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Horn sentences in related lattices

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CHAPTER I

INTRODUCTION

By a *Horn sentence* or, in other words, a *conditional lattice identity* we mean a universally quantified

$$(p_1 \le q_1)\ \&\ \dots\ \&\ (p_t \le q_t) \Longrightarrow p \le q \tag{1}$$

lattice sentence where $t \ge 0$ and $p_1, q_1, \dots, p_t, q_t, p, q$ are lattice terms. Note that replacing "$\le$" by "$=$" in (1) at all occurrences leads to an equivalent notion modulo lattice theory. The most important case is $t = 0$, when we speak of a *lattice identity*.

The importance and, in part, the beauty of lattice theory is in the fact that many algebraic and other mathematical structures are accompanied by so-called *related lattices*. These related lattices consist of certain objects connected with the escorted structures. In our case the related lattices will consist of congruences, certain congruences, quasiorders, submodules and coalitions. Among these related lattices the congruence lattices are the most important ones.

The congruence lattice of an algebra $A = \langle A; F\rangle$ will be denoted by $\mathrm{Con}(A)$. It is often reasonable to consider the congruence lattices of a whole class of algebras rather than just a single algebra. For a class $\mathcal{V}$ of similar algebras let $\mathrm{Con}(\mathcal{V})$ denote the class $\{\mathrm{Con}(A)\colon A \in \mathcal{V}\}$ of lattices, and let $\underline{\mathbf{Con}}(\mathcal{V})$ stand for the lattice variety generated by $\mathrm{Con}(\mathcal{V})$. If $\mathcal{V}$ is a variety (of an arbitrary type) then $\underline{\mathbf{Con}}(\mathcal{V})$ is called the *congruence variety* of $\mathcal{V}$. This notion came to existence in the early seventies. Its birth is due to the fact that not every lattice variety is a congruence variety, cf. Nation [Na1][1], and some important properties of a variety $\mathcal{V}$ depend only on the congruence variety of $\mathcal{V}$.

For example, by Baker [Ba1], the distributivity of $\underline{\mathbf{Con}}(\mathcal{V})$ is sufficient to ensure that each finite algebra in the variety $\mathcal{V}$ have a finite base for its equational theory. Another influence of the distributivity of $\underline{\mathbf{Con}}(\mathcal{V})$, in other words the *congruence distributivity*, on the original variety $\mathcal{V}$ is captured by a famous result of Jónsson [Jo2] which asserts that under this assumption any subdirectly irreducible algebra in a subvariety of $\mathcal{V}$ generated by some $\mathcal{U} \subseteq \mathcal{V}$ is already in $HSP_u\mathcal{U}$. The appearance of commutator theory gave a similar importance to congruence modularity.

A considerable part of the present work is devoted to congruence varieties and their generalizations.

In **Chapter II** we deal with lattice identities, as special Horn sentences, in congruence varieties. We define a consequence relation among lattice identities

[1]The references are arranged in lexicographic order by the key-strings given in brackets, like Na1 in the above case. Lower-case letters precedes the corresponding upper-case ones, e.g. a<A<b<B, etc.

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as follows. Let Σ be a set of lattice identities and let μ be an additional lattice identity. We say that Σ *implies* μ *in congruence varieties*, if for any variety $\mathcal{V}$ of algebras such that every member of Σ holds in the congruence variety $\underline{\mathbf{Con}}(\mathcal{V})$ the identity μ also holds in $\underline{\mathbf{Con}}(\mathcal{V})$. Let $\Sigma \models_c \mu$ denote the fact that Σ implies μ in congruence varieties; for a singleton $\Sigma = \{\sigma\}$ we will write $\sigma \models_c \mu$ rather than $\{\sigma\} \models_c \mu$. The notation $\Sigma \models \mu$ (and $\sigma \models \mu$) will stand for the usual consequence relation among lattice identities. I.e., $\Sigma \models \mu$ means that for any lattice L if each member of Σ holds in L then so does μ.

