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MAL'CEV CONDITIONS FOR HORN SENTENCES WITH CONGRUENCE PERMUTABILITY

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1. Introduction

From the general description of properties of varieties characterizable by Mal'cev conditions (cf. Taylor [12] and Neumann [10]) it follows (Jónsson [9, Theorem 2.16]) that the satisfaction of a Horn sentence in the congruence lattices of n-permutable varieties can be characterized by Mal'cev conditions. Yet, no concrete Mal'cev conditions have been known in general (for exceptions cf. [1, 2, 3]). The aim of the present note is to give an algorithm which associates a suitable concrete Mal'cev condition with any (lattice) Horn sentence.

By a (universal lattice) Horn sentence we mean a first order formula of the form

$$\chi:\ (\forall\alpha_0)\dots(\forall\alpha_s)\left(\bigwedge_{i=1}^{k}(p_i \leqq q_i) \Rightarrow p \leqq q\right)$$

where p_i, q_i, p, and q are lattice terms over the set $S=\{\alpha_0, \alpha_1, \dots, \alpha_s\}$ of variables, and $k\geqq 0$. Let χ of the above form be fixed throughout, and let us fix an integer $n\geqq 2$, too. A variety **V** of algebras is said to be n-permutable if $\alpha\vee\beta=\alpha\circ\beta\circ\alpha\circ\beta\circ\alpha\circ\dots$ (with n factors) holds for congruences of its members.

Given a variety **V**, the class of lattices embeddable into congruence lattices of its members will be denoted by $\mathbf{SCV}=\mathbf{S}\{\mathrm{Con}(A)\colon A\in\mathbf{V}\}=\{L\colon L\leqq\mathrm{Con}(A)$ for some $A\in\mathbf{V}\}$. The quest for our Mal'cev conditions can be motivated by the following.

PROPOSITION 1.1 (cf. Hutchinson [6] for varieties of modules). *For any n-permutable variety* **V**, **SCV** *is a quasivariety, i.e., a class of lattices definable by a set of Horn sentences.*

In [8] rings are classified according to lattice identities being satisfied in congruence lattices of the corresponding modules. This classification is based on (strong) Mal'cev conditions supplied by the Wille—Pixley algorithm [11, 13]. Hutchinson [7] has shown that this classification can be properly refined if we use Horn sentences rather than only identities, which gives another motivation for our investigations.

2. Proof of Proposition 1.1

Let Σ be the set of all Horn formulae that hold in **CV** or, equivalently, in **SCV**, and let Σ hold in a lattice L, too. We have to show that $L\in\mathbf{SCV}$. Choose a well-ordering of L, i.e., $L=\{a_\alpha\colon \alpha<\gamma\}$ and over the set $X=\{x_\alpha\colon \alpha<\gamma\}$ of variables let D be the operation table (or diagram statement) of L. I.e., D is the conjunction of all

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inequalities of the forms $x_\delta \leqq x_\varepsilon$, $x_\mu \wedge x_\nu \leqq x_\eta$, $x_\zeta \leqq x_\varrho \vee x_\xi$ that hold in L under the substitution $X \to L$, $x_\alpha \mapsto a_\alpha$ $(\alpha < \gamma)$. Note that D holds for a system $\{b_\alpha : \alpha < \gamma\}$ of elements in a lattice A iff $L \to A$, $a_\alpha \mapsto b_\alpha$ is a homomorphism. Now let $a_\delta \not\leqq a_\varepsilon$ in L and consider the universally quantified generalized Horn sentence $D \Rightarrow x_\delta \leqq x_\varepsilon$. Then $D \Rightarrow x_\delta \leqq x_\varepsilon$ does not hold in L and we claim that it does not hold in **SCV** either. Suppose indirectly that $D \Rightarrow x_\delta \leqq x_\varepsilon$ still hold in **SCV** and, following the method of Taylor's [12], consider the similarity type $\tau = \sigma \cup \{r_\alpha : \alpha < \gamma\}$ where σ is the similarity type of **V** and r_α are binary relation symbols. Let Δ_1 be the set of all identities holding in **V** and let Δ_2 be the set of first order τ-formulae that express that r_α are congruences and satisfy D. (This is possible. If, e.g., $a_\zeta \leqq a_\varrho \vee a_\xi$ holds in L then consider the formula $(\forall y_0)(\forall y_n)(y_0 r_\zeta y_n \Rightarrow (\exists y_1) \dots (\exists y_{n-1})(y_0 r_\varrho y_1 \,\&\, y_1 r_\xi y_2 \,\&\, y_2 r_\varrho y_3 \,\&\, \&\, y_3 r_\xi y_4 \,\&\, \dots y_n))$, cf. also Taylor [12].) Since $D \Rightarrow x_\delta \leqq x_\varepsilon$ holds in **CV**, we have that $(\forall y)(\forall z)$ $(y r_\delta z \Rightarrow y r_\varepsilon z)$ is a consequence of $\Delta_1 \cup \Delta_2$. By the compactness theorem it is a consequence of $\Delta_1 \cup \Delta_2'$ for some finite $\Delta_2' \subseteq \Delta_2$, too. Therefore there exists a finite part D' of D (i.e., D' is only a finite conjunction of some members occurring in D) such that $D' \Rightarrow x_\delta \leqq x_\varepsilon$ holds in **CV**. But $D' \Rightarrow x_\delta \leqq x_\varepsilon$ is a Horn sentence. Hence it holds in L, which is a contradiction (choose the substitution $x_\alpha \mapsto a_\alpha$).

