*Acta Sci. Math.*, 43 (1981), 267—272

# A Mal'cev type condition for the semi-distributivity of congruence lattices

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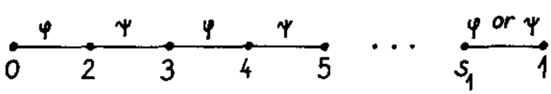
**1. Introduction.** A variety of algebras is said to be *congruence semi-distributive* if in the congruence lattices of its algebras the *semi-distributive law*,

$$(\forall\varphi)(\forall\psi)(\forall\eta)(\varphi\vee\psi=\varphi\vee\eta\Rightarrow\varphi\vee\psi=\varphi\vee(\psi\wedge\eta)),$$

holds. JÓNSSON [4, Problem 2.18] and GUMM [3] ask whether there exists a weak Mal'cev condition that characterizes congruence semi-distributivity of varieties. Now, to characterize congruence semi-distributivity of varieties, we intend to present a Mal'cev type condition, which is somewhat weaker than a weak Mal'cev condition in the sense of JÓNSSON [4].

**2. A Mal'cev type condition.** First, for any integers $n\geqq 1$ and $s_1, \dots, s_n>1$ we define a graph $G(s_1, \dots, s_n)$ whose vertices are the integers $0, 1, \dots, k(s_1, \dots, s_n)$. The edges of $G(s_1, \dots, s_n)$ will be denoted by ordered pairs $(i, j)$ with $i<j$, and will be coloured by the elements of $\Gamma=\{\varphi, \psi, \eta\}$. (The pair $(i, j)$ without providing $i<j$ can mean the edge $(j, i)$ for $j<i$.)

Let $k(s_1)=s_1$ and define $G(s_1)$ as follows:




(the colours $\varphi$ and $\psi$ alternate).

Suppose $G(s_1, \dots, s_n)$ is already defined and consider the following linear ordering of the edges of $G(s_1, \dots, s_n)$:

$$(i_1, j_1)<(i_2, j_2) \quad \text{iff either} \quad i_1<i_2 \quad \text{or} \quad i_1=i_2 \quad \text{and} \quad j_1<j_2.$$

Suppose $n$ is odd (even, respectively). Let

$$(i_0, j_0)<(i_1, j_1)<\dots<(i_t, j_t)$$

Received October 8, 1979.

be the $\psi$-coloured ($\eta$-coloured, resp.) edges of $G(s_1, \ldots, s_n)$ whose endpoints cannot be connected by a path which consists of edges coloured by the elements of $\Gamma \setminus \{\psi\}$ ($\Gamma \setminus \{\eta\}$, resp.). Now we construct the graph $G(s_1, \ldots, s_n, s_{n+1})$ by adding new vertices and new edges as follows:

(i) we add $(t+1)(s_{n+1}-1)$ new vertices, i.e.

$$k(s_1, \ldots, s_n, s_{n+1}) = k(s_1, \ldots, s_n)+(t+1)(s_{n+1}-1),$$

and for any $r$, $0 \leqq r \leqq t$,

(ii) denoting $k(s_1, \ldots, s_n)$ by $k$ we add the edges

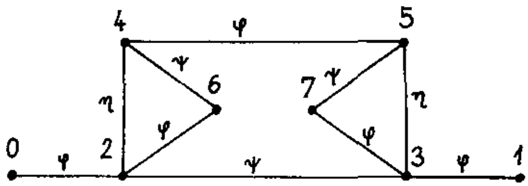
$$(i_r, k+r(s_{n+1}-1)+1);$$
$$(k+r(s_{n+1}-1)+q, k+r(s_{n+1}-1)+q+1), \quad 1 \leqq q \leqq s_{n+1}-2;$$
$$(k+r(s_{n+1}-1)+s_{n+1}-1, j_r),$$

among which

$$(i_r, k+r(s_{n+1}-1)+1);$$
$$(k+r(s_{n+1}-1)+q, k+r(s_{n+1}-1)+q+1), \quad 1 \leqq q \leqq s_{n+1}-2, \quad q \text{ even};$$
$$(k+r(s_{n+1}-1)+s_{n+1}-1, j_r), \quad \text{provided } s_{n+1} \text{ is odd},$$

are coloured by $\eta$ ($\varphi$, resp.) and the others are coloured by $\varphi$ ($\psi$, resp.).

For example, $G(3, 3, 2)$ is the following graph.




