A HORN SENTENCE FOR INVOLUTION LATTICES OF QUASIORDERS

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Abstract. The quasiorders of a set A form a lattice $\mathrm{Quord}(A)$ with an involution $\rho \mapsto \rho^{-1} = \{\langle x, y\rangle \colon \langle y, x\rangle \in \rho\}$. Some results in [1] and Chajda and Pinus [2] lead to the problem whether every lattice with involution can be embedded in $\mathrm{Quord}(A)$ for some set A. Using the author's approach to the word problem of lattices (cf. [3]), which also applies for involution lattices, it is shown that the answer is negative.

Introduction

A triplet $L = \langle L; \leq, {}^{-1}\rangle$ is called an *involution lattice* or a *lattice with involution* if ${}^{-1}\colon L \to L$ is a lattice automorphism such that $(x^{-1})^{-1} = x$ holds for all $x \in L$. The involution is called trivial if $x^{-1} = x$ for all $x \in L$. To present a natural example, let us consider a set A. A binary relation $\rho \subseteq A^2$ is called a *quasiorder* of A if ρ is reflexive and transitive. Defining $\rho^{-1} = \{\langle x, y\rangle \colon \langle y, x\rangle \in \rho\}$ as usual, the set $\mathrm{Quord}(A)$ of quasiorders of A becomes an involution lattice $\mathrm{Quord}(A) = \langle \mathrm{Quord}(A); \subseteq, {}^{-1}\rangle$. The sublattice $\{\rho \in \mathrm{Quord}(A)\colon \rho^{-1} = \rho\}$ is just the equivalence lattice of A.

Many involution lattices are known to be embeddable in some $\mathrm{Quord}(A)$. E.g., all distributive involution lattices by [1], all lattices with trivial involution by Whitman [7], and many more (e.g., $\mathrm{Quord}(B)$ for $|B| \geq 3$). These facts and Chajda and Pinus [2, Problem 2] naturally raise the question whether any involution lattice can be embedded in some $\mathrm{Quord}(A)$. The aim of the present note is to give a negative answer. The algorithm and computer program for the word problem of lattices, which is used in the proof below, is worth separate mentioning. (This algorithm is applicable to involution lattices as well.)

On the set $\{x, y, z, t, u, v, w\}$ of variables let us define the following involution lattice terms

$$\begin{aligned}
s_1 &= (z \vee u) \wedge (u^{-1} \vee x \vee z^{-1} \vee t^{-1}),\\
s_2 &= (y \vee w) \wedge (y^{-1} \vee x \vee v^{-1} \vee w^{-1}),\\
s_3 &= (y \vee s_1) \wedge (u^{-1} \vee x \vee z^{-1} \vee t^{-1}),\\
s_4 &= (u \vee s_2) \wedge (y^{-1} \vee x \vee v^{-1} \vee w^{-1}).
\end{aligned}$$

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Theorem. *The Horn sentence*

$$x \le y \vee u \;\&\; y \le z \vee t \;\&\; u \le v \vee w \Longrightarrow x \le s_3 \vee s_4 \vee z^{-1} \vee w^{-1}$$

holds in Quord(A) for any set A but does not hold in all involution lattices.

Proof. Let χ be an arbitrary Horn sentence for involution lattices. Then, w.l.o.g., χ is of the form

$$p_1(x, x^{-1}) \le q_1(x, x^{-1}) \;\&\; \dots \;\&\; p_t(x, x^{-1}) \le q_t(x, x^{-1}) \Longrightarrow p(x, x^{-1}) \le q(x, x^{-1})$$

where $x = \langle x_1, x_2, \dots, x_n\rangle$, $x^{-1} = \langle x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}\rangle$ and p_i, q_i, p, q are lattice terms. Let $y = \langle y_1, y_2, \dots, y_n\rangle$, and consider the lattice Horn sentence $\hat{\chi}$:

$$p_1(x, y) \le q_1(x, y) \;\&\; p_1(y, x) \le q_1(y, x) \;\&\; \dots \;\&\; p_t(x, y) \le q_t(x, y) \\ \;\&\; p_t(y, x) \le q_t(y, x) \Longrightarrow p(x, y) \le q(x, y).$$

Claim 1. *χ holds in all involution lattices iff $\hat{\chi}$ holds in all lattices.*

Suppose $\hat{\chi}$ holds in all lattices, L is an involution lattice, $a \in L^n$ and $p_i(a, a^{-1}) \le q_i(a, a^{-1})$ for $i = 1, 2, \dots, t$. Denoting a^{-1} by b we obtain

$$p_i(b, a) = p_i(a, b)^{-1} \le q_i(a, b)^{-1} = q_i(b, a),$$

whence the premise of $\hat{\chi}$ holds for $\langle a, b\rangle$ and $p(a, a^{-1}) = p(a, b) \le q(a, b) = q(a, a^{-1})$ follows.

