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JORDAN–HÖLDER CONDITION WITH SUBSEMILATTICES OF COALITION LATTICES

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Dedicated to Professor László Leindler on his 70th birthday

Abstract. Given a finite partially ordered set P, for subsets or, in other words, *coalitions* X, Y of P let $X \leq Y$ mean that there exists an injection $\varphi : X \to Y$ such that $x \leq \varphi(x)$ for all $x \in X$. This definition turns the set $\mathcal{L}(P)$ of all subsets of P into a partially ordered set. When no two comparable elements of P has an upper bound in P then $\mathcal{L}(P)$ is a lattice, the so-called *coalition lattice* of P. Using the structure theorem introduced in [2], coalition lattices will be shown to satisfy the Jordan-Hölder condition; not only in the classical sense but also in a stronger form which is related to certain subsemilattices. In other words, certain subsemilattices, which are coalition lattices with respect to their own ordering, are neatly positioned in the original lattice. An overview of former results on coalition lattices is also given.

1. Introduction and overview

Given a partially ordered set P, *always assumed to be finite*, the subsets of P will be called *coalitions* of P. Let $\mathcal{L}(P)$ denote the set of coalitions, i.e. subsets, of P. For $X, Y \in \mathcal{L}(P)$, a map $\varphi : X \to Y$ is called an *extensive map* if φ is injective and $x \leq \varphi(x)$ for all $x \in X$. Let $X \leq Y$ mean that there exists an extensive map $X \to Y$; this definition turns $\mathcal{L}(P)$ into a partially ordered set. This concept, with roots in game theory and human decision making, was introduced in [1] with a detailed motivation. When $\mathcal{L}(P)$ happens to be a lattice then it is called a *coalition lattice*, the coalition lattice of P. For a general reference on lattices the reader can resort to Grätzer [5].

If incomparable elements of P have no common upper bounds, i.e., if for all $a, b, c \in P$, $a \leq c$ and $b \leq c$ imply $a \leq b$ or $b \leq a$, then P is called a *forest*. The connected components of a forest are called the *trees* of P. Clearly, trees

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are the maximal subsets having a least element. It is relatively easy to show that if $\mathcal{L}(P)$ is a lattice then P is a forest. The converse is also true.

Theorem 1. [1] *For a finite poset P, $\mathcal{L}(P)$ is a lattice if and only if P is a forest.*

Related lattices, like lattices of congruences or subalgebras, are lattices usually by a trivial reason: one of the lattice operations is easy to describe. However, this is far from being true for coalition lattices. There are five different proofs of Theorem 1 so far, and four of them describe one of the lattice operations. Since most of these proofs reveal something interesting on coalition lattices, we give a short account on them. We will disregard from the fact that some of these proofs are for a more general statement where P is a quasi ordered set.

In the first proof, cf. [1], for a forest P and for $A_1, \ldots, A_k \in \mathcal{L}(P)$ let $M := \{b_1 \wedge \ldots \wedge b_k : b_1 \in A_1, \ldots, b_k \in A_k$, and the infimum $b_1 \wedge \ldots \wedge b_k$ exists in $P\}$. If M is empty (in particular when one of the A_i is empty) then $\bigwedge_{i=1}^k A_i = \emptyset$. If M is nonempty then choose a maximal element $c = a_1 \wedge \ldots \wedge a_k$ in M where the a_i belong to A_i such that, for every i, $c \in A_i \Longrightarrow c = a_i$. Let $A_i' := A_i \setminus \{a_i\}$ for $i = 1, \ldots, k$, $P' := P \setminus \{c\}$, and put $C' := \bigwedge_{i=1}^k A_i'$ in $\mathcal{L}(P')$. Then $\bigwedge_{i=1}^k A_i = C' \cup \{c\}$ in $\mathcal{L}(P)$.

Now let P_1 be the set of maximal elements of P, and for $i \geq 2$ let P_i be the set of maximal elements of $P \setminus (P_1 \cup \cdots \cup P_{i-1})$. The second proof, cf. Theorem 1 in [1], defines the join $C = \bigvee_{i=1}^k A_i$ in $\mathcal{L}(P)$ by inductively constructing $C \cap (P_1 \cup \cdots \cup P_i)$.

