A HORN SENTENCE IN COALITION LATTICES

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Abstract. Given a finite partially ordered set P, for subsets or, in other words, coalitions X, Y of P let $X \le Y$ mean that there exists an injection φ: $X \to Y$ such that $x \le \varphi(x)$ for all $x \in X$. The set $\mathcal{L}(P)$ of all subsets of P equipped with this relation is a partially ordered set. When $\mathcal{L}(P)$ is a lattice, it is called a coalition lattice. A recursive construction of coalition lattices is given. Using this construction, which can be of separate interest, it is shown that not every lattice is embeddable in coalition lattices.

1. Introduction

Given a finite partially ordered set $P = \langle P, \le \rangle$, the set of all subsets, alias coalitions, of P will be denoted by $\mathcal{L}(P)$. For $X, Y \in \mathcal{L}(P)$, a map φ: $X \to Y$ is called an *extensive map* if φ is injective and for every $x \in X$ we have $x \le \varphi(x)$. Let $X \le Y$ mean that there exists an extensive map $X \to Y$; this definition turns $\mathcal{L}(P)$ into a partially ordered set $\mathcal{L}(P) = \langle \mathcal{L}(P), \le \rangle$. When $\mathcal{L}(P)$ is a lattice then it is called a *coalition lattice*. This concept, with roots in game theory and the mathematics of human decision making, was introduced in [1] with a detailed motivation.

For undefined terminology the reader is referred to Grätzer [3]. Even without explicit mentioning, all sets occurring in this paper except Section 2 are assumed to be finite.

A partially ordered set P is called *upper bound free*, in short UBF, if for any $a, b, c \in P$ we have

$$((a \le c) \;\&\; (b \le c)) \implies ((a \le b) \text{ or } (b \le a)).$$

The equivalence classes of the equivalence generated by $\le_P$ will be called the *components* of P. If P is an UBF poset and has only one component then P is called a *tree*. A poset is called a *forest* if its components are trees. Clearly, a finite poset is a forest iff it is UBF. For $a \in P$ we will use the notation $(a] = \{x \in P\colon x \le a\}$. A poset P is a forest iff $(a]$ is a chain for every $a \in P$.

The main result of [1] asserts that, for a finite partially ordered set P, $\mathcal{L}(P)$ is a lattice iff P is a forest. The meet in this lattice is described by

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Lemma A. *([1]) Let P be a forest, $k \geq 2$, and for $A_1, \ldots, A_k \in \mathcal{L}(P)$ let $M = \{b_1 \wedge \ldots \wedge b_k\colon b_1 \in A_1, \ldots, b_k \in A_k$, and the infimum $b_1 \wedge \ldots \wedge b_k$ exists in $P\}$. If M is empty (in particular when one of the A_i is empty) then $\bigwedge_{i=1}^k A_i = \emptyset$. If M is non-empty then choose a maximal element $c = a_1 \wedge \ldots \wedge a_k$ in M where the a_i belong to A_i such that, for every i, $c \in A_i \Longrightarrow c = a_i$. Let $A_i' = A_i \setminus \{a_i\}$ for $i = 1, \ldots, k$, $P' = P \setminus \{c\}$, and put $C' = \bigwedge_{i=1}^k A_i'$ in $\mathcal{L}(P')$. Then $\bigwedge_{i=1}^k A_i = C' \cup \{c\}$ in $\mathcal{L}(P)$.*

The following result, which follows directly from [1, (5) in the proof of Prop. 2], will be also used in the sequel.

Lemma B. *If $X \leq Y$ in a coalition lattice and $|X| = |Y|$ then there is an extensive map $X \to Y$ which acts identically on $X \cap Y$.*

It is shown in [1] that the lattice $\mathcal{L}(P)$ is distributive iff it is modular iff all trees of the forest P are chains. On the other hand, it is not known yet whether coalition lattices generate the variety of all lattices. Developing a constructive way to build an arbitrary coalition lattice from smaller ones, it will be shown that the quasivariety generated by coalition lattices does not include all lattices. The construction producing a new lattice from two given lattices in Section 2 may be of separate interest.