Therefore $D \Rightarrow x_\delta \leqq x_\varepsilon$ does not hold in **CV**, whence there is an algebra $A_{\delta,\varepsilon} \in \mathbf{V}$ and a lattice homomorphism $\varphi_{\delta,\varepsilon} : L \to \mathrm{Con}\,(A_{\delta,\varepsilon})$ for which $\varphi_{\delta,\varepsilon}(a_\delta) \not\leqq \varphi_{\delta,\varepsilon}(a_\varepsilon)$. Let Γ denote the set $\{(\delta, \varepsilon) : a_\delta \not\leqq a_\varepsilon$ in L, $\delta, \varepsilon < \gamma\}$, let $\prod_{(\delta,\varepsilon)\in\Gamma} A_{\delta,\varepsilon} = A \in \mathbf{V}$, and consider the map $\varphi : L \to \mathrm{Con}\,(A)$, $a_\alpha \mapsto \varphi(a_\alpha) = \{(f, g) \in A^2 : f(\delta, \varepsilon)\, \varphi_{\delta,\varepsilon}(a_\alpha)\, g(\delta, \varepsilon)$ for all $(\delta, \varepsilon) \in \Gamma\}$. We claim that φ is an embedding. It is easy to see that φ preserves the meet. Since $\varphi(a) \vee \varphi(b) \leqq \varphi(a \vee b)$ $(a, b \in L)$ follows from φ being monotone, the converse inequality has to be shown. Suppose $(f, g) \in \varphi(a \vee b)$. Then, for all $(\delta, \varepsilon) \in \Gamma$, $(f(\delta, \varepsilon), g(\delta, \varepsilon)) \in \varphi_{\delta,\varepsilon}(a \vee b) = \varphi_{\delta,\varepsilon}(a) \vee \varphi_{\delta,\varepsilon}(b)$. By n-permutability we can choose elements $h^0_{\delta,\varepsilon}, \dots, h^n_{\delta,\varepsilon} \in A_{\delta,\varepsilon}$ such that $f(\delta, \varepsilon) = h^0_{\delta,\varepsilon}$, $g(\delta, \varepsilon) = h^n_{\delta,\varepsilon}$, $(h^i_{\delta,\varepsilon}, h^{i+1}_{\delta,\varepsilon}) \in \varphi_{\delta,\varepsilon}(a)$ for $i < n$, i even, and $(h^i_{\delta,\varepsilon}, h^{i+1}_{\delta,\varepsilon}) \in \varphi_{\delta,\varepsilon}(b)$ for $i < n$, i odd. Defining $h^0, h^1, \dots, h^n \in A$ by $h^i(\delta, \varepsilon) = h^i_{\delta,\varepsilon}$ we obtain $f = h^0$, $g = h^n$, $(h^i, h^{i+1}) \in \varphi(a)$ for i even, and $(h^i, h^{i+1}) \in \varphi(b)$ for i odd. Thus $(f, g) \in \varphi(a) \vee \varphi(b)$, and φ is a homomorphism. Finally, for $(\delta, \varepsilon) \in \Gamma$, $\varphi_{\delta,\varepsilon}(a_\delta) \not\leqq \varphi_{\delta,\varepsilon}(a_\varepsilon)$ implies $\varphi(a_\delta) \not\leqq \varphi(a_\varepsilon)$, whence φ is injective. The proof, which uses only that **V** is axiomatizable, n-permutable, and closed under products, is complete.

3. The Mal'cev condition and its necessity

If we presented the main theorem before having proved anything from it, it could seem too complicated at the first sight. Hence the proof of its necessity part will preceed the main theorem. However, the necessary notations and concepts will be emphasized in this section.

To simplify our forthcoming notations, if $Q = \{z_0, z_1, \dots, z_s\}$ is a set (or a system) and d is an $(s+1)$-ary term over Q then let $d(Q)$, $d(y : y \in Q)$, and $d(z_i : i \leqq s)$ stand for $d(z_0, z_1, \dots, z_s)$. If no (well-)ordering of Q is given then any *fixed* one can be considered. Therefore our Mal'cev conditions will be unique only modulo the choices of these orderings, but we shall get appropriate conditions at all choices.

First for any lattice term $d = d(\alpha : \alpha \in S)$ we define a graph $G(d) = G_n(d)$ associated with d. The edges of $G(d)$ will be coloured by the variables $\alpha \in S$, and two distinguished vertices, the so-called left and right endpoints, will have special roles.

In figures these endpoints will be always placed on the left-hand side and on the right-hand side, respectively. An α-coloured edge connecting the vertices x and y will be often denoted by (x, α, y). Before defining $G(d)$ we introduce two kinds of operations for graphs. We obtain the *parallel connection* of graphs G_1 and G_2 by taking disjoint copies of G_1 and G_2 and identifying their left (right, resp.) endpoints (Figure 1). By

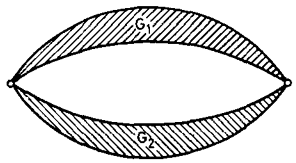

Fig. 1

taking disjoint graphs $H_1, H_2, \dots, H_n$ such that $H_i \cong G_1$ for i odd and $H_i \cong G_2$ for i even, and identifying the right endpoint of H_i and the left endpoint of H_{i+1} for $i = 1, 2, \dots, n-1$ we obtain the *serial connection* (of length n) of the graphs G_1 and G_2. (The left endpoint of H_1 and the right one of H_n are the endpoints of the serial connection, cf. Figure 2.)

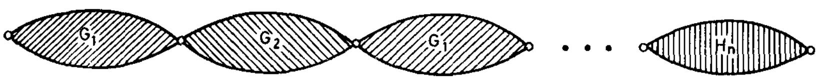

Fig. 2

Now, if d is a variable then let $G(d)$ be the graph consisting of two vertices (the endpoints) and a single d-coloured edge connecting them. Let $G(d_1 \wedge d_2)$ be the parallel connection of $G(d_1)$ and $G(d_2)$ while $G(d_1 \vee d_2)$ is defined to be the serial connection (of length n) of $G(d_1)$ and $G(d_2)$.

For an algebra A, a lattice term $d(\alpha\colon \alpha \in S)$, $a_0, a_1 \in A$ and congruences $\mathrm{con}(\alpha)$ $(\alpha \in S)$ of A we say that a_0 and a_1 *can be connected by* $G(d)$ *in* A if there is a map φ (referred to as *connecting map*) from the vertex set of $G(d)$ into A such that a_0 and a_1 are the images of the left and right endpoint, respectively, and for any $\alpha \in S$ and α-coloured edge (x, α, y) we have $(\varphi(x), \varphi(y)) \in \mathrm{con}(\alpha)$. If it is necessary we can emphasize that the *colour* α *is represented by the congruence* $\mathrm{con}(\alpha)$.

The following statement follows by a straightforward induction from definitions, therefore its proof will be omitted.