Let v be a *maximal element* of P, and let $\breve{P} := P \setminus \{v\}$. To make the previous description of $C = \bigvee_{i=1}^k A_i$ easier, the third proof, cf. Corollary 1 in [3], reduces the problem to $\mathcal{L}(\breve{P})$. For $X \in \mathcal{L}(P)$, if $\{a : a \in X$ and $a \leq v\}$ is nonempty then it has a unique largest element c_X, and we put $\breve{X} := X \setminus \{c_X\}$. Put $\breve{X} := X$ if $\{a : a \in X$ and $a \leq v\}$ is empty. Let $\breve{C}$ be the join of the $\breve{A}_1, \ldots, \breve{A}_k$ in $\mathcal{L}(\breve{P})$. If $A_1, \ldots, A_k \in \mathcal{L}(\breve{P})$ then $C = \breve{C}$, otherwise $C = \breve{C} \cup \{v\}$.

The fourth proof, given by Davidson and Grätzer [4], starts with a tree P, rather than a forest, and, contrary to the previous approach, now let $m = 0$, *the least element* of P. With the notations $\breve{P} := P \setminus \{0\}$ and $\breve{X} := X \setminus \{0\}$ for $X \in \mathcal{L}(P)$, if $A, B \in \mathcal{L}(P)$ then let $C := \breve{A} \vee \breve{B}$ in $\mathcal{L}(\breve{P})$. Then $A \vee B$ in $\mathcal{L}(P)$ is either C or $C \cup \{0\}$; it is C iff $A \leq C$ and $B \leq C$ in $\mathcal{L}(P)$. This settles the case when P is a tree, and the rest follows from the following easy statement.

Proposition 1. [1] *Let $T_1, \ldots, T_n$ be the connected components of a finite poset P. Then $\mathcal{L}(P) \cong \prod_{i=1}^n \mathcal{L}(T_i)$.*

Before giving any information on the fifth proof, now some other properties of coalition lattices will be recalled.

Proposition 2. [1] *For any finite poset P, $\mathcal{L}(P) \to \mathcal{L}(P)$, $X \mapsto P \setminus X$ is a dual automorphism.*

Notice that once we have a method to calculate one of the lattice operations in $\mathcal{L}(P)$ then, in virtue of the above statement, we can use the de Morgan laws to calculate the other one. The various ways to compute the lattice operations make it relatively easy to derive the last part of the following statement.

Proposition 3. *Let P be a finite forest.*

(A) [1] *$\mathcal{L}(P)$ is distributive iff it is modular iff every tree component of P is a chain.*

(B) [2] *M_3, the five element nondistributive modular lattice, cannot be embedded in $\mathcal{L}(P)$.*

(C) [3] *If Q is another forest with $\mathcal{L}(P) \cong \mathcal{L}(Q)$ then $P \cong Q$.*

(D) [3] *$\{X \in \mathcal{L}(P) : P \setminus X \le X\}$, i.e. the set of the so-called* winning coalitions*, is a dual ideal of $\mathcal{L}(P)$.*

(E) [3] *$\mathcal{L}(P)$ satisfies the Jordan–Hölder chain condition, i.e., any two maximal chains of $\mathcal{L}(P)$ have the same length.*

(F) [2] *For $A_1, \ldots, A_n \in \mathcal{L}(P)$ we have $\bigcap_{i=1}^n A_i \subseteq \bigwedge_{i=1}^n A_i$ and $\bigvee_{i=1}^n A_i \subseteq \bigcup_{i=1}^n A_i$.*

The former proof for (E) does not remember to Section 2 of the present paper, where a stronger statement will be proved. In connection with (B) we notice that, except for the Jordan–Hölder chain condition, the most important lattice properties seem to fail in $\mathcal{L}(P)$ in general. This will be demonstrated by Figure 1, which is the coalition lattice of $T = \{a, b, c, u, v\}$ where a is the least elements, b, c, v are the maximal elements, and $a \prec b$, $a \prec u$, $u \prec c$ and $u \prec v$ is the list of all covering pairs. Notice that $\{a_1, \ldots, a_k\} \in \mathcal{L}(T)$ is denoted by $a_1 \ldots a_k$ in the figure.