The aforementioned result of Nation [Na1] says that $\lambda \models_c \mu$ may hold without $\lambda \models \nu$. Thus, $\models_c$ is quite different from $\models$. Concerning the consequence relation $\models_c$ we are interested in two problems: compactness and decidability. In both cases the problem will be settled only for a particular case, where this relation will be shown to be compact and decidable. We say that $\models_c$ is *compact* at a lattice identity μ if for any set Σ of lattice identities $\Sigma \models_c \mu$ implies the existence of a finite subset Δ of Σ such that $\Delta \models_c \mu$. If, for a given μ, there is an algorithm which, for every finite set Σ of lattice identities, decides whether $\Sigma \models_c \mu$ or not then we say that $\models_c$ is *decidable* at μ.

The first result on the compactness of $\models_c$ is a very deep result of Day and Freese [DF1] which states that $\models_c$ is compact at the modular law. Using this result the consequence relation in congruence varieties was soon shown to be compact at distributivity, cf. [Cz12], and decidable at modularity and distributivity, cf. [CF1]. As a common generalization of distributivity and modularity we will define certain lattice identities in Chapter II. These identities will be called *diamond identities*. Diamond identities are in connection with András Huhn's theory of n-distributive lattices and also with von Neumann's coordinatization theory of modular lattices. The consequence relation $\models_c$ will be shown to be compact and decidable at any diamond identity. In fact, this will be shown in two different ways. One of the proofs is relatively simple, thanks to commutator theory. The other proof is much longer but permits useful generalizations.

Chapter II also contains some other applications of diamond identities as well. A $\sigma \models_c \mu$ statement is called *nontrivial* if $\sigma \not\models \mu$. There are many early results stating nontrivial $\sigma \models_c \mu$ results. We mention Nation [Na1], Day and Freese [DF1], Freese, Herrmann and Huhn [FHH1] and Jónsson [Jo1] only; note that [Jo1] gives a good overview on results of this kind. All these results are located at distributivity or modularity in the sense that either σ or μ is the distributive or modular law. That time this could have led to the false feeling that any nontrivial $\models_c$ result must be somehow in connection with congruence distributivity or modularity. By the help of diamond identities we will give infinitely many nontrivial $\sigma \models_c \mu$ results, which are far from distributivity, modularity and each other in a very strong sense.

The rest of Chapter II generalizes the notion of congruence varieties in two directions. Firstly, we consider lattice varieties $\underline{\mathbf{Con}}(\mathcal{V})$ without requiring that $\mathcal{V}$ is a variety, we only assume that $\mathcal{V}$ is closed under finite subdirect powers. Note that already some of the above-mentioned $\models_c$ results, e.g., Freese and Jónsson [FJ1] and Freese, Herrmann and Huhn [FHH1], were proved under this weaker assumption. Further, [Cz4] was devoted to a systematic study of one kind of these generalized congruence varieties. Secondly, we will consider structures, where relations are also allowed, not only algebras. Then there are several variants of the notion of congruence relations; we will deal with relative congruences, $*$-congruences and order congruences. Relative congruences have been extensively studied even for algebras, not only for structures. For example, Dziobiak [Dz1] and Nurakunov [Nu1] have found some Mal'cev type conditions characterizing distributivity of relative congruences of a quasivariety, and even commutator theory is now extended for relative congruences of a quasivariety by Kearnes and McKenzie [KM1]. Chapter II combines the above-mentioned two ways of generalizations and surprisingly lot of $\models_c$ results will be extended to these more general situations. Of course, still diamond identities will play the central role.

It is a longstanding open problem whether the congruence varieties form a lattice under inclusion or not. On the other hand, some generalized congruence varieties do form a (complete) lattice, cf. [Cz4]. Thus, it would be interesting to know if the ℓ-congruence varieties from [Cz4] or, more generally, those which could be naturally defined in connection with the several kinds of $\models_c$ relations in Chapter II are the same or not. Equivalently, the problem is whether $\models_c$ coincides with its generalizations. Unfortunately, no progress has been achieved on this problem for fifteen years, since [Cz4]. So, we do not know if $\models_c$ and its various generalizations in Chapter II are pairwise distinct or they are all the same.