For $\pi \in \Gamma$ define $\pi(s_1, \ldots, s_n)$ to be the equivalence relation on the vertex set of $G(s_1, \ldots, s_n)$ generated by $\{(i, j)\colon (i, j)$ is an edge of $G(s_1, \ldots, s_n)$ coloured by $\pi\}$.

For $m \geqq 1$ let $U(m, s_1, \ldots, s_n)$ denote the following strong Mal'cev condition: There exist $k(s_1, \ldots, s_n)+1$-ary terms $f_0, f_1, \ldots, f_m$ such that the identities

$$f_0(x_i\colon i \leqq k) = x_0, \quad f_m(x_i\colon i \leqq k) = x_1,$$
$$f_j(x_{i\varphi}\colon i \leqq k) = f_{j+1}(x_{i\varphi}\colon i \leqq k) \quad \text{for } j \text{ even}, \quad 0 \leqq j \leqq m-1,$$
$$f_j(x_{i\psi}\colon i \leqq k) = f_{j+1}(x_{i\psi}\colon i \leqq k) \quad \text{for } j \text{ odd}, \quad 0 \leqq j \leqq m-1, \quad \text{and}$$
$$f_j(x_{i\eta}\colon i \leqq k) = f_{j+1}(x_{i\eta}\colon i \leqq k) \quad \text{for } j \text{ odd}, \quad 0 \leqq j \leqq m-1$$

hold, where $k=k(s_1, \ldots, s_n)$ and $f_j(x_i\colon i \leqq k)$ stands for $f_j(x_0, x_1, \ldots, x_k)$. Here, for $\pi \in \Gamma$ and $i \leqq k(s_1, \ldots, s_n)$, $i\pi$ denotes the smallest integer $j$ $(0 \leqq j \leqq k(s_1, \ldots, s_n))$ for which $(i, j) \in \pi(s_1, \ldots, s_n)$.

Now we can formulate the following

Theorem. *For any variety* **V** *of algebras the following two conditions are equivalent:*

(i) **V** *is congruence semi-distributive;*

(ii) *For any infinite sequence* $\mathbf{s}=(s_1, s_2, s_3, \ldots)$ *of integers* $(s_i>1, i=1, 2, 3, \ldots)$ *there exist integers* $m, n\geqq 1$ *such that* $U(m, s_1, \ldots, s_n)$ *holds in* **V**.

**3. The proof of the Theorem.** In order to prove our theorem, we need several statements.

(i) *implies* (ii). Let **V** be a congruence semi-distributive variety of similarity type $\tau$. Suppose $\mathbf{s}=(s_1, s_2, \ldots)$ is an infinite sequence of integers $s_i>1$ $(i=1, 2, \ldots)$. Note that for $n<t$, $G(s_1, \ldots, s_n)$ is a subgraph of $G(s_1, \ldots, s_t)$, i.e., for $i, j\leqq \leqq k(s_1, \ldots, s_n)$ $(i, j)$ is a $\pi$-coloured edge in $G(s_1, \ldots, s_n)$ iff it is a $\pi$-coloured edge in $G(s_1, \ldots, s_t)$. Let $G(\mathbf{s})$ be the direct union of the graphs $G(s_1, \ldots, s_n)$ $(n\geqq 1)$ and let $X=X(\mathbf{s})=\{0, 1, 2, \ldots\}$ denote the vertex set of $G(\mathbf{s})$. For $\pi\in\Gamma$ let $\pi(\mathbf{s})= =\bigcup_{n=1}^{\infty}\pi(s_1, \ldots, s_n)$.

Claim 1. *For* $n\leqq t$ *and* $\pi\in\Gamma$, *both* $\pi(s_1, \ldots, s_t)$ *and* $\pi(\mathbf{s})$ *restricted to* $\{0, 1, \ldots, k(s_1, \ldots, s_n)\}$ *are* $\pi(s_1, \ldots, s_n)$.

This claim is an easy consequence of the definitions. By Claim 1, for any $\pi\in\Gamma$, $\pi(\mathbf{s})$ is an equivalence relation. Since

$$\varphi(s_1, \ldots, s_n)\vee\psi(s_1, \ldots, s_n)\subseteq\varphi(s_1, \ldots, s_{n+1})\vee\eta(s_1, \ldots, s_{n+1})\subseteq$$
$$\subseteq\varphi(s_1, \ldots, s_{n+2})\vee\psi(s_1, \ldots, s_{n+2})$$

is also obvious from our definitions, we have

Claim 2. $\varphi(\mathbf{s})\vee\psi(\mathbf{s})=\varphi(\mathbf{s})\vee\eta(\mathbf{s})$ *in the lattice of equivalence relations on* $X$.