Kira Adaricheva rased the question what nice lattice properites, other than (E) above, hold in coalition lattices. Unfortunately we have some negative results only. For example, coalition lattices are not join (equivalently, meet) *semidistributive* in general. Indeed, for $X = \{c, v\}$, $Y = \{a, c, u\}$ and $Z = \{a, u, v\}$ in $\mathcal{L}(T)$ we have $X \vee Y = X \vee Z = \{a, c, v\}$ but $X \vee (Y \wedge Z) = X$. (Notice that the same holds in $\mathcal{L}(T \setminus \{b\})$.) Since $\{a, b, u\} \wedge \{b, c\} = \{b, u\} = \{a, b\} \vee \{a, u\}$ in $\mathcal{L}(T)$, and also in $\mathcal{L}(T \setminus \{v\})$, coalition lattices fail to satisfy the *Whitman condition* in general. Since $\{a, b, u\} \prec \{a, b, c\}$ but $\{a, b, u\} \vee \{b, v\} \not\prec \{a, b, c\} \vee \{b, v\}$, $\mathcal{L}(T)$ is *not semimodular*. The following *problem*, mentioned already in [2], *is still open*: do coalition lattices satisfy a nontrival lattice identity?

Now we recall a construction from [2], which will be used heavily in the next section. Since this construction produces a lattice L from given lattices

