wedge \gamma_n)$, $\gamma_{n+1} = \gamma \vee (\alpha \wedge \beta_n)$. Denoting $U(m, G_n(\beta_n))$

by $U(m, n)$ and letting $\mathbf{G}(\beta_\infty)$ be equal to the union of all $\mathbf{G}(\beta_n)$, $2 \le n < \omega$, we can formulate our main result:

THEOREM. *For any variety* $\mathbf{V}$ *of algebras the following three conditions are equivalent*:

(i) $\mathbf{V}$ *is congruence meet semi-distributive*;

(ii) *For any integer* $m \ge 2$ *there exists an even* $n \ge 2$ *such that the strong Mal'cev condition* $U(m, n)$ *holds in* $\mathbf{V}$;

(iii) $U(m, G)$ *holds in* $\mathbf{V}$ *for infinitely many* $m \ge 2$ *and appropriate (depending on* m) $G \in \mathbf{G}(\beta_\infty)$.

Moreover (ii) *is a weak Mal'cev condition in Jónsson's sense* [4], *i.e.* $U(m, n)$ *implies* $U(m, n + 2)$ *for all* m, n.

3. PROOF OF THE THEOREM. Since (ii) implies (iii) trivially, (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(i) have to be shown. While the latter requires almost the same argument that Wille [11] and Pixley [9] used, the implication (i)$\Rightarrow$(ii) needs a different approach.

Given congruences $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ of an algebra A, $a_0, a_1 \in A$, a ternary lattice term p, and $G \in \mathbf{G}(p)$, we say that a_0, a_1 can be *connected by the graph* G *in* A if there are further elements $a_i \in A$ for $i \in \{2, 3, \ldots, s\}$, where $\{0, 1, \ldots, s\}$ is the vertex set of G with endpoints 0 and 1, such that $(a_i, a_j) \in \bar{\pi}$ holds for all $\pi \in \{\alpha, \beta, \gamma\}$ and π-coloured edge of G connecting i and j. The following statement follows from the general description of the join of congruences $\Theta \vee \Psi = \bigcup(\Theta \circ \Psi \circ \Theta \circ \ldots \text{ (k factors)}: k < \omega)$ and from reflexivity, thus the proof will be omitted.

Claim 1. Let $A, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, a_0, a_1$, and p be as above. If $(a_0, a_1) \in p(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ then there exists a natural number k_0 such that for all $k \ge k_0$ a_0 and a_1 can be connected by the graph $G_k(p)$ in A. Conversely, if a_0 and a_1 can be connected by some member of $\mathbf{G}(p)$ in A then $(a_0, a_1) \in p(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$.

The following assertion will be also needed.

Claim 2. Given a variety $\mathbf{V}$, $m \ge 2$ and an equivalence π of $\{0, 1, \ldots, m\}$. Let $\bar{\pi}$ denote the congruence generated by $\{(x_i, x_j) : (i, j) \in \pi\}$ in the free algebra $F_{\mathbf{V}}(x_0, x_1, \ldots, x_m)$. If for m-ary $\mathbf{V}$-terms f and g $(f(x_0, x_1, \ldots, x_m), g(x_0, x_1, \ldots, x_m)) \in \bar{\pi}$ then the identity $f(x_{0\pi}, x_{1\pi}, \ldots, x_{m\pi}) = g(x_{0\pi}, x_{1\pi}, \ldots, x_{m\pi})$ holds throughout $\mathbf{V}$.

Proof. Extend the map $x_i \mapsto x_{i\pi}$ $(i = 0, 1, \ldots, m)$ to an endomorphism φ of $F_{\mathbf{V}}(x_0, x_1, \ldots, x_m)$. Since $\bar{\pi} \subseteq \operatorname{Ker} \varphi$ we obtain $f(x_{0\pi}, x_{1\pi}, \ldots, x_{m\pi}) = f(x_0\varphi, \ldots, x_m\varphi) = f(x_0, \ldots, x_m)\varphi = g(x_0, \ldots, x_m)\varphi = g(x_0\varphi, \ldots, x_m\varphi) = g(x_{0\pi}, x_{1\pi}, \ldots, x_{m\pi})$, yielding the assertion.