PROPOSITION 3.1. *Consider an algebra A with n-permutable congruences, $a_0, a_1 \in A$, a lattice term $d(\alpha\colon \alpha \in S)$ and congruences* $\mathrm{con}(\alpha)$ $(\alpha \in S)$ *of A. Then* $(a_0, a_1) \in d(\mathrm{con}(\alpha)\colon \alpha \in S)$ *if and only if a_0 and a_1 can be connected by $G(d)$ in A.*

We also need the following.

PROPOSITION 3.2. *Let* $\mathbf{V}$ *be an n-permutable variety, $a, b \in A \in \mathbf{V}$, and let* $\mathrm{con}(a, b)$ *be the congruence of A generated by (a, b). Then, for $x, y \in A$,* $(x, y) \in \mathrm{con}(a, b)$ *if and only if there are translations (i.e., unary algebraic functions) $t_0, \dots, t_{n-2}$ of A such that $x = t_0(a)$, $y = t_{n-2}(b)$, and $t_i(b) = t_{i+1}(a)$ for all i, $0 \leqq i < n-2$.*

PROOF. It is easy to check that the relation $\Psi=\{(x, y)\in A^2\colon t(a)=x$ and $t(b)=y$ for some translation $t\}$ is reflexive and compatible. By applying a theorem of Hagemann [4] (cf. also the Remark in [5]) we obtain that $\Phi=\Psi\circ\Psi\circ\ldots\circ\Psi$ ($n-1$ factors) is already a congruence. Now $(a, b)\in\Phi\subseteq\mathrm{con}\,(a, b)$ completes the proof.

In order to formulate an evident corollary to Proposition 3.2 precisely *we define two functions,* $v\colon \{1, 2, 3, \ldots\}\to\{1, 2, 3, \ldots\}$ and $k\colon \{(u, i)\colon 1\leqq i\leqq v(u)\}\to\{1, 2, 3, \ldots\}$, via induction. Put $v(1)=k(1, 1)=1$ and assume that $v(u)$, $k(u, 1)$, $k(u, 2), \ldots, k(u, v(u))$ have already been defined. Then let $v(u+1)=v(u)+1+v(u)+1+v(u)+1+\ldots$ (with n summands), $k(u+1, i)=k(u, j)$ for $1\leqq j\leqq v(u)$ and $j\equiv i \bmod (v(u)+1)$, and $k(u+1, i)=u+1$ for $i\equiv 0 \bmod (v(u)+1)$.

COROLLARY 3.3. *Given an n-permutable variety* $\mathbf{V}$, $A\in\mathbf{V}$, $(x, y)\in A^2$ *and* $H=\{(a_1, b_1), \ldots, (a_u, b_u)\}\subseteq A^2$, *we have* $(x, y)\in\mathrm{con}\,(H)=\bigvee\{\mathrm{con}\,(a_i, b_i)\colon 1\leqq i\leqq u\}$ *if and only if there are translations* t_{ij} $(1\leqq i\leqq v(u), 0\leqq j\leqq 2)$ *such that* $x=t_{10}(a_{k(u,1)})$, $y=t_{v(u),n-2}(b_{k(u,v(u))})$, $t_{i,n-2}(b_{k(u,i)})=t_{i+1,0}(a_{k(u,i+1)})$ *for* $1\leqq i<v(u)$, *and* $t_{ij}(b_{k(u,i)})=t_{i,j+1}(a_{k(u,i)})$ *for* $1\leqq i\leqq v(u)$ *and* $0\leqq j<n-2$.

Now let us fix an n-permutable variety $\mathbf{V}$ of similarity type σ and consider the Horn sentence χ from the Introduction. Following the customary way of finding Mal'cev conditions we intend to define suitable congruences of a free $\mathbf{V}$-algebra for which the premise of χ holds.

Continue the list $p_1\leqq q_1, \ldots, p_k\leqq q_k$ cyclicly by repeating its elements, i.e., for $k<i<\omega$ p_i and q_i are defined to be p_j and q_j, respectively, provided $i\equiv j \bmod (k)$. For a lattice term $d=d(\alpha\colon \alpha\in S)$ let $V(d)$, $W(d)$ and $E(d)$ denote the *vertex* set of $G(d)$, the set of inner vertices of $G(d)$ (vertex set *without* the endpoints), and the *edge* set of $G(d)$, respectively. Note that $0\leqq|W(d)|=|V(d)|-2$ and $E(d)\subseteq V(d)\times S\times V(d)$.

Now let $X_0=Y_0=V(p)$, with $x_0, x_1\in V(p)$ the endpoints, and put $A_0=F_{\mathbf{V}}(X_0)$, the free $\mathbf{V}$-algebra generated by X_0. For $\alpha\in S$ let $\mathrm{con}_0\,(\alpha)$ be the congruence of A_0 generated by $\{(x, y)\colon (x, \alpha, y)$ is an edge of $G(p)\}$. By Proposition 3.1 we have $(x_0, x_1)\in p\,(\mathrm{con}_0\,(\alpha)\colon \alpha\in S)$. To make the premise of χ hold we shall improve this construction in countably many steps.

Suppose $A_{i-1}=F_{\mathbf{V}}(X_{i-1})$ and $\mathrm{con}_{i-1}\,(\alpha)$ $(\alpha\in S)$ have already been defined for some $1\leqq i<\omega$. For a lattice term $d(\alpha\colon \alpha\in S)$ let $\mathrm{con}_{i-1}(d)$ stand for $d(\mathrm{con}_{i-1}(\alpha)\colon \alpha\in S)$. Put $Y_i=\{i\}\times N\times\mathrm{con}_{i-1}(p_i)\times W(q_i)$ (where N is the set of natural numbers) and $X_i=X_{i-1}\cup Y_i=X_{i-1}\mathring{\cup} Y_i=Y_0\mathring{\cup}\ldots\mathring{\cup} Y_{i-1}\mathring{\cup} Y_i$ (where $\mathring{\cup}$ stands for disjoint union). Note that $\{i\}$ and N are for making Y_i disjoint from X_{i-1} and for making Y_i infinite (if not empty). For $\alpha\in S$ let $\mathrm{con}_i\,(\alpha)$ be the congruence of A_i generated by $\mathrm{con}_{i-1}(\alpha)\cup\{((i, j, a, b, x), (i, j, a, b, y))\colon j\in N,\ (a, b)\in\mathrm{con}_{i-1}(p_i),\ x, y\in W(q_i)$, and $(x, \alpha, y)\in E(q_i)\}\cup\{(a, (i, j, a, b, z))\colon j\in N,\ (a, b)\in\mathrm{con}_{i-1}(p_i)$, and (left endpoint, α, $Z)\in E(q_i)\}\cup\{((i, j, a, b, z), b)\colon j\in N,\ (a, b)\in\mathrm{con}_{i-1}(p_i)$, and ($z$, α, right endpoint)$\in E(q_i)\}$ where (i, j, a, b, x) stands for $(i, j, (a, b), x)\in Y_i$. Observe that, by Proposition 3.1, $\mathrm{con}_{i-1}(p_i)\subseteq q_i(\mathrm{con}_i(\alpha)\colon \alpha\in S)=\mathrm{con}_i\,(q_i)$.