Now consider $F(X)$, the free algebra in **V** generated by $X$. For any $\pi\in\Gamma$ let $\hat{\pi}$ denote the congruence of $F(X)$ generated by the relation $\pi(\mathbf{s})$. Claim 2 together with the well-known descriptions of the join of congruences and the congruence generated by a relation (cf. GRÄRZER [2], Lemma 2 and Theorem 4 in § 10, Chapter 1) immediately imply

Claim 3. $\hat{\varphi}\vee\hat{\psi}=\hat{\varphi}\vee\hat{\eta}$ *in the congruence lattice of* $F(X)$.

Since $0\equiv 1$ $(\hat{\varphi}\vee\hat{\psi})$ in $F(X)$, from Claim 3 and from the assumption made on **V** we obtain $0\equiv 1$ $(\hat{\varphi}\vee(\hat{\psi}\vee\hat{\eta}))$. Therefore there are elements $a_0, \ldots, a_m$ in $F(X)$ such that

$$(1)\qquad \begin{aligned} &a_0=0, \quad a_m=1\\ &a_j\equiv a_{j+1}(\hat{\varphi}) \quad \text{for } j \text{ even}\\ &a_j\equiv a_{j+1}(\hat{\psi}) \quad \text{for } j \text{ odd}\\ &a_j\equiv a_{j+1}(\hat{\eta}) \quad \text{for } j \text{ odd.}\end{aligned}$$

Since $X$ generates $F(X)$, there is a finite subset of $X$ that generates a subalgebra containing all the $a_j$ $(0 \leqq j \leqq m)$. Hence there are $n \geqq 1$ and $(k(s_1, \ldots, s_n)+1)$-ary terms in $\mathbf{V}$ such that

$$a_j = f_j(i\colon i \leqq k(s_1, \ldots, s_n)) \tag{2}$$

holds for all $j \leqq m$ in $\mathbf{V}$. For $\pi \in \Gamma$ let

$$X_\pi = \{x_i\colon i \in X \text{ and } (i, j) \in \pi(\mathbf{s}) \text{ implies } i \leqq j\}$$

and define an onto mapping $g_\pi\colon X \to X_\pi$ by $ig_\pi = x_j$ iff $j = \min\{t\colon (t, i) \in \pi(\mathbf{s})\}$. Let us denote by $W(Y)$ and $W(Y_\pi)$ the absolutely free algebras of type $\tau$ generated by $Y = \{y_i\colon i \in X\}$ and $Y_\pi = \{y_i\colon x_i \in X_\pi\}$, respectively. Let $F(X_\pi)$ be the free $\mathbf{V}$-algebra generated by $X_\pi$. We consider the natural homomorphisms $u\colon W(Y) \to F(X)$ and $v\colon W(Y_\pi) \to F(X_\pi)$ defined by $y_i u = i$ $(i \in X)$ and $y_i v = x_i$ $(x_i \in X_\pi)$. Let $p\colon F(X) \to F(X_\pi)$ and $q\colon W(Y) \to W(Y_\pi)$ be the unique homomorphisms for which $ip = ig_\pi$ $(i \in X)$ and $y_i q = y_{ig_\pi}$ $(i \in X)$. Then the diagram

$$\begin{array}{ccc} W(Y) & \xrightarrow{\;q\;} & W(Y_\pi) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ F(X) & \xrightarrow{\;p\;} & F(X_\pi) \end{array}$$

commutes. Furthermore, we have $\hat{\pi} \subseteq \operatorname{Ker} p$ since $\pi(\mathbf{s}) = \operatorname{Ker} g_\pi$.