2. A lattice construction

Let L_i be a complete lattice with bounds 0_i and 1_i, $i = 1, 2$, and let $\emptyset \neq S_i \subseteq L_i$ such that $1_1 \in S_1$, $0_2 \in S_2$, S_1 is closed under arbitrary meets and S_2 is closed under arbitrary joins. Note that the S_i are necessarily complete lattices under the ordering inherited from L_i but they need not be sublattices. Let $\psi\colon S_1 \to S_2$ be a lattice isomorphism. Associated with the quintuplet $\langle L_1, L_2, S_1, S_2, \psi\rangle$, we intend to define a lattice $L = L(L_1, L_2, S_1, S_2, \psi)$ as follows. Let L be the disjoint union of L_1 and L_2. For $x, y \in L$ we put $x \leq y$ iff one of the following three possibilities holds:

- $x, y \in L_1$ and $x \leq y$ in L_1;
- $x, y \in L_2$ and $x \leq y$ in L_2;
- $x \in L_1$, $y \in L_2$ and there exists a $z \in S_1$ such that $x \leq z$ in L_1 and $\psi(z) \leq y$ in L_2.

Proposition 1. *$L = L(L_1, L_2, S_1, S_2, \psi) = \langle L(L_1, L_2, S_1, S_2, \psi), \leq\rangle$ defined above is a complete lattice.*

Proof. It is straightforward to check that $\langle L, \leq\rangle$ is a partially ordered set with least element $0 = 0_1$ and greatest element $1 = 1_2$. To avoid confusion, $\bigwedge$, $\leq_1$, $\wedge_2$, $\bigvee_{S_1}$, etc. will denote the meet in L, the relation in L_1, the binary meet in L_2, the join in S_1, etc., respectively. Of course, $\bigwedge_{S_1} = \bigwedge_1$ and $\bigvee_{S_2} = \bigvee_2$.

Now we intend to show that any nonempty subset of L has a supremum. We start with a particular case. Let $\emptyset \neq A \subseteq L_1$ and $b = \bigvee_1 A$. We claim that b is a supremum of A in L as well. Clearly, $b \in L_1$ is an upper bound of A. Assume that $c \in L$ is another upper bound of A in L. We may suppose that $c \in L_2$, for otherwise $b \leq_1 c$ yields $b \leq c$ prompt. Then for each $a \in A$ there is a $z_a \in S_1$ such that $a \leq_1 z_a$ and $\psi(z_a) \leq c$. We have $b = \bigvee_1\{a\colon a \in A\} \leq_1 \bigvee_1\{z_a\colon a \in A\} \leq_1 \bigvee_{S_1}\{z_a\colon a \in A\}$ and $\psi(\bigvee_{S_1}\{z_a\colon a \in A\}) = \bigvee_{S_2}\{\psi(z_a)\colon a \in A\} = \bigvee_2\{\psi(z_a)\colon a \in A\} \leq_2 c$, whence $b \leq c$. Therefore b is the join of A in L.