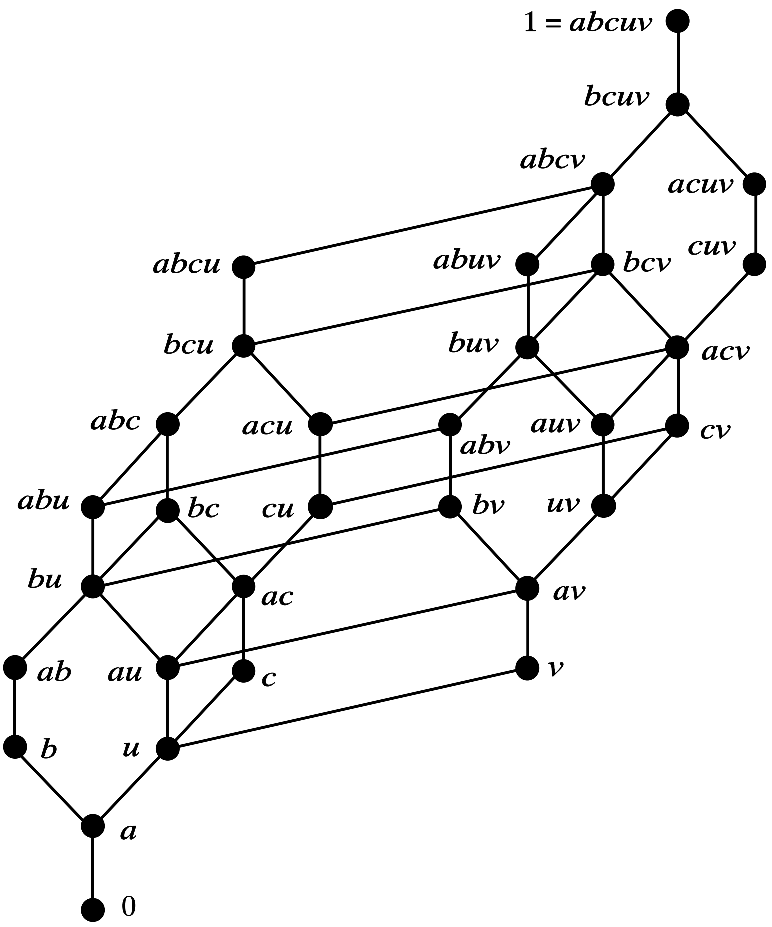

FIGURE 1

L_1 and L_2 such that $|L| = |L_1| + |L_2|$, we will call L as a *sum* of L_1 and L_2. For definition, let L_i be a complete lattice with bounds $0_i, 1_i$, $i = 1, 2$. Let S_i be a nonempty subset of L_i such that S_1 is closed with respect to arbitrary meets and S_2 is closed with respect to arbitrary joins. In particular, $1_1 \in S_1$ and $0_2 \in S_2$. In short, we say that S_1 is a $\bigwedge$-subsemilattice of L_1 and S_2 is a $\bigvee$-subsemilattice of L_2. Notice that the S_i are necessarily complete lattices under the ordering inherited from L_i but they need not be sublattices in general. Let $\psi : S_1 \to S_2$ be an order isomorphism (or, equivalently, a lattice isomorphism). We define the sum L of L_1 and L_2, in notation $L = L_1 +_\psi L_2$ as follows. (Notice that ψ determines its domain, S_1, and its range, S_2. Hence, in contrast to Proposition 1 of [2], S_1 and S_2 are not included in the notation $L_1 +_\psi L_2$.) Let L be the disjoint union of L_1 and L_2. For $x, y \in L$ let $x \leq y$

mean that either $x \leq y$ in some of the L_i or $x \in L_1$, $y \in L_2$ and $x \leq z$ and $\psi(z) \leq y$ for some $z \in S_1$. Pictorially, in the finite case this means that we add the $z \prec \psi(z)$ edges, $z \in S_1$, to the union of the Hasse diagrams of L_1 and L_2. Notice that the particular case $|S_1| = |S_2| = 1$ gives the so-called ordinal sum of L_1 and L_2.

Proposition 4. [2] *$L = L_1 +_\psi L_2$ is a complete lattice, in which L_1 is a prime ideal and L_2 is a prime filter.*

The importance of $L_1 +_\psi L_2$ in our investigations comes from the following statement.

Theorem 2. [2] *Let v be a maximal element of a finite forest P, let $u \in P$ and suppose that v covers u in P. Put $L_1 := \{X \in \mathcal{L}(P) : v \notin X\}$, $L_2 := \{X \in \mathcal{L}(P) : v \in X\}$, $S_1 := \{X \in L_1 : u \in X\}$, $S_2 := \{X \in L_2 : u \notin X\}$, and $\psi : S_1 \to S_2$, $X \mapsto (X \setminus \{u\}) \cup \{v\}$. Then $L_1 = \mathcal{L}(P \setminus \{v\}) \cong L_2$, all conditions in connection with Proposition 4 are fulfilled, and $\mathcal{L}(P) = L_1 +_\psi L_2$.*

Notice that *it is not known* if there is a nontrivial lattice variety $\mathcal{V}$ such that $L_1 +_\psi L_2 \in \mathcal{V}$ for all $L_1, L_2 \in \mathcal{V}$. In virtue of the above theorem, this problem is related to the problem mentioned before the definition of $L_1 +_\psi L_2$.

Now the fifth proof of Theorem 1, which is implicit in [2], is a trivial induction using Theorem 2 and the fact that $\mathcal{L}(P)$ is a lattice when P is an antichain.

2. More about coalition lattices

Lemma 1. *If $X \leq Y$ in $\mathcal{L}(P)$ then there is an extensive map $X \to Y$ which acts identically on $X \cap Y$.*

Proof. Choose an extensive map $\varphi : X \to Y$ such that $k := |\{z \in X \cap Y : \varphi(z) = z\}|$ is maximal. By way of contradiction, suppose $k < |X \cap Y|$. Then $u < \varphi(u)$ for some $u \in X \cap Y$. If $u \notin \varphi(X)$ then define $\psi : X \to Y$, $u \mapsto u$ and $x \mapsto \varphi(x)$ for $x \neq u$. If $u = \varphi(w)$ for some $w \in X$ then define $\psi : X \to Y$, $u \mapsto u$, $w \mapsto \varphi(u)$ and $x \mapsto \varphi(x)$ for $x \notin \{u, w\}$. In both cases, ψ is an extensive map with $|\{z \in X \cap Y : \varphi(z) = z\}| > k$, a contradiction. □

For an element w of a finite forest P let $J(w) := \{X \in \mathcal{L}(P) : w \notin X\}$ and $M(w) := \{X \in \mathcal{L}(P) : w \in X\}$.