Finally, let X and $A=F_{\mathbf{V}}(X)$ be the directed unions of X_i and A_i $(i<\omega)$, respectively. Then, for $\alpha\in S$, $\mathrm{con}(\alpha)=\bigcup\{\mathrm{con}_i\,(\alpha)\colon i<\omega\}$ is a congruence. From Proposition 3.1 it follows that $d(\mathrm{con}\,(\alpha)\colon \alpha\in S)=\bigcup\{d(\mathrm{con}_i\,(\alpha)\colon \alpha\in S)\colon i<\omega\}$, for any lattice term d. Hence we obtain that, for $1\leqq j\leqq k$ and $(a, b)\in p_j\,(\mathrm{con}\,(\alpha)\colon \alpha\in S)$, $(a, b)\in p_j\,(\mathrm{con}_{i-1}(\alpha)\colon \alpha\in S)=\mathrm{con}_{i-1}(p_j)=\mathrm{con}_{i-1}(p_i)\subseteq\mathrm{con}_i(q_i)=\mathrm{con}_i(q_j)=$

$=q_j(\mathrm{con}_i(\alpha)\colon \alpha\in S)\subseteq q_j(\mathrm{con}(\alpha)\colon \alpha\in S)$ for some $i<\omega$, $i\equiv j \bmod (k)$. I.e., the premise equations of χ hold for $\mathrm{con}(\alpha)$, $\alpha\in S$.

Suppose that χ holds in **CV**. Then we have $(x_0, x_1)\in p(\mathrm{con}(\alpha)\colon \alpha\in S)\subseteq \subseteq q(\mathrm{con}(\alpha)\colon \alpha\in S)$, from which we will derive our Mal'cev condition.

For a cardinal $\varkappa>0$ and a lattice term $d(\alpha\colon \alpha\in S)$ let $G(\varkappa\times d)$ denote the disjoint union of $\varkappa$ copies of $G(d)$. Let $E(\varkappa\times d)$ and $V(\varkappa\times d)$ stand for the edge set and for the vertex set of $G(\varkappa\times d)$, respectively. The graph $G(\varkappa\times d)$ has $2\varkappa$ endpoints. However, $\varkappa$ pairs of endpoints will have special roles, the two components c_0, c_1 of any such pair (c_0, c_1) are the left and right endpoints, respectively, of the same copy of $G(d)$. Let us also agree that $E(d_1)\mathring{\cup} E(\varkappa\times d_2)$ stands for the edge set of $G(d_1)\mathring{\cup} \mathring{\cup} G(\varkappa\times d_2)$, etc. The set of inner (not endpoint) vertices of $G(\varkappa\times d)$ is denoted by $W(\varkappa\times d)$.

Now from $(x_0, x_1)\in q(\mathrm{con}(\alpha)\colon \alpha\in S)$ we obtain that x_0 and x_1 can be connected by $G(q)$ in A, using a connecting map $\varphi\colon V(q)\to A$ and representing a colour α by $\mathrm{con}(\alpha)$. By finiteness, there exists an m, $2\leqq m<\omega$, such that $\varphi(V(q))\subseteq A_m$ and each colour $\alpha\in S$ is represented by $\mathrm{con}_m(\alpha)$.

Consider the map $\varrho\colon V(p)\mathring{\cup}\mathring{\bigcup}\{V(\varkappa_i\times q_i)\colon 1\leqq i<\omega\}\to A$ where $\varkappa_i=|N|\cdot \cdot|\mathrm{con}_{i-1}(p_i)|$, $\varrho(x)=(i, j, a, b, x)\in Y_i$ if x is an inner vertex of the $(j, (a, b))$-th copy of $G(q_i)$, $\varrho(x)=a$ ($=b$, resp.) if x is the left (right, resp.) endpoint in the $(j, (a, b))$-th copy of $G(q_i)$ for some i, j, and $\varrho(x)=x$ if $x\in V(p)$. Observe that the restriction of ϱ to $V(p)\mathring{\cup}\mathring{\bigcup}\{W(\varkappa_i\times q_i)\colon 1\leqq i<\omega\}$ is injective and maps onto X. Further, $\mathrm{con}_i(\alpha)$ is generated by $\{(\varrho(a), \varrho(b))\colon (a, \alpha, b)\in E(p)\mathring{\cup} E(\varkappa_1\times q_1)\mathring{\cup}\dots\mathring{\cup} E(\varkappa_i\times q_i)\}$.