Now, by (1) and (2), the identities $f_0(x_i\colon i \leqq k(s_1, \ldots, s_n)) = x_0$ and

$$f_m(x_i\colon i \leqq k(s_1, \ldots, s_n)) = x_1$$

are evidently satisfied in $\mathbf{V}$. For the rest of the identities in $U(m, s_1, \ldots, s_n)$, let $\pi \in \Gamma$ and let $a_j \equiv a_{j+1}(\hat{\pi})$ be one of the formulae listed in (1). Denoting $k(s_1, \ldots, s_n)$ by $k$, we can compute:

$$f_j(x_{i\pi}\colon i \leqq k) = f_j(y_{i\pi} v\colon i \leqq k) = f_j(y_{ig_\pi} v\colon i \leqq k), \quad \text{by Claim 1,}$$

$$= f_j(y_i qv\colon i \leqq k)$$

$$= f_j(y_i up\colon i \leqq k), \quad \text{by the commutativity of the diagram,}$$

$$= f_j(ip\colon i \leqq k) = f_j(i\colon i \leqq k)p = a_j p = a_{j+1} p, \quad \text{since } \hat{\pi} \subseteq \operatorname{Ker} p,$$

$$= f_{j+1}(i\colon i \leqq k)p = f_{j+1}(ip\colon i \leqq k) = f_{j+1}(y_i up\colon i \leqq k)$$

$$= f_{j+1}(y_i qv\colon i \leqq k), \quad \text{by the commutativity of the diagram,}$$

$$= f_{j+1}(y_{ig_\pi} v\colon i \leqq k)$$

$$= f_{j+1}(y_{i\pi} v\colon i \leqq k), \quad \text{by Claim 1,}$$

$$= f_{j+1}(x_{i\pi}\colon i \leqq k).$$

Therefore the identity $f_j(x_{i\pi}: i \leqq k) = f_{j+1}(x_{i\pi}: i \leqq k)$ holds in $F(X_\pi)$, whence it holds in $\mathbf{V}$ as well. Hence $\mathbf{V}$ satisfies (ii).

*To prove the converse*, let $\mathbf{V}$ be a variety satisfying (ii). Let $\hat\varphi, \hat\psi, \hat\eta$ be congruences of an algebra $A$ in $\mathbf{V}$ such that $\hat\varphi \vee \hat\psi = \hat\varphi \vee \hat\eta$. We have to show that $\hat\varphi \vee \hat\psi \subseteq \hat\varphi \vee (\hat\psi \wedge \hat\eta)$, which is clearly equivalent to $\hat\varphi \vee \hat\psi = \hat\varphi \vee (\hat\psi \wedge \hat\eta)$. Let $a_0, a_1$ be arbitrary elements of $A$ that are congruent modulo $\hat\varphi \vee \hat\psi$. We define an infinite sequence $\mathbf{s}$ and assign an element $a_i$ in $A$ to each vertex $i$ of $G(\mathbf{s})$ by means of induction. Let $s_1 \geqq 2$ be the smallest integer for which $(a_0, a_1) \in \hat\varphi \circ \hat\psi \circ \hat\varphi \circ \hat\psi \circ \dots$ ($s_1$ factors) and let us choose elements $a_j$ $(2 \leqq j \leqq k(s_1) = s_1)$ from $A$ such that

$$
\begin{aligned}
&(a_0, a_2) \in \hat\varphi, \\
&(a_j, a_{j+1}) \in \hat\varphi \quad \text{for } j \text{ odd}, \quad 3 \leqq j < s_1, \\
&(a_{s_1}, a_1) \in \hat\varphi \quad \text{provided } s_1 \text{ is odd}, \\
&(a_j, a_{j+1}) \in \hat\psi \quad \text{for } j \text{ even}, \ 2 \leqq j < s_1, \\
&(a_{s_1}, a_1) \in \hat\psi, \quad \text{provided } s_1 \text{ is even}.
\end{aligned}
$$

Then $G(s_1)$ and the elements chosen from $A$ have the following property:

(3) if the graph has an edge $(i, j)$ coloured by $\pi$ then $(a_i, a_j) \in \hat\pi$.

Suppose $s_1, \dots, s_n$ and $a_j$ $(j \leqq k(s_1, \dots, s_n))$ are already defined. Then let $s_{n+1} \geqq 2$ be the smallest integer such that $G(s_1, \dots, s_n, s_{n+1})$ has property (3) with appropriate further elements $a_i \in A$ $(k(s_1, \dots, s_n) < i \leqq k(s_1, \dots, s_n, s_{n+1}))$ associated with the new vertices. There exist such an integer $s_{n+1}$ and such elements $a_i \in A$, since whenever we have elements $b$, $c$, $d$ and $e$ in $A$ with $(b, c) \in \hat\psi$ and $(d, e) \in \hat\eta$ then, by $\hat\psi \subseteq \hat\varphi \vee \hat\eta$ and $\hat\eta \subseteq \hat\varphi \vee \hat\psi$, there are integers $t, t'$ such that $(b, c) \in \hat\varphi \circ \hat\eta \circ \hat\varphi \circ \hat\eta \circ \dots$ ($t$ factors) and $(d, e) \in \hat\varphi \circ \hat\psi \circ \hat\varphi \circ \hat\psi \circ \dots$ ($t'$ factors).