Lemma 2. *$J(w)$ is a $\bigvee$-subsemilattice and $M(w)$ is a $\bigwedge$-subsemilattice of $\mathcal{L}(P)$. Moreover, $J(w) = \mathcal{L}(P \setminus \{w\}) \cong M(w)$, and $\alpha_w : J(w) \to M(w)$, $X \mapsto X \cup \{w\}$ is an isomorphism.*

Proof. The first part comes from Proposition 3 (F) while the second one follows from Lemma 1. □

Now let $\mathcal{C} = \{A = C_0 \prec C_1 \prec C_2 \prec \cdots \prec C_n = B\}$ be a maximal chain in the interval $[A, B] \subseteq \mathcal{L}(P)$. The *length* of $\mathcal{C}$, denoted by $\ell(\mathcal{C})$, is n. The *w-length* of $\mathcal{C}$ is

$$\ell_w(\mathcal{C}) := |\{i : \text{ either } C_i, C_{i+1} \in J(w) \text{ or } C_i, C_{i+1} \in M(w)\}|.$$

We also define

$$\ell_w^{JM}(\mathcal{C}) := |\{i : C_i \in J(w),\ \ C_{i+1} \in M(w)\}|,$$
$$\ell_w^{MJ}(\mathcal{C}) := |\{i : C_i \in M(w),\ \ C_{i+1} \in J(w)\}|.$$

We will use $\widehat{\ell}_w$ to denote any of ℓ, ℓ_w, ℓ_w^{JM} and ℓ_w^{MJ}. If $\widehat{\ell}_w(\mathcal{C})$ is the same for all maximal chains in $[A, B]$, and only if, then this common value will be denoted by $\widehat{\ell}_w[A, B]$. If $\widehat{\ell}_w[0, 1]$ makes sense then so does $\widehat{\ell}_w[A, B]$ for any $A \le B \in \mathcal{L}(P)$. Based on the notations of this section, now we can formulate our main result.

Theorem 3. *Let P be a finite forest, and let $w, z \in P$. Then for any $\widehat{\ell}_w \in \{\ell, \ell_w, \ell_w^{JM}, \ell_w^{MJ}\}$ we have*

(1) for any two maximal chains $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{L}(P)$, $\widehat{\ell}_w(\mathcal{C}) = \widehat{\ell}_w(\mathcal{D})$;

(2) $\widehat{\ell}_w[A, \alpha_z(A)] = \widehat{\ell}_w[B, \alpha_z(B)]$ for any $A, B \in J(z)$.

Besides generalizing Proposition 3 (E), this theorem gives a lot of information how certain subsemilattices, which are coalition lattices of a smaller size, are positioned in $\mathcal{L}(P)$. In connection with (2) we note that, for $A, B \in J(z)$, $|[A, \alpha_z(A)]| = |[B, \alpha_z(B)]|$ is not always true; this is exemplified by Figure 1 with $z = \{c\}$, $A = \emptyset$ and $B = \{a\}$. Notice that the theorem would fail with the obviously defined ℓ_w^{JJ} and ℓ_w^{MM}; for example, we have two maximal chains, say $\mathcal{C}$ and $\mathcal{D}$, in the interval $[\{a\}, \{b, u\}]$ in Figure 1 such that $\ell_{\{b\}}^{MM}(\mathcal{C}) = 2$, $\ell_{\{b\}}^{MM}(\mathcal{D}) = 0$, $\ell_{\{b\}}^{JJ}(\mathcal{C}) = 0$ and $\ell_{\{b\}}^{JJ}(\mathcal{D}) = 2$.

Proof. Clearly, for a maximal chain $\mathcal{C}$ in $[A, B]$, $\ell_w^{JM}(\mathcal{C}) + \ell_w^{MJ}(\mathcal{C}) = \ell(\mathcal{C}) - \ell_w(\mathcal{C})$ and $\ell_w^{JM}(\mathcal{C}) - \ell_w^{MJ}(\mathcal{C}) \in \{-1, 0, 1\}$ depending on $|\{A\} \cap J(w)|$ and $|\{A, B\} \cap J(w)|$. Therefore ℓ and ℓ_w determine ℓ_w^{JM} and ℓ_w^{MJ}, and it suffices to prove (1) and even (2) only for $\widehat{\ell}_w \in \{\ell, \ell_w\}$. From now on let us agree that each constituent of the proof (i.e., condition, formula, assertion, etc.) which includes $\widehat{\ell}_w$ is understood as the conjunction of two constituents, one with ℓ and the other with ℓ_w. In other words, $\widehat{\ell}_w$ will always be understood as $\forall \widehat{\ell}_w \in \{\ell, \ell_w\}$ even without quantifying it explicitly. For example, (1) will mean $\ell(\mathcal{C}) = \ell(\mathcal{D})$ and $\ell_w(\mathcal{C}) = \ell_w(\mathcal{D})$ for any two maximal chains $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{L}(P)$.