Since $G(q)$ is finite, there are finite cardinals $u_1', u_2', \dots, u_{m-1}', u_m$ and a restriction ψ of ϱ such that for any $(f, \alpha, g)\in E(q)$ $(\varphi(f), \varphi(g))$ is in the congruence of A_m generated by $\{(\psi(x), \psi(y))\colon (x, \alpha, y)\in E(p)\mathring{\cup} E(u_1'\times q_1)\mathring{\cup}\dots\mathring{\cup} E(u_{m-1}'\times q_{m-1})\mathring{\cup} \mathring{\cup} E(u_m\times q_m)\}$ and for any $f\in V(q)$ $\varphi(f)$ is in the subalgebra generated by $\psi(V(p)\mathring{\cup} \mathring{\cup} W(u_1'\times q_1)\mathring{\cup}\dots\mathring{\cup} W(u_{m-1}'\times q_{m-1}\mathring{\cup} W(u_m\times q_m))$. If (a, b) is a pair of endpoints in $G(u_i'\times q_i)$, for $1\leqq i\leqq m$ (where $u_i'=u_m$ in case of $i=m$), then $(\psi(a), \psi(b))\in \in\mathrm{con}_{i-1}(p_i)$. Hence, by Proposition 3.1, there is a (multiple connecting) map η: $V(u_1'\times p_1)\mathring{\cup}\dots\mathring{\cup} V(u_{m-1}'\times p_{m-1})\mathring{\cup} V(u_m\times p_m)\to A_{m-1}$, representing the colour of any $(a, \alpha, b)\in E(u_i'\times p_i)$ by $\mathrm{con}_{i-1}(\alpha)$, such that ψ and η are "*compatible*". I.e., for any $1\leqq i\leqq m$ and $1\leqq j\leqq u_i'$, if (a, b) and (c, d) are the j-th pairs of endpoints in $G(u_i'\times p_i)$ and $G(u_i'\times q_i)$, respectively (i.e., they are taken from the j-th copies of $G(p_i)$ and $G(q_i)$), then $\eta(a)=\psi(c)$ and $\eta(b)=\psi(d)$.

Similarly, there are $u_{m-1}\geqq u_{m-1}'$, $u_1''\geqq u_1', \dots, u_{m-2}''\geqq u_{m-2}'$ and we can extend ψ and η such that, besides the previous properties, for any $(f, \alpha, g)\in E(u_m\times p_m)$ $(\eta(f), \eta(g))$ belongs to the congruence generated by $\{(\psi(a), \psi(b))\colon (a, \alpha, b)\in E(p)\mathring{\cup} \mathring{\cup} E(u_1''\times q_1)\mathring{\cup}\dots\mathring{\cup} E(u_{m-2}''\times q_{m-2})\mathring{\cup} E(u_{m-1}\times q_{m-1})\}$ and for any $f\in V(u_m\times p_m)$ $\eta(f)$ is in the subalgebra generated by $\psi(V(p)\mathring{\cup} W(u_1''\times q_1)\mathring{\cup}\dots\mathring{\cup} W(u_{m-2}''\times q_{m-2})\mathring{\cup} \mathring{\cup} W(u_{m-1}\times q_{m-1}))$. And so on, finally we obtain $u_1, u_2, \dots, u_m$ such that, besides the earlier properties, for $1\leqq i\leqq m$, $(f, \alpha, g)\in E(u_i\times p_i)$ and $h\in V(u_i\times p_i)$ the congruence generated by $\{(\psi(a), \psi(b))\colon (a, \alpha, b)\in E(p)\mathring{\cup} E(u_1\times q_1)\mathring{\cup}\dots\mathring{\cup} E(u_{i-1}\times q_{i-1})\}$ contains $(\eta(f), \eta(g))$ and $\eta(h)$ is in the subalgebra generated by $\psi(V(p)\mathring{\cup} W(u_1\times q_1)\mathring{\cup}\dots \dots\mathring{\cup} W(u_{i-1}\times q_{i-1}))$.

Without loss of these properties any u_i can be enlarged. Indeed, we can define ψ and η for the (u_i+1)-th copy of $V(q_i)$ and $V(p_i)$ as follows. Let $\eta(V(p_i))=\{x_0\}\subseteq \subseteq X_0$, $\psi(\text{left endpoint})=\psi(\text{right endpoint})=\psi(x_0)=x_0$, and $\psi(a)=(i, j, x_0, x_0, a)$ if $a\in W(q_i)$ and $j\in N$ is the smallest index not occurring before.

Therefore $u_1=u_2=\ldots=u_m=m$ can be assumed (to enlarge m it suffices to enlarge $u_{m+1}=0$) and we obtain:

$$\psi\colon\ V(p)\mathring{\cup}V(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}V(m\times q_m)\to A_m,$$

$$\eta\colon\ V(m\times p_1)\mathring{\cup}\ldots\mathring{\cup}V(m\times p_m)\to A_{m-1},$$

ψ restricted to $V(p)\mathring{\cup}W(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}W(m\times q_m)$ is injective, ψ and η are compatible, $\psi(x_0)=x_0$, $\psi(x_1)=x_1$, $\eta(V(m\times p_i))$ is a subset of the subalgebra of A_{i-1} generated by $\psi(V(p)\mathring{\cup}W(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}W(m\times q_{i-1}))$, $\varphi(V(q))$ is in the subalgebra of A_m generated by $\psi(V(p)\mathring{\cup}W(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}W(m\times q_m))$, for $(f, \alpha, g)\in E(m\times p_i)$ $(\eta(f), \eta(g))$ belongs to the congruence generated by $\{(\psi(a), \psi(b))\colon (a, \alpha, b)\in \in E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_{i-1})\}$, for $(f, \alpha, g)\in E(q)$ $(\varphi(f), \varphi(g))$ belongs to the congruence generated by $\{(\psi(a), \psi(b))\colon (a, \alpha, b)\in E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots \ldots\mathring{\cup}E(m\times q_m)\}$, $\psi(W(m\times q_i))\subseteq Y_i$, and ψ restricted to $V(p)=X_0$ is the identity map.

Now any element of $A=F_V(X)$ can be written in the form $f(x\colon x\in X)$ where f is a σ-term over X, i.e. it is an element of the absolute free σ-algebra $F_\sigma(X)$. Hence we can consider appropriate maps (not homomorphisms!) $\bar\varphi\colon V(q)\to F_\sigma(X)$, $\bar\psi\colon V(p)\mathring{\cup}V(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}V(m\times q_m)\to F_\sigma(X)$, and $\bar\eta\colon V(m\times p_1)\mathring{\cup}\ldots\mathring{\cup}V(m\times p_m)\to \to F_\sigma(X)$ such that $\varphi(a)=\bar\varphi(a)(x\colon x\in X)$, $\psi(b)=\bar\psi(b)(x\colon x\in X)$, and $\eta(c)= =\bar\eta(c)(x\colon x\in X)$ hold in A for any admissible choice of a, b, and c. The right hand sides of these equations will be abbreviated by $a(X)$, $b(X)$, and $c(X)$, respectively.