Let $m$ and $n$ be the integers that exist by (ii) for the sequence $\mathbf{s} = (s_1, s_2, s_3, \dots)$ constructed above and let $f_0, f_1, \dots, f_m$ be $(k(s_1, \dots, s_n)+1)$-ary terms satisfying the identities of $U(m, s_1, \dots, s_n)$ throughout $\mathbf{V}$. Let $k$ stand for $k(s_1, \dots, s_n)$. It remains to show that

$$
\begin{aligned}
&f_j(a_i: i \leqq k) \equiv f_{j+1}(a_i: i \leqq k)\ (\hat\varphi) \quad \text{for } j \text{ even}, \\
(4) \qquad &f_j(a_i: i \leqq k) \equiv f_{j+1}(a_i: i \leqq k)\ (\hat\psi) \quad \text{for } j \text{ odd and} \\
&f_j(a_i: i \leqq k) \equiv f_{j+1}(a_i: i \leqq k)\ (\hat\eta) \quad \text{for } j \text{ odd}.
\end{aligned}
$$

Indeed, then $(a_0, a_1) = (f_0(a_i: i \leqq k),\ f_m(a_i: i \leqq k)) \in \hat\varphi \circ (\hat\psi \wedge \hat\eta) \circ \hat\varphi \circ (\hat\psi \wedge \hat\eta) \circ \dots$ ($m$ factors) $\subseteq \hat\varphi \vee (\hat\psi \wedge \hat\eta)$, completing the proof. Since $(a_i, a_{i\pi}) \in \hat\pi$ $(\pi \in \Gamma)$ follows from (3), for $j$ even we can compute:

$$f_j(a_i: i \leqq k)\, \hat\varphi\, f_j(a_{i\varphi}: i \leqq k) = f_{j+1}(a_{i\varphi}: i \leqq k)\, \hat\varphi f_{j+1}(a_i: i \leqq k).$$

Hence $f_j(a_i\colon i\leqq k)\equiv f_{j+1}(a_i\colon i\leqq k)$ $(\phi)$ holds for $j$ even and the rest of (4) follows similarly. The proof of the Theorem is complete.

**4. Concluding remarks.** In this section we mention some statements concerning congruence semi-distributivity. The proofs are omitted because they are easy but most of them would require a long formulation.

A variety **V** is said to be *n-permutable* $(n\geqq 2)$ if $\varphi\vee\psi=\varphi\circ\psi\circ\varphi\circ\psi\circ\ldots$ ($n$ factors) holds for any congruences $\varphi$ and $\psi$ of any algebra in **V**. It is easy to see that the method we used yields the following result, too.

Proposition 1. *An m-permutable variety* **V** *is congruence semi-distributive if and only if* $U(m, m, \ldots, m)$ *(where m occurs* $n+1$ *times) holds in V for some* $n\geqq 1$.

Making use of Claim 1 it can be shown that whenever $U(m, s_1, \ldots, s_n)$ holds in a variety **V** then $U(m+1, s_1, \ldots, s_n)$ and $U(m, s_1, \ldots, s_n, s_{n+1})$ hold in **V** as well. Therefore, condition (ii) in the Theorem is equivalent to:

(iii) For any infinite sequence $\mathbf{s}=(s_1, s_2, s_3, \ldots)$ of integers $s_i\geqq 2$ $(i=1, 2, 3, \ldots)$ there exists an integer $n\geqq 1$ such that $U(n, s_1, \ldots, s_n)$ holds in **V**.

In some varieties the terms and identities are easy to handle. For example, it is not hard to check that there are no $m, n\geqq 2$ for which $U\left(m, 3, 2, \ldots, \frac{1}{2}(5-(-1)^n)\right)$ holds in the variety of semilattices. Therefore the variety of semilattices is not congruence semi-distributive. However, as it was shown by Papert [5], it is congruence dually semi-distributive.

As a non-trivial example of varieties satisfying the conditions of the Theorem we can mention Polin's variety **P**. Indeed, as it was shown by Day and Freese [1], **P** is congruence semi-distributive, but it is even not congruence modular.

Acknowledgement. The author wishes to thank H. P. Gumm for helpful discussions.

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