Now the proof of Theorem 3 goes via induction on $|P|$.

Case 1: P is an antichain. Now $\mathcal{L}(P)$ is a Boolean lattice, the usual power set lattice, and we clearly have $\ell(\mathcal{C}) = |P|$ and $\ell_w(\mathcal{C}) = |P| - 1$ for any maximal chain $\mathcal{C}$. Further, $\ell[A, \alpha_z(A)] = 1$ and $\ell_w[A, \alpha_z(A)] = |\{w, z\}| - 1$ for $A \in J(z)$.

Case 2: P is not an antichain and there are $u, v \in P$ such that $u \prec v$, v is a maximal element of P, and $v \notin \{w, z\}$. Let us observe that (1) and (2) imply

(3) if $A, B \in J(z)$ with $A \le B$ then $\widehat{\ell}_w[\alpha_z(A), \alpha_z(B)] = \widehat{\ell}_w[A, B]$.

Indeed, (1) allows us to apply $\widehat{\ell}_w$ to intervals. Since α_z is monotone, we have $A \le \alpha_z(A) \le \alpha_z(B)$. From (1) we obtain $\widehat{\ell}_w[A, \alpha_z(A)] + \widehat{\ell}_w[\alpha_z(A), \alpha_z(B)] = \widehat{\ell}_w[A, \alpha_z(B)] = \widehat{\ell}_w[A, B] + \widehat{\ell}_w[B, \alpha_z(B)]$, whence (3) follows by (2).

Now recall the notations from Theorem 2. In particular, $L_1 = J(v)$, $L_2 = M(v)$, $\alpha_v : L_1 \to L_2$, $X \mapsto X \cup \{v\}$ is an isomorphism and $L_1 = \mathcal{L}(P \setminus \{v\})$. By the induction hypothesis, (1), (2), and therefore (3) as well, hold in L_1. Using the isomorphism α_v and $v \notin \{w, z\}$ we see that (1), (2) and (3) hold in L_2, too. We will use the previous notations $J(w)$, α_z, etc. for L_i with a subscript: $J_i(w)$, α_{iz}, etc. For $A \in J_1(z)$ we have $\widehat{\ell}_w[A, \alpha_{1z}(A)] = \widehat{\ell}_w[\alpha_v(A), \alpha_v(\alpha_{1z}(A))] = \widehat{\ell}_w[\alpha_v(A), \alpha_{2z}(\alpha_v(A))]$, for α_v is an isomorphism and $v \notin \{w, z\}$. Since $\alpha_z = \alpha_{1z} \cup \alpha_{2,z}$ and (2) holds in L_1 and L_2, we conclude that (2) holds in $\mathcal{L}(P) = L_1 +_\psi L_2$ as well.

To show (1) for $\mathcal{L}(P)$, let $\mathcal{D}$ be a maximal chain in $\mathcal{L}(P)$ including $F = P \setminus \{v\}$, the top of L_1, and therefore $\psi(F) = P \setminus \{u\}$, too. Let $\mathcal{C}$ be an arbitrary maximal chain in $\mathcal{L}(P)$. Since L_1 is an ideal and L_2 is a filter of $\mathcal{L}(P)$, we conclude from the description of $L_1 +_\psi L_2$ that there is a unique $A \in \mathcal{C} \cap L_1$ such that $\psi(A) \in \mathcal{C} \cap L_2$, $A \prec \psi(A)$, and for each $X \in \mathcal{C}$ either $X \le A$ and $X \in L_1$ or $X \ge \psi(A)$ and $X \in L_2$.