The aim of **Chapter III** is to present an example of a non-selfdual modular congruence variety. It was about fifteen years ago that Day and Freese [DF1] proved the existence of a non-selfdual congruence variety. However, the example they gave was not modular. On the other hand, the list of *known* modular congruence varieties (cf. [HC1]) did not changed for over more than a decade. This list consisted of congruence varieties of module varieties over rings, which are self-dual by Hutchinson [HC1, Thm. 7] and [Ht4]. A few years ago Pálfy and Szabó [PS1] and [PS2] showed that the congruence varieties of certain group varieties are distinct from any congruence variety of modules. This raises the problem whether every modular congruence variety is self-dual, cf. Pálfy and Szabó [PS2, Problem 4.2] for a slightly different formulation. In Chapter III we solve this problem in negative. One direction of the proof is done by computer; this seems to be necessary because solving (and even constructing) a system of 130 linear equations for 108 unknowns by hand would be neither reasonable nor reliable.

In **Chapter IV** another consequence relations among lattice identities, denoted by $\models_c^{\text{const}}$, will be considered. Like the consequence relations defined in Chapter II, this relation will also be a a generalization of the consequence relation $\models_c$ for congruence varieties. This generalization consists of two ingredients. Firstly, like in Chapter II, instead of varieties $\mathcal{V}$ we start from classes subject to weaker closedness stipulations. Secondly, the satisfaction of a lattice identity $p(x_1, \dots, x_n) \le q(x_1, \dots, x_n)$ for congruences of algebras in $\mathcal{V}$ will not be required everywhere, just only at a constant given by a nullary operation. Let A be an algebra with a nullary operation symbol e in its type. We say that the congruences of A satisfy the identity $p(x_1, \dots, x_n) \le q(x_1, \dots, x_n)$ at e if for any congruences $\alpha_1, \dots, \alpha_n$ of A the $p(\alpha_1, \dots, \alpha_n)$-class of e is included in the $q(\alpha_1, \dots, \alpha_n)$-class of e. This definition is due to Chajda [Ch1].

Now let Σ be a set of lattice identities and let λ be another lattice identity. We write $\Sigma \models_c^{\text{const}} \lambda$ to denote that for any class $\mathcal{V}$ which is closed under subalgebras and finite direct powers and has a nullary operation symbol e in its type if every member of Σ holds for congruences of any $A \in \mathcal{V}$ at e then so does λ.

While the consequence relations in Chapter II were all stronger than the usual $\models$ relation in the sense that $\Sigma \models \lambda$ always implied them, now the case becomes different. Indeed, an example of lattice identities λ and μ with $\lambda \models \mu$ but $\lambda \not\models_c^{\text{const}} \mu$ will be given in Proposition 4.12. Yet, in spite of this weakness of the $\models_c^{\text{const}}$ relation, Theorem 4.4. will give nontrivial

$$\sigma \models_c^{\text{const}} \text{distributivity}$$

results for a large class of lattice identities σ, including, e.g., András Huhn's n-distributive law from [Hn1]. Since $\Sigma \models_c^{\text{const}} \lambda$ always implies $\Sigma \models_c \lambda$, our $\models_c^{\text{const}}$ result is a generalization of its classical counterpart in Nation [Na1].

In [Ch1], Chajda has given a Mal'cev condition which characterizes those varieties $\mathcal{V}$ with a constant e for which the distributive law holds for congruences of every $A \in \mathcal{V}$ at e. Combining his approach with the Wille – Pixley algorithm, cf. [Wi1] and [Pi1], we will give a weak Mal'cev condition for an arbitrary lattice identity in the similar sense. This is particularly useful in some cases. E.g., the numbers 130 and 108 (the sizes of a system of linear equations) mentioned before (in connection with Chapter III) would have been considerably larger without this result.

Chapter V is devoted for quasivarieties of submodule lattices. Since quasivarieties are just the classes definable by Horn sentences, this chapter goes well with the title of this work. Moreover, it is not a surprise that some Horn sentences, denoted by $\chi(m,p)$, will play a central role in one of our proofs. For a ring R with unit the class of lattices embeddable in the submodule lattices of R-modules is known to be a quasivariety, cf. Makkai and McNulty [MM1]. This quasivariety will be denoted by

$$\mathcal{L}(R) = IS\{\text{Sub}({}_RM)\colon\ {}_RM \text{ is an } R\text{-module}\}.$$

Here I and S are the operators of forming isomorphic copies and subalgebras, respectively. Since $\mathcal{L}(R) = IS\{\text{Con}({}_RM)\colon\ {}_RM$ is an R-module$