Claim 3. Given congruences $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ of an algebra A, define $\bar{\beta}_\infty = \beta_\infty(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and $\bar{\gamma}_\infty = \gamma_\infty(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ to be $\bigcup(\beta_n(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) : n<\omega)$ and $\bigcup(\gamma_n(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) : n<\omega)$, respectively. Then $\bar{\beta}_\infty$ and $\bar{\gamma}_\infty$ are congruences. Furthermore, denoting $\beta_n(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and $\gamma_n(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ by $\bar{\beta}_n$ and $\bar{\gamma}_n$, respectively, we have $\bar{\beta}_n \subseteq \bar{\beta}_{n+1}$, $\bar{\gamma}_n \subseteq \bar{\gamma}_{n+1}$ for all n and $\bar{\alpha} \wedge \bar{\beta}_\infty = \bar{\alpha} \wedge \bar{\gamma}_\infty$. If $\bar{\alpha} \wedge \bar{\beta} = \bar{\alpha} \wedge \bar{\gamma}$ then $\bar{\beta} = \bar{\beta}_n = \bar{\beta}_\infty$ and $\bar{\gamma} = \bar{\gamma}_n = \bar{\gamma}_\infty$ for all n.

Proof. The inclusions are trivial for $n = 0$. If they hold for $n - 1$ then $\bar{\beta}_n = \bar{\beta} \vee (\bar{\alpha} \wedge \bar{\gamma}_{n-1}) \subseteq \bar{\beta} \vee (\bar{\alpha} \wedge \bar{\gamma}_n) = \bar{\beta}_{n+1}$, and $\bar{\gamma}_n \subseteq \bar{\gamma}_{n+1}$ follows similarly. Therefore $\bar{\beta}_\infty$ and $\bar{\gamma}_\infty$ are congruences. If $(x, y) \in \bar{\alpha} \wedge \bar{\beta}_\infty$ then we have $(x, y) \in \bar{\alpha} \wedge \bar{\beta}_n \subseteq \bar{\alpha} \wedge (\bar{\gamma} \vee (\bar{\alpha} \wedge \bar{\beta}_n)) = \bar{\alpha} \wedge \bar{\gamma}_{n+1} \subseteq \bar{\alpha} \wedge \bar{\gamma}_\infty$, thus $\bar{\alpha} \wedge \bar{\beta}_\infty = \bar{\alpha} \wedge \bar{\gamma}_\infty$ by symmetry. The rest is a trivial induction.

(i)$\Rightarrow$(ii): Suppose $\underset{\sim}{V}$ is a congruence $SD_\wedge$ variety, $m \geq 2$ and consider the congruences $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ of $F_V(x_0, x_1, \dots, x_m)$ generated by $\{(x_i, x_j) : (i, j) \in \alpha(m)\}$, $\{(x_i, x_j) : (i, j) \in \beta(m)\}$ and $\{(x_i, x_j) : (i, j) \in \gamma(m)\}$, respectively. Let us adopt the abbreviations $\bar{\beta}_n, \bar{\beta}_\infty, \bar{\gamma}_n, \bar{\gamma}_\infty$ from Claim 3. Since $(x_0, x_m) \in \alpha(m) \cap (\beta(m) \circ \gamma(m) \circ \beta(m) \circ \gamma(m) \circ \dots) \subseteq \bar{\alpha} \wedge (\bar{\beta} \circ \bar{\gamma} \circ \bar{\beta} \circ \bar{\gamma} \circ \dots) \subseteq \bar{\alpha} \wedge (\bar{\beta} \vee \bar{\gamma}) \subseteq \bar{\alpha} \wedge (\bar{\beta}_\infty \vee \bar{\gamma}_\infty)$ (with $m - 1$ factors occurring), $SD_\wedge$ and Claim 3 yield $(x_0, x_m) \in \bar{\alpha} \wedge \bar{\beta}_\infty$. Therefore there exists an even integer $n \geq 2$ such that $(x_0, x_m) \in \beta_n(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$. Therefore, by Claim 1, there exists $k \geq n$ such that x_0 and x_m can be connected by $G_k(\beta_n)$ in $F_{\underset{\sim}{V}}(x_0, x_1, \dots, x_m)$. We can assume that $k = n$. (We have $(0, m) \in \alpha(m)$ whence, by repeating the "end-point" elements x_0 and x_m, x_0 and x_m can be connected by $G_k(\beta_{n+2})$, $G_k(\beta_{n+4})$, etc.). Now we have elements a_i in $F_{\underset{\sim}{V}}(x_0, \dots, x_m)$ associated with the vertices f_i of $G_n(\beta_n)$. But $a_i = f_i(x_0, x_1, \dots, x_m)$ for some terms f_i whence, by Claim 2, it follows that $U(m, G_n(\beta_n)) = U(m, n)$ holds in $\underset{\sim}{V}$.