Conversely, assume that χ holds in all involution lattices, L is a lattice, $a, b \in L^n$, and $p_i(a, b) \le q_i(a, b)$, $p_i(b, a) \le q_i(b, a)$ for $i = 1, \dots, t$. The congruence ϑ generated by

$$\{\langle p_i(x, y), p_i(x, y) \wedge q_i(x, y)\rangle \colon 1 \le i \le t\} \quad \cup \\ \{\langle p_i(y, x), p_i(y, x) \wedge q_i(y, x)\rangle \colon 1 \le i \le t\}$$

in the free lattice $F = F(x_1, \dots, x_n, y_1, \dots, x_n)$ is clearly included in the kernel of the lattice homomorphism φ: $F \to L$, $x_i \mapsto a_i$, $y_i \mapsto b_i$. Consider the automorphism ψ: $F \to F$, $x_i \mapsto y_i$, $y_i \mapsto x_i$. Then ψ preserves ϑ, for it preserves the set generating ϑ. Therefore the map κ: $F/\vartheta \to F/\vartheta$, $[u]\vartheta \mapsto [\psi(u)]\vartheta$ is a lattice automorphism. Thus we can consider F/ϑ as an involution lattice where $v^{-1} = \kappa(v)$. Then $([x_i]\vartheta)^{-1} = [y_i]\vartheta$, and the premise of χ holds for $[x]\vartheta$. From the assumption on χ we infer $p([x]\vartheta, [y]\vartheta) = p([x]\vartheta, ([x]\vartheta)^{-1}) \le q([x]\vartheta, ([x]\vartheta)^{-1}) = q([x]\vartheta, [y]\vartheta)$, and the canonical lattice homomorphism $F/\vartheta \to L$, $[u]\vartheta \mapsto \varphi(u)$ yields $p(a, b) \le q(a, b)$. This proves Claim 1.

Therefore, to decide if χ holds in all involution lattices, it suffices to deal with $\gamma = \hat{\chi}$. There are several known algorithms for the word problem of lattices, cf. Dean [4], Evans [5], McKinsey [6], and [3]; we have chosen [3], which seems to give the simplest and fastest algorithm. To point out that the closure operator T is needed only for a few subsets and can be determined fast, we cite the algorithm given in [3]. First we have to reduce γ into an equivalent "canonical" form

$$\bigwedge M_1 \le \bigvee J_1 \;\&\; \dots \;\&\; \bigwedge M_r \le \bigvee J_r \Longrightarrow \bigwedge M \le z$$

such that $M_1, \dots, M_r$, $J_1, \dots, J_r$, M are subsets of the set $X = \{z_1, z_2, \dots, z_s\}$ of variables occurring in γ and $z \in X$. For $j = 0, 1, 2, \dots$ we define a map T_j from $X \cup \{M\}$ to the power set of X by the following induction. Let $T_0(M) = M$, $T_0(u) = \{u\}$ for $u \in X$, and let

$$T_{j+1}(u) := T_j(u) \cup \bigcup_{\substack{0<i\le r\\ M_i\subseteq T_j(u)}} \bigcap_{v\in J_i} T_j(v)$$

for $u \in X \cup \{M\}$. By finiteness, there is a (smallest) j'such that $T_{j'+1}(u) = T_{j'}(u)$ holds for all $u \in X \cup \{M\}$. In other words, $T_{j'}$ is the smallest T such that

$$v \in T(v), \quad M \subseteq T(M) \quad \text{and} \quad T(u) = T(u) \cup \bigcup_{\substack{0<i\le r\\ M_i\subseteq T(u)}} \bigcap_{v\in J_i} T(v)$$

holds for all $u \in X \cup \{M\}$ and $v \in X$, and this formula leads to a more effective algorithm than the previous one. Now γ holds in all lattices iff $z \in T_{j'}(z_0)$.