Since $X \prec \psi(X)$ for any $X \in S_1 = M_1(u)$, $\widehat{\ell}_w[A, \psi(A)] = \widehat{\ell}_w[F, \psi(F)]$. Further, using the isomorphism α_v, $v \notin \{w, z\}$, and (3) in L_2 with u instead of z, we conclude that for any $X, Y \in S_1$ with $X \le Y$ we have $\widehat{\ell}_w[X, Y] = \widehat{\ell}_w[\alpha_v(X), \alpha_v(Y)] = \widehat{\ell}_w[\alpha_{2u}^{-1}(\alpha_v(X)), \alpha_{2u}^{-1}(\alpha_v(Y))] = \widehat{\ell}_w[\psi(X), \psi(Y)]$. Hence, using (1), (2) and (3) within L_1 or L_2, we can compute:

$$\begin{gathered}
\widehat{\ell}_w(\mathcal{C}) = \widehat{\ell}_w[0, A] + \widehat{\ell}_w[A, \psi(A)] + \widehat{\ell}_w[\psi(A), 1] = \\
\widehat{\ell}_w[0, A] + \widehat{\ell}_w[F, \psi(F)] + \widehat{\ell}_w[\psi(A), \psi(F)] + \widehat{\ell}_w[\psi(F), 1] = \\
\widehat{\ell}_w[0, A] + \widehat{\ell}_w[A, F] + \widehat{\ell}_w[F, \psi(F)] + \widehat{\ell}_w[\psi(F), 1] = \\
\widehat{\ell}_w[0, F] + \widehat{\ell}_w[F, \psi(F)] + \widehat{\ell}_w[\psi(F), 1] = \widehat{\ell}_w(\mathcal{D}),
\end{gathered}$$

proving (1) for $\mathcal{L}(P)$.

Case 3: P is not an antichain, $u \prec v$ in P, v is a maximal element of P, and $v \in \{w, z\}$. Let $(i)_\ell$ denote the condition (i) with ℓ instead of $\widehat{\ell}_w$. E.g., $(1)_\ell$ is the Jordan–Hölder condition.

Since Case 2 is applicable with another w, $(1)_\ell$ holds in $\mathcal{L}(P)$. If $\mathcal{C}$ is a maximal chain in $\mathcal{L}(P)$ then the description of $L_1 +_\psi L_2$ gives $\ell_v(\mathcal{C}) = \ell(\mathcal{C}) - 1$. This and Case 2 give that (1) holds in $\mathcal{L}(P)$.

Now let $z = v$ and $A \in L_1 = J(z)$. Then $\ell[\emptyset, A] = \ell[\alpha_v(\emptyset), \alpha_v(A)]$, since α_v is an isomorphism. If $w \neq v$ then α_v maps $J_1(w)$ resp. $M_1(w)$ to $J_2(w)$ resp. $M_2(w)$, whence $\ell_w[\emptyset, A] = \ell_w[\alpha_v(\emptyset), \alpha_v(A)]$. If $w = v$ then $\ell_w[\emptyset, A] = \ell[\emptyset, A]$ and $\ell_w[\alpha_v(\emptyset), \alpha_v(A)] = \ell[\alpha_v(\emptyset), \alpha_v(A)]$. So $\widehat{\ell}_w[\emptyset, A] = \widehat{\ell}_w[\alpha_v(\emptyset), \alpha_v(A)]$ for all $w \in P$. On the other hand, (1) gives $\widehat{\ell}_w[\emptyset, \alpha_v(\emptyset)] + \widehat{\ell}_w[\alpha_v(\emptyset), \alpha_v(A)] = \widehat{\ell}_w[\emptyset, A] + \widehat{\ell}_w[A, \alpha_v(A)]$, whence (2) holds in $\mathcal{L}(P)$.

Finally, let $z \neq v$. Then $w = v$. Case 2 yields that $(2)_\ell$ holds in $\mathcal{L}(P)$. Notice that both L_1 and L_2 are closed with respect to α_z, and ℓ_v restricted to L_i, $i = 1, 2$, coincides with ℓ. Hence (2) holds in $\mathcal{L}(P)$. $\square$

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