(iii)$\Rightarrow$(i): Now suppose $a_0, a_1 \in A \in \underset{\sim}{V}$, $\underset{\sim}{V}$ is a variety satisfying (iii), $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are congruences of A, $\bar{\alpha} \wedge \bar{\beta} = \bar{\alpha} \wedge \bar{\gamma}$, and $(a_0, a_1) \in \bar{\alpha} \wedge (\bar{\beta} \vee \bar{\gamma})$. Then there are elements $b_0, b_1, \dots, b_m \in A$ such that $a_0 = b_0$, $a_1 = b_m$, $(b_0, b_m) \in \bar{\alpha}$, $(b_i, b_{i+1}) \in \bar{\beta}$ for i even, and $(b_i, b_{i+1}) \in \bar{\gamma}$ for i odd. From (iii) we have a graph $G \in \underset{\sim}{G}(\beta_\infty)$, and thus $G \in G(\beta_n)$ for some n, such that $U(m, G)$ holds in $\underset{\sim}{V}$. We claim that via assigning $f_i(b_0, b_1, \dots, b_m) \in A$ to all vertices f_i of G b_0 and b_m are connected by G in A. Really, if two vertices, f_i and f_j, are connected by a π-coloured edge in G, $\pi \in \{\alpha, \beta, \gamma\}$, then $f_i(b_0, b_1, \dots, b_m)\bar{\pi} f_i(b_{0\pi}, b_{1\pi}, \dots, b_{m\pi}) = f_j(b_{0\pi}, b_{1\pi}, \dots, b_{m\pi})\bar{\pi} f_j(b_0, b_1, \dots, b_m)$. Hence Claims 1 and 3 yield $(a_0, a_1) = (b_0, b_m) \in \beta_m(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = \bar{\beta}$, yielding (i).

Finally suppose $U(m, n)$ holds in a variety $\underset{\sim}{V}$ via the terms $f_0, f_1, f_2, \dots$. To satisfy $U(m, G_n(\beta_{n+2}))$ in $\underset{\sim}{V}$ we can associate the same terms $f_0, f_1, f_2, \dots$ with the vertices of a subgraph S, $S \cong G_n(\beta_n)$, and associate the projections on x_0 and x_m with the other vertices of $G_n(\beta_{n+2})$. Having $U(m, G_n(\beta_{n+2}))$ satisfied, by repeating terms appropriately one can define terms for $U(m, G_{n+2}(\beta_{n+2})) = U(m, n+2)$.

4. COROLLARIES. In Jónsson and Rival's paper [5] a sequence of lattice identities ε_n was produced with the property that an arbitrary lattice variety is meet semi-distributive if and only if ε_n holds in it for some $n<\omega$. (Note that the proof of Theorem 6.1 in [5] yields this result, which we cite in a slightly modified form.) Furthermore, Day [2] showed that $\underset{\sim}{K}_n$, the n-th Polin variety, is congruence meet and join semi-distributive, congruence (n+2)-permutable, and ε_{2n} holds in its congruence lattices. (For $n = 2$ Day and Freese [3, Theorem 7.1] have proved more, namely, even ε_2 holds in the congruence lattices of $\underset{\sim}{K}_2 = \underset{\sim}{P}$, the original Polin variety.) Denoting the lattice identity $\alpha\wedge(\beta\vee\gamma)\leq\beta_n$ by ε_n we can present a similar observation.

COROLLARY 1. Given a congruence m-permutable variety $\underset{\sim}{V}$, $\underset{\sim}{V}$ is congruence meet semi-distributive iff there exists $n<\omega$ such that the identity ε_n holds in the congruence lattices of $\underset{\sim}{V}$, or equivalently, iff $U(m, n)$ holds in $\underset{\sim}{V}$ for some $n<\omega$.

<u>Proof</u>. If $\underset{\sim}{V}$ is congruence $SD_\wedge$ then, by our Theorem, $U(m, n)$ holds in it for some n. But what was really shown in the proof of Theorem is that if $U(m, n)$ holds in a variety with m-permutable congruences then its congruence lattices satisfy ε_n. Conversely, if $\alpha\wedge\beta = \alpha\wedge\gamma$ for elements α, β, γ of an arbitrary lattice, then an easy induction yields $\beta_n(\alpha,\beta,\gamma) = \beta$ and $\gamma_n(\alpha,\beta,\gamma) = \gamma$ for all $n<\omega$. Thus ε_n implies $\alpha\wedge(\beta\vee\gamma)\leq\alpha\wedge\beta_n(\alpha,\beta,\gamma) = \alpha\wedge\beta$, the meet semi-distributivity, in any lattice.

It is worth mentioning that the dual statement also holds, i.e. we have the following:

<u>Observation</u>. Let $\underset{\sim}{V}$ be a congruence m-permutable variety of algebras. Then $\underset{\sim}{V}$ is congruence join semi-distributive if and only if there exists an $n<\omega$ such that ε_n^*, the dual of ε_n, holds in the congruence lattices of $\underset{\sim}{V}$.