Let $X_0', X_1', \ldots, X_m'$ stand for $X_0=V(p)=\psi(V(p))$, $\psi(V(p)\mathring{\cup}W(m\times q_1))$, … …, $\psi(V(p)\mathring{\cup}W(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}W(m\times q_m))$, respectively. We have seen that, for $1\leqq i\leqq m$,

$$\text{(1)}\qquad \begin{cases}\bar\psi(V(m_i\times q_i)\setminus W(m_i\times q_i))\subseteq F_\sigma(X_{i-1}'),\\ \bar\eta(V(m\times p_i))\subseteq F_\sigma(X_{i-1}')\quad\text{and}\quad \bar\varphi(V(q))\subseteq F_\sigma(X_m').\end{cases}$$

Further, for $1\leqq i\leqq m$,

$$\text{(2)}\qquad \begin{cases}\text{if}\quad (f, \alpha, g)\in E(m\times p_i)\quad\text{then}\quad (f(X_{i-1}'), g(X_{i-1}'))\\ \text{belongs to the congruence of}\quad A_{i-1}\quad\text{generated by}\\ \{(a(X_{i-1}'), b(X_{i-1}'))\colon\ (a, \alpha, b)\in E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_{i-1})\}.\end{cases}$$

Similarly,

$$\text{(2')}\qquad \begin{cases}\text{if}\quad (f, \alpha, g)\in E(q)\quad\text{then}\quad (f(X_m'), g(X_m'))\quad\text{belongs to}\\ \text{the congruence of}\quad A_m\quad\text{generated by}\\ \{(a(X_m'), b(X_m'))\colon\ (a, \alpha, b)\in E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_m)\}.\end{cases}$$

Now we intend to introduce a condition which is similar to a strong Mal'cev (cf. Jónsson [9] or Taylor [12]) condition. Suppose we have two graphs G_1, G_2 with edge sets $E_i \subseteq V(G_i) \times S \times V(G_i)$, and $\pi\colon V(G_1) \to F_\sigma(Y)$, $\tau\colon V(G_2) \to F_\sigma(Y)$ are maps for some set Y. The condition to be introduced will be denoted by $T(E_1, \pi, E_2, \tau, Y)$ and we want, e.g., (2) to be equivalent to $T(E(p) \mathring{\cup} E(m\times q_1) \mathring{\cup} \dots \mathring{\cup} E(m\times q_{i-1}), \bar{\psi}, E(m\times p_i), \bar{\eta}, X'_{i-1})$. For $\alpha \in S$ let E_α stand for $\{(a, \alpha, b)\colon (a, \alpha, b) \in E_1\}$ and let, say, $E_\alpha = \{(a_i, \alpha, b_i)\colon 1 \leqq i \leqq u\}$. Let z be a variable symbol, $z \notin Y$, and consider an edge $(f, \alpha, g) \in E_2$. Then $T_{f,\alpha,g}$ is defined to be the following condition:

"There are terms $t_{ij}(z, Y) = t_{ij}(z; x\colon x \in Y)$, for $1 \leqq i \leqq v(u)$ and $0 \leqq j \leqq n-2$, such that the identities

$$\tau(f)(x\colon x\in Y) = t_{10}(\pi(a_{k(u,1)})(x\colon x\in Y);\ x\colon x\in Y)$$

$$\text{or shortly } \tau(f)(Y) = t_{10}(\pi(a_{k(u,1)})(Y), Y),$$

$$\tau(g)(Y) = t_{v(u),n-2}(\pi(b_{k(u,v(u))})(Y), Y),\ t_{i,n-2}(\pi(b_{k(u,i)})(Y), Y) =$$

$$= t_{i+1,0}(\pi(a_{k(u,i+1)})(Y), Y) \quad \text{for} \quad 1 \leqq i < v(u),$$

and

$$t_{ij}(\pi(b_{k(u,i)})(Y), Y) = t_{i,j+1}(\pi(a_{k(u,i)})(Y), Y) \quad \text{for} \quad 1 \leqq i \leqq v(u)$$

and $0 \leqq j < n-2$ hold."

Now let $T(E_1, \pi, E_2, \tau, Y)$ *be the conjunction of all* $T_{f,\alpha,g}$ where $\alpha \in S$ and $(f, \alpha, g) \in E_2$.

Since any translation of A_{i-1} is of the form $t(z, X_{i-1})$ for some σ-term t and identities holding for the free generators of A_{i-1} also hold in **V**, from Corollary 3.3 and (2) we can conclude that $T(E(p) \mathring{\cup} E(m\times q_1) \mathring{\cup} \dots \mathring{\cup} E(m\times q_{i-1}), \bar{\psi}, E(m\times p_i), \bar{\eta}, X_{i-1})$ holds in **V**. By identifying any variable $y \in X_{i-1} \setminus X'_{i-1}$ with $x_0 \in X_0$ we obtain that, for $1 \leqq i \leqq m$,

(3) $\quad T(E(p) \mathring{\cup} E(m\times q_1) \mathring{\cup} \dots \mathring{\cup} E(m\times q_{i-1}), \bar{\psi}, E(m\times p_i), \bar{\eta}, X'_{i-1})$ holds in **V**.

Similarly, from (2′) we obtain that

(3′) $\quad T(E(p) \mathring{\cup} E(m\times q_1) \mathring{\cup} \dots \mathring{\cup} E(m\times q_m), \bar{\psi}, E(q), \bar{\varphi}, X'_m)$ holds in **V**.

Conversely, from Corollary 3.3 it is easy to check that

(4) $\quad \begin{cases} \text{if } T(E_1, \pi, E_2, \tau, Y) \text{ holds in } \mathbf{V}, \text{ and } \zeta\colon F_\sigma(Y) \to A \\ \text{is a homomorphism then for any } (f, \alpha, g,) \in E_2 \\ (\zeta(\tau(f)), \zeta(\tau(g))) \text{ belongs to the congruence of } A \\ \text{generated by } \{(\zeta(\pi(a)), \zeta(\pi(b)))\colon (a, \alpha, b) \in E_1\}, \end{cases}$

Now we can define our Mal'cev condition. (We shall write X_i, $W(m\times q_i)$, $\varphi, \dots$ instead of $X'_i, \psi\,(W(m\times q_i)), \bar{\varphi}, \dots$).