Now let $\emptyset \neq C \subseteq L$. Then $C = A_1 \cup A_2$ with $A_i \subseteq L_i$. We claim that C has a supremum in L. The case $A_2 = \emptyset$ has just been settled. If $A_1 = \emptyset$ then $\bigvee_2 A_2$ is clearly the supremum of C in L. Therefore we assume that $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Let $b_i = \bigvee_i A_i$. By the previous arguments we have $b_i = \bigvee A_i$. Consider the element $t = \bigwedge_1 \{z \in S_1\colon\ b_1 \leq_1 z\} = \bigwedge_{S_1} \{z \in S_1\colon\ b_1 \leq_1 z\} \in S_1$ and let $c = \psi(t) \vee_2 b_2$. Since $b_1 \leq_1 t$, c is an upper bound of b_1 and b_2, whence it is an upper bound of C in L. Suppose $d \in L$ is another upper bound of C. Then d is an upper bound of the A_i and therefore also of the b_i, $i = 1, 2$. Hence $b_2 \leq_2 d$ and there is a $u \in S_1$ such that $b_1 \leq_1 u$ and $\psi(u) \leq_2 d$. The choice of t yields $t \leq_1 u$, whence $\psi(t) \leq_2 \psi(u)$. Consequently, $c = \psi(t) \vee_2 b_2 \leq_2 \psi(u) \vee_2 b_2 \leq_2 d$, implying $c \leq d$. I.e., $c = \bigvee C$. We have shown that each nonempty subset of L has a supremum. Since L has a least element, or using duality, it follows that L is a complete lattice.

When S_1 is a principal dual ideal of L_1 and S_2 is a principal ideal of L_2 then our construction resembles the Hall – Dilworth gluing (cf. [2] or [3, page 31]) with the difference that we do not identify S_1 and S_2.

3. Results on coalition lattices

Theorem 1. *Let P be a finite forest, v a maximal element of P, $u \in P$, and suppose that v covers u in P. Let $L_1 := \{X \in \mathcal{L}(P)\colon\ v \notin X\}$, $L_2 := \{X \in \mathcal{L}(P)\colon\ v \in X\}$, $S_1 := \{X \in L_1\colon\ u \in X\}$, $S_2 := \{X \in L_2\colon\ u \notin X\}$, and $\psi\colon\ S_1 \to S_2$, $X \mapsto (X \setminus \{u\}) \cup \{v\}$. Then L_1 is a prime ideal and L_2 is a dual prime ideal of $\mathcal{L}(P)$, both L_1 and L_2 are isomorphic to $\mathcal{L}(P \setminus \{v\})$, the conditions of Section 2 are fulfilled, and $\mathcal{L}(P)$ is exactly the lattice $L(L_1, L_2, S_1, S_2, \psi)$.*

Note that a rather special case of Theroem 1, when P is a chain, implicitly occurs in [1]. The first conspicuous use of Theorem 1 is that we can easily draw the diagram of $\mathcal{L}(P)$ for a tree P, provided it has not too many elements. A more serious consequence is

Corollary 1. *Each coalition lattice can be obtained from the two-element lattice by the construction of Section 2 (i.e. forming lattices $L(L_1, L_2, S_1, S_2, \psi)$ from L_1 and L_2 with appropriate S_1, S_2 and ψ) and forming direct products of finitely many lattices in a finite number of steps.*

Consider the lattice Horn sentence

$$(x \wedge y = x \wedge z = y \wedge z \ \&\ x \vee y = x \vee z = y \vee z) \Longrightarrow x = y,$$

which we denote by χ.

Corollary 2. *χ holds in every coalition lattice. In other words, the five-element nondistributive modular lattice, M_3, cannot be embedded in a coalition lattice.*

Proofs

Proof of Theorem 1. It is easy to see that $L_1 = (P \setminus \{v\}]$ and $L_2 = [\{v\})$. Thus, being complementary subsets of $\mathcal{L}(P)$, L_1 is a prime ideal and L_2 is a dual prime ideal. $L_1 \cong \mathcal{L}(P \setminus \{v\})$ hardly needs any proof. To show $L_2 \cong \mathcal{L}(P \setminus \{v\})$ let us consider the map $\alpha\colon\ \mathcal{L}(P \setminus \{v\}) \to L_2$, $X \mapsto X \cup \{v\}$. Then α is bijective and $X \leq Y$ implies $\alpha(X) \leq \alpha(Y)$. Conversely, if $\alpha(X) \leq \alpha(Y)$ then take an extensive map $\beta\colon\ \alpha(X) \to \alpha(Y)$. Since v is a maximal element, $\beta(v) = v$. So the restriction of β to $X = \alpha(X) \setminus \{v\}$ is an $X \to Y$ map and $X \le Y$ follows. Thus, $L_2 \cong \mathcal{L}(P \setminus \{v\})$.