<u>Proof</u>. By duality, ε_n^* implies join semi-distributivity (in any lattice). Consider the lattice terms $u_n = u_n(\alpha,\beta,\gamma)$ and $v_n = v_n(\alpha,\beta,\gamma)$ defined by the following induction: $u_0 = \alpha\wedge\beta$, $v_0 = \alpha\wedge\gamma$, $u_{n+1} = \alpha\wedge(\beta\vee v_n)$, $v_{n+1} = \alpha\wedge(\gamma\vee u_n)$, and let κ_n denote the identity $\alpha\wedge(\beta\vee\gamma)\leq u_n$. We obtain $u_n = \alpha\wedge\beta_n$ and $v_n = \alpha\wedge\gamma_n$, whence ε_n and κ_n (and thus ε_n^* and κ_n^* as well) are equivalent in any lattice. Now, if $\underset{\sim}{V}$ is m-permutable and congruence join semi-distributive then, by [1, Proposition 1] $U(m, m, \dots, m)$ (defined there, m occours $n+1$ times) holds in $\underset{\sim}{V}$ sor some $n<\omega$. Therefore, as it is implicit in [1] (cf. also Pixley [9]), κ_n^* holds in the congruence lattices of $\underset{\sim}{V}$.

Before formulating our last observation we define some (recursively defined) Mal'cev conditions occurring in (iii) more explicitly. Let $G_3(\beta_{m-1}) + G_3(\beta_{m-1}) \in \underset{\sim}{G}(\beta_\infty)$ denote the serial connection of length two of two disjoint copies of $G_3(\beta_{m-1})$ for m odd. Then $U(m, G_3(\beta_{m-1}) + G_3(\beta_{m-1}))$ is the following condition (cf. Figure 3 where $m = 3$):

"There exist $(m+1)$-ary terms f_i, f^i, g_i, g^i for $0 \le i \le m-1$ such that, denoting $\pi(m)$ by π and $h(x_{0\pi}, x_{1\pi}, x_{2\pi}, \dots, x_{m\pi})$ by $h(\pi)$, the following identities

$$f_i(\beta) = f_{i+1}(\beta),\ f^i(\beta) = f^{i+1}(\beta),\ g^i(\beta) = g^{i+1}(\beta),\ g_i(\beta) = g_{i+1}(\beta) \quad \text{for } 0 \le i < m-1,\ i \text{ even},$$

$$f_i(\gamma) = f_{i+1}(\gamma),\ f^i(\gamma) = f^{i+1}(\gamma),\ g^i(\gamma) = g^{i+1}(\gamma),\ g_i(\gamma) = g_{i+1}(\gamma) \quad \text{for } 0 < i < m-1,\ i \text{ odd},$$

$$f_i(\alpha) = f^i(\alpha),\ g^i(\alpha) = g_i(\alpha) \quad \text{for } 0 \le i \le m-1,$$

$$f_{m-1}(\beta) = f^{m-1}(\beta),\ g^{m-1}(\beta) = g_{m-1}(\beta),\ f^0(x_0, x_1, \dots, x_m) = g^0(x_0, x_1, \dots, x_m),$$

$$f_0(x_0, x_1, \dots, x_m) = x_0, \text{ and } g_0(x_0, x_1, \dots, x_m) = x_m$$

hold".

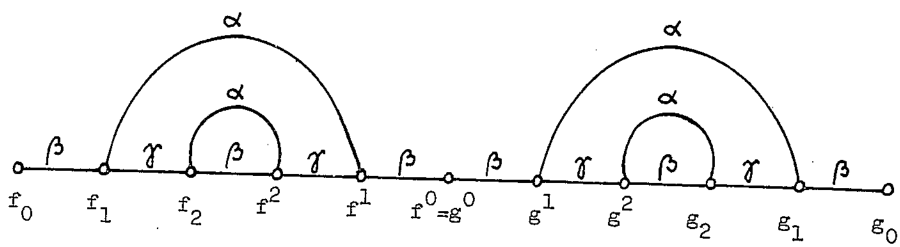

Figure 3

COROLLARY 2 (Papert [8]). The variety of semilattices is congruence meet semi-distributive.

<u>Proof</u>. For $i = 0, 1, \dots, m-1$ consider the semilattice terms $f_i = f_i(x_0, x_1, \dots, x_m) = x_0x_1x_2 \dots x_i$, $f^i = f_ix_m$, $g_i = x_mx_{m-1}x_{m-2} \dots x_{m-i}$, and $g^i = x_0g_i$. Since these terms satisfy the identities prescribed in $U(m, G_3(\beta_{m-1}) + G_3(\beta_{m-1}))$ for all odd $m > 1$, our Theorem completes the proof.

Note that essentially these terms from $U(m, G_3(\beta_{m-1}) + G_3(\beta_{m-1}))$ were used by Nation [6] in proving congruence $SD_\wedge$ for semilattices.

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JATE, Bolyai Institute

Szeged, Aradi vértanúk tere 1.

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