A variety **V** *of type* σ *is said to satisfy the (strong Mal'cev) condition* $U(n, \chi, m)$ if and only if

"Considering the pairwise disjoint sets $V(p)$, $W(m\times q_1), \dots, W(m\times q_m)$ and denoting $V(p)$ and $V(p) \mathring{\cup} W(m\times q_1) \mathring{\cup} \dots \mathring{\cup} W(m\times q_i)$ by X_0 and X_i, respectively, there are maps

$\varphi\colon V(q) \to F_\sigma(X_m)$,

$\psi\colon V(p) \mathring{\cup} V(m\times q_1) \mathring{\cup} \dots \mathring{\cup} V(m\times q_m) \to F_\sigma(X_m)$, and

η: $V(m\times p_1)\mathring{\cup}\ldots\mathring{\cup}V(m\times p_m)\to F_\sigma(X_{m-1})$ such that the following five conditions hold:

(a) For $1\leqq i\leqq m$ $\psi(V(m\times q_i)\setminus W(m\times q_i))$ and $\eta(V(m\times p_i))$ are subsets of $F_\sigma(X_{i-1})$, and ψ restricted to X_m is the identity map;

(b) For $1\leqq i\leqq m$ $T(E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_{i-1}), \psi, E(m\times p_i), \eta, X_{i-1})$ holds in **V**;

(c) $T(E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_m), \psi, E(q), \varphi, X_m)$ holds in **V**;

(d) η and ψ are compatible in the sense that for any $1\leqq i, j\leqq m$, if (c_0, c_1) and (e_0, e_1) are the j-th pairs of endpoints in $G(m\times p_i)$ and $G(m\times q_i)$, respectively, then the "endpoint" identities $\eta(c_0)(x\colon x\in X_{i-1})=\psi(e_0)(x\colon x\in X_{i-1})$ and $\eta(c_1)(x\colon x\in X_{i-1})=\psi(e_1)(x\colon x\in X_{i-1})$ hold in **V**; and

(e) φ and ψ are compatible in the sense that for the endpoints x_0, x_1 and f_0, f_1 of $G(p)$ and $G(q)$, respectively, the identities $\varphi(f_i)(x\colon x\in X_m)=\psi(x_i)=x_i$, $i=0, 1$, hold in **V**."

We have shown that if χ holds in **CV** then there exists an m, $2\leqq m<\omega$, such that the (strong Mal'cev) condition $U(n, \chi, m)$ holds in **V**.

4. Main theorem and its sufficiency

THEOREM. *For any n-permutable variety* **V**, *a lattice Horn sentence χ is satisfied in the congruence lattices of members of* **V** *if and only if $U(n, \chi, m)$ holds in* **V** *for some m, $2\leqq m<\omega$.*

To prove the sufficiency assume $U(n, \chi, m)$. Let con (α), $\alpha\in S$, be congruences of an algebra $B\in\mathbf{V}$ such that $p_i(\text{con}(\alpha)\colon \alpha\in S)\leqq q_i(\text{con}(\alpha)\colon \alpha\in S)$ for all $i\geqq 1$. Let $(a_0, a_1)\in p(\text{con}(\alpha)\colon \alpha\in S)$. We have to show that $(a_0, a_1)\in q(\text{con}(\alpha)\colon \alpha\in S)$ as well.

For a map γ, γx will stand for $\gamma(x)$. Via induction we define homomorphisms ζ_i: $F_\sigma(X_i)\to B$ for $i=0, 1, \ldots, m$ such that ζ_i is an extension of ζ_{i-1}, $\zeta_i\psi$ and $\zeta_i\eta$ are (multiple) connecting maps* (i.e. if (a, α, b) and (c, α, d) are edges in the corresponding domains then $(\zeta_i\psi a, \zeta_i\psi b)\in\text{con}(\alpha)$ and $(\zeta_i\eta c, \zeta_i\eta d)\in\text{con}(\alpha)$), and for the endpoints x_0, x_1 of $V(p)$ $\zeta_i\psi x_0=a_0$ and $\zeta_i\psi x_1=a_1$ hold.

By Proposition 3.1, a_0 and a_1 can be connected by $G(p)$ in B. Choose a connecting map ϱ: $V(p)=X_0\to B$ and extend it to a homomorphism ζ_0: $F_\sigma(X_0)\to B$. Then $\zeta_0\psi=\varrho_0$ is clearly a connecting map. Since $T(E(p), \psi, E(m\times p_1), \eta, X_0)$ holds in **V**, (4) applies and we have that $\{(\zeta_0\psi a, \zeta_0\psi b)\colon (a, \alpha, b)\in E(p)\}$ generates a congruence, say α_0, which collapses $\zeta_0\eta f$ and $\zeta_0\eta g$ for any $(f, \alpha, g)\in E(m\times p_1)$. But $\alpha_0\subseteqq\alpha$ follows from $\zeta_0\psi$ being connecting, whence $\zeta_0\eta$ is (multiple) connecting, too.

Now assume that $\zeta_0\subseteqq\zeta_1\subseteqq\ldots\subseteqq\zeta_{i-1}$ $(0<i\leqq m)$ have been defined appropriately. Since $\zeta_{i-1}\eta$ is a (multiple) connecting map, Proposition 3.1 yields that $(\zeta_{i-1}\eta f, \zeta_{i-1}\eta g)\in p_i(\text{con}(\alpha)\colon \alpha\in S)\subseteqq q_i(\text{con}(\alpha)\colon \alpha\in S)$ holds for any pair (f, g) of endpoints in $G(m\times p_i)$. Therefore, by Proposition 3.1 again, there is a (multiple) connecting map ϱ_i: $V(m\times q_i)\to B$ such that ϱ_i and $\zeta_{i-1}\eta$ (restricted to $G(m\times p_i)$) are compatible. Now consider the map $X_i\to B$, $x\mapsto\zeta_{i-1}x$ if $x\in X_{i-1}$ and $x\mapsto\varrho_i x$ if $x\in W(m\times q_i)$,

* In this context ψ and η will be restricted to $V(p)\mathring{\cup}V(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}V(m\times q_i)$ and, if $i<m$, to $V(m\times p_1)\mathring{\cup}\ldots\mathring{\cup}V(m\times p_{i+1})$, respectively.