Now we claim that, for any coalitions $A_1, \dots , A_k \in \mathcal{L}(P)$,

$$\bigwedge_{i=1}^{k} A_i \supseteq \bigcap_{i=1}^{k} A_i. \tag{1}$$

Using Lemma A, this will be shown via an induction on $|A_1| + \ldots + |A_n|$. If $\bigcap_{i=1}^{k} A_i$ is empty then there is nothing to prove. Suppose that $d \in \bigcap_{i=1}^{k} A_i$. If d is a maximal element of M given in the lemma then choosing $c = d$ we obtain $d \in \bigwedge_{i=1}^{k} A_i$. If d is not maximal in M then choose a maximal element $c \in M$ such that $d < c$. Then $d < c \le a_i$ for the a_i occurring in the lemma. So $d \in A'_i$ and $d \in P'$. By the induction hypothesis we obtain $d \in \bigcap_{i=1}^{k} A'_i \subseteq \bigwedge_{i=1}^{k} A'_i$ and $d \in \bigwedge_{i=1}^{k} A_i$ follows from the lemma. (1) has been proved.

It follows instantly from (1) that $S_1 \subseteq L_1$ is closed under meets. Clearly, $1_{L_1} = P \setminus \{v\} \in S_1$ and $0_{L_2} = \{v\} \in S_2$. It is known, cf. [1, Prop. 2 and the comment after it] that

$$A_1 \vee \ldots \vee A_k = \overline{\overline{A}_1 \wedge \ldots \wedge \overline{A}_k}.$$

Combining this with (1) we easily obtain

$$\bigvee_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{k} A_i.$$

Hence it follows that $S_2 \subseteq L_2$ is closed with respect to joins.

Now we intend to show that ψ is a lattice isomorphism. ψ is clearly bijective. First let us assume that $X \le Y$ in S_1 and $|X| = |Y|$. By Lemma B there is an extensive $\alpha\colon\ X \to Y$ with $\alpha(u) = u$. Clearly, $(\alpha \setminus \{\langle u, u\rangle\}) \cup \{\langle v, v\rangle\}$ is an extensive $\psi(X) \to \psi(Y)$ map, yielding $\psi(X) \le \psi(Y)$. Now let $X, Y \in S_1$ be arbitrary with $X \le Y$. If $u \notin \alpha(X)$ then we can replace α by $(\alpha \setminus \{\langle u, \alpha(u)\rangle\}) \cup \{\langle u, u\rangle\}$, which is also an extensive $X \to Y$ map. This way we can assume that $Y_1 = \alpha(X)$ contains u. Since $X \le Y_1$ and $|X| = |Y_1|$, the previous argument gives $\psi(X) \le \psi(Y_1)$ and we conclude the desired $\psi(X) \le \psi(Y)$ from $\psi(Y_1) \le \psi(Y)$. Hence ψ is monotone. Suppose now that $\psi(X) \le \psi(Y)$ and let $\beta\colon\ \psi(X) \to \psi(Y)$ be an extensive map. Since $v \in \psi(X)$ is maximal in P, $\beta(v) = v$. Hence $(\beta \setminus \{\langle v, v\rangle\}) \cup \{\langle u, u\rangle\}$ is an extensive $X \to Y$ map, whence $X \le Y$. Thus, ψ is an isomorphism.

What we have shown so far says that the lattice construction of Section 2 makes sense in our case. The base set of $L = L(L_1, L_2, S_1, S_2, \psi)$ and that of $\mathcal{L}(P)$ are identical, but we have to show that they possess the same partial order. Since $Z \le \psi(Z)$ for every $Z \in S_1$, it follows easily that if $X \le Y$ in L then $X \le Y$ in $\mathcal{L}(P)$. The converse implication will be derived less easily.

Suppose $X \le Y$ in $\mathcal{L}(P)$; we have to show the same relation in L. Since v is a fixed point of any extensive map, $X \in L_2$ and $Y \in L_1$ is impossible. The cases $\{X, Y\} \subseteq L_1$ and $\{X, Y\} \subseteq L_2$ are trivial.

Consequently, we can assume that $X \in L_1$ and $Y \in L_2$. Let us fix an extensive map $\alpha\colon\ X \to Y$; we have to show the existence of a $Z \in S_1$ such that $X \le Z$ and $\psi(Z) \le Y$.

First we deal with the case $v \notin \alpha(X)$. If $u \notin X$ then let $Z = X \cup \{u\} \geq X$ and the extensive map $\alpha \cup \{\langle v, v\rangle\}\colon \psi(Z) \to Y$ yields $\psi(Z) \leq Y$. If $u \in X$ then put $Z = X$ and consider the extensive map $(\alpha \setminus \{\langle u, \alpha(u)\rangle) \cup \{\langle v, v\rangle\}\colon \psi(Z) \to Y$, which gives $\psi(Z) \leq Y$.

From now on we assume that $v \in \alpha(X)$, say $\alpha(b) = v$. Since $u \prec v$, $b \neq v$, and b and u are comparable by $b, u \in (v]$, we conclude $b \leq u$. If $u \notin X$ then let $Z = (X \setminus \{b\}) \cup \{u\}$; clearly $X \leq Z$ and the extensive map $(\alpha \setminus \{\langle b, v\rangle\}) \cup \{\langle v, v\rangle\}\colon \psi(Z) \to Y$ yields $\psi(Z) \leq Y$. Thus, we suppose that $u \in X$. We can also assume that $b = u$, for otherwise, by $b < u < v$, we could consider the extensive map

$$X \to Y, \quad x \mapsto \begin{cases} v = \alpha(b), & \text{if } x = u, \\ \alpha(u), & \text{if } x = b, \\ \alpha(x), & \text{otherwise} \end{cases}$$

instead of α. Now we put $Z = X$ and the map $(\alpha \setminus \{\langle u, v\rangle\}) \cup \{\langle v, v\rangle\}\colon \psi(Z) \to Y$ yields $\psi(Z) \leq Y$. $\square$

Proof of Corollary 1. If $|P| = 1$ then $|\mathcal{L}(P)| = 2$ and the statement holds. Suppose $|P| > 1$ and the corollary holds for all forests with less than $|P|$ elements. If there is a pair $\langle u, v\rangle$ of elements in P such that v is a maximal element and $u \prec v$ then Theorem 1 applies. Otherwise P is an antichain, $X \leq Y$ in $\mathcal{L}(P)$ is equivalent to $X \subseteq Y$, and $\mathcal{L}(P)$ is the $|P|$th direct power of the two-element lattice. $\square$

Proof of Corollary 2. Since M_3 cannot be embedded in the two-element lattice, in virtue of Corollary 1 it suffices to show that this property is preserved under the construction of Section 2 and direct products. Suppose M_3 is embedded in a direct product $\prod_{i\in I} L_i$ but it cannot be embedded in the direct components L_i. Let $\pi_j\colon \prod_{i\in I} L_i \to L_j$ denote the jth projection. Since $\pi_j(M_3) \ncong M_3$ and M_3 is a simple lattice, $\pi_j(M_3)$ is a singleton for every $j \in I$, a contradiction. Thus, direct products preserve χ. Now suppose that M_3 is embedded in $L = L(L_1, L_2, S_1, S_2, \psi)$. Since $L = L_1 \cup L_2$ and M_3 has three atoms, there is an $i \in \{1, 2\}$ such that L_i contains at least two atoms of M_3. Since L_i is an ideal or a dual ideal of L, $M_3 \subseteq L_i$. Thus, if χ holds in L_1 and L_2 then it also holds in L. $\square$

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