Based on the algorithm described above, the author has developed a Turbo Pascal program for personal computers (Borland's Turbo Pascal version 4.0 — 7.0), which reduces γ to a canonical form and tests if γ holds in all lattices or not. (The program is available from the author upon request.) If χ is the Horn sentence in the Theorem and $\gamma = \hat{\chi}$ then $s = 34$, $r = 74$, and it takes about a second for this program to manifest that γ does not hold in all lattices. Therefore χ does not hold in all involution lattices.

Now, in order to show the other statement of the Theorem, let A be a set, x, y, z, t, u, v, $w \in \mathrm{Quord}(A)$, suppose that the premise of the Horn sentence χ in the Theorem holds for these quasiorders, and let $\langle a_0, a_1\rangle \in x$. Since $x \le y \vee u$, there are elements $a_0 = b_0, b_1, \dots, b_n = a_1$ in A such that $\langle b_i, b_{i+1}\rangle \in y$ for i even and $\langle b_i, b_{i+1}\rangle \in u$ for i odd. By reflexivity, we may assume that $n \ge 4$ and n is even. Since $y \le z \vee t$ and $u \le v \vee w$, there are elements $b_i = c_{i0}, c_{i1}, \dots, c_{ik} = b_{i+1}$ for i even and $b_i = d_{i0}, d_{i1}, \dots, d_{ik} = b_{i+1}$ for i odd such that $k \ge 4$, k is even, k does not depend on i, $\langle c_{ij}, c_{i,j+1}\rangle \in z$ for j even, $\langle c_{ij}, c_{i,j+1}\rangle \in t$ for j odd, $\langle d_{ij}, d_{i,j+1}\rangle \in v$ for j even, and $\langle d_{ij}, d_{i,j+1}\rangle \in w$ for j odd. Now, for $i = 1, 3, 5, \dots, n-3$, $b_i\ u\ b_{i+1} = c_{i+1,0}\ z\ c_{i+1,1}$ and $b_i = c_{i-1,k}\ t^{-1}\ c_{i-1,k-1}\ z^{-1}\ c_{i-1,k-2}\ t^{-1}\ c_{i-1,k-3}\ z^{-1} \dots z^{-1}\ c_{i-1,0} = b_{i-1}\ u^{-1}\ b_{i-2} = c_{i-3,k}\ t^{-1}\ c_{i-3,k-1}\ z^{-1}\ c_{i-3,k-2}\ t^{-1}\ c_{i-3,k-3}\ z^{-1} \dots z^{-1}\ c_{i-3,0} = b_{i-3} \dots b_0 = a_0\ x\ a_1 = b_n\ u^{-1}\ b_{n-1} = c_{n-2,k}\ t^{-1}\ c_{n-2,k-1}\ z^{-1}\ c_{n-2,k-2}\ t^{-1}\ c_{n-2,k-3}\ z^{-1} \dots z^{-1}\ c_{n-2,0} = b_{n-2}\ u^{-1}\ b_{n-3} \dots b_{i+2} = c_{i+1,k}\ t^{-1}\ c_{i+1,k-1}\ z^{-1}\ c_{i+1,k-2}\ t^{-1}\ c_{i+1,k-3}\ z^{-1} \dots t^{-1}\ c_{i+1,1}$, whence we conclude $\langle b_i, c_{i+1,1}\rangle \in s_1$. The rest of the following four formulas follow similarly:

$$\begin{aligned}
\langle b_i, c_{i+1,1}\rangle &\in s_1 &&\text{for } i = 1, 3, 5, \dots, n-3;\\
\langle d_{i-1,k-1}, b_{i+1}\rangle &\in s_2 &&\text{for } i = 2, 4, 6, \dots, n-2;\\
\langle b_{i-1}, c_{i+1,1}\rangle &\in s_3 &&\text{for } i = 1, 3, 5, \dots, n-3; \quad \text{and}\\
\langle d_{i-1,k-1}, b_{i+2}\rangle &\in s_4 &&\text{for } i = 2, 4, 6, \dots, n-2.
\end{aligned}$$

In virtue of these formulas we obtain $a_0 = b_0\ s_3\ c_{21}\ z^{-1}\ c_{20} = b_2\ s_3\ c_{41}\ z^{-1}\ c_{40} = b_4 \dots b_{n-2} = d_{n-3,k}\ w^{-1}\ d_{n-3,k-1}\ s_4\ b_n = a_1$, yielding $\langle a_0, a_1\rangle \in s_3 \vee s_4 \vee z^{-1} \vee w^{-1}$. Hence χ holds in $\mathrm{Quord}(A)$. $\square$

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