and extend it to a homomorphism $\zeta_i\colon F_\sigma(X_i)\to B$. Clearly ζ_i extends ζ_{i-1}. Hence to show that $\zeta_i\psi$ is a connecting map it suffices to check that ϱ_i and the restriction of $\zeta_i\psi$ to $V(m\times q_i)$ coincide. If $x\in W(m\times q_i)$ then $\zeta_i\psi x=\zeta_i x=\varrho_i x$ while, by (d), for the j-th pairs (c_0, c_1) and (e_0, e_1) of endpoints in $G(m\times p_i)$ and $G(m\times q_i)$ we have $\zeta_i\psi e_0=\zeta_{i-1}\psi e_0=\zeta_{i-1}\eta c_0=\varrho_i e_0$, and the same for e_1. Since $\zeta_i\eta$ and $\zeta_{i-1}\eta$ coincide on $V(m\times p_1)\mathring{\cup}\ldots\mathring{\cup}V(m\times p_i)$; it suffices to show that (if $i<m$) $\zeta_i\eta$ restricted to $V(m\times p_{i+1})$ is connecting. Since $T(E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_i), \psi, E(m\times p_{i+1}), \eta, X_i)$ holds in $\mathbf{V}$, (4) applies and we have that the congruence α_i generated by $\{(\zeta_i\psi a, \zeta_i\psi b)\colon (a, \alpha, b)\in E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_i)\}$ collapses $\zeta_i\eta f$ and $\zeta_i\eta g$ for any $(f, \alpha, g)\in E(m\times p_{i+1})$. But we have $\alpha_i\subseteq\alpha$ from $\zeta_i\psi$ being a connecting map, whence $\zeta_i\eta$ (restricted to $V(m\times p_{i+1})$) is a connecting map either.

Now put $\zeta=\zeta_m$ and denote α_m the congruence generated by $\{(\zeta\psi a, \zeta\psi b)\colon (a, \alpha, b)\in E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_m)\}$. From (c) and (4) we obtain that $\zeta\varphi\colon V(q)\to B$ is a connecting map which represents α by α_m. Since $\zeta\psi$ is also a connecting map, $\alpha_m\subseteq\operatorname{con}(\alpha)$. Hence $\zeta\varphi$ represents α by $\operatorname{con}(\alpha)$ as well. From Proposition 3.1 it follows that $(\zeta\varphi f_0, \zeta\varphi f_1)\in q((\operatorname{con}(\alpha)\colon \alpha\in S)$. Now by making use of (e) we have $(\zeta\varphi f_0, \zeta\varphi f_1)=(\zeta\psi x_0, \zeta\psi x_1)=(a_0, a_1)$, which completes the proof of the sufficiency.

5. Concluding remarks

The usual form of a strong Mal'cev condition is (roughly saying, for precise definition cf., e.g., Jónsson [9]) the following: "There are certain finitary terms $f_0(X), \ldots, f_s(X)$ for which certain prescribed identities hold". If any prescribed identity is of the form $f_i(\mu(x)\colon x\in X)=f_j(\nu(x)\colon x\in X)$ or $f_i(\mu(x)\colon x\in X)=y$ for some maps $\mu\colon X\to X$ and $\nu\colon X\to X$ and $y\in X$ then the strong Mal'cev condition will be called linear. While the Wille–Pixley algorithm [11, 13] yields linear Mal'cev conditions, ours does not. (However, the $U(n, \chi, m)$ in Theorem is a strong Mal'cev condition.)

Now we intend to point out that our algorithm cannot be improved so that $U(n, \chi, m)$ in Theorem be linear in general.

The reason for this is the following. For a ring R with 1 let $\mathbf{M}(R)$ denote the variety of unitary left R-modules. Hutchinson [7] has produced two rings, R_1 and R_2, such that though $\mathbf{CM}(R_1)$ and $\mathbf{CM}(R_2)$ satisfy the same lattice identities, they satisfy different sets of Horn sentences. But, as it is implicit in [8], if $\mathbf{CM}(R_1)$ and $\mathbf{CM}(R_2)$ satisfy the same lattice identities then $\mathbf{M}(R_1)$ and $\mathbf{M}(R_2)$ satisfy the same linear strong Mal'cev conditions.

The condition $(\exists m)\,(U(n, \chi, m))$ in our Theorem does not seem to be a Mal'cev condition yet. To make it a real Mal'cev condition we have to prescribe appropriate linear orders of the occurring edge sets. Let the orderings of $E(p)$, $E(q)$, $E(p_i)$, and $E(q_i)$, $1\leqq i\leqq k$, be fixed arbitrarily. The linear orders of $E^m=E(p)\mathring{\cup}E(m\times q_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times q_m)$ and ${}^mE=E(m\times p_1)\mathring{\cup}\ldots\mathring{\cup}E(m\times p_m)$ will be defined via induction. Since $E^0=E(p)$ and ${}^0E=\emptyset$, their orderings are unique. Having the linear orders of E^m and ${}^mE$ defined put $E^{m+1}=E^m+E(q_1)+\ldots+E(q_m)+E(q_{m+1})+\ldots+E(q_{m+1})$ and ${}^{m+1}E={}^mE+E(p_1)+\ldots+E(p_m)+E(p_{m+1})+\ldots+E(p_{m+1})$. Here $+$ stands for the ordinal sum, whence, e.g., any element of E^m precedes all the elements of $E^{m+1}\setminus E^m$. Further, let us also agree that if E_1 has a fixed linear ordering in the definition of $T(E_1, \pi, E_2, \tau, Y)$ then $(a_i, \alpha, b_i)\in E_\alpha$ precedes $(a_{i+1}, \alpha, b_{i+1})$ for $i=1, 2, \ldots, u-1$.

Now $(\exists m)\,(U(n,\chi,m))$ has become a Mal'cev condition. To show that $U(n,\chi,m)$ implies $U(n,\chi,m+1)$ (in any variety and for all $m\geqq 2$) indeed, we can repeat some earlier terms, consider some projections, and introduce irrelevant variables. The long technical details will be omitted.

Added in proof (April 5, 1984). Recently it has appeared that Proposition 1.1 is a special case of Herrmann and Poguntke's [14, Thm. 6]. Yet, our approach is different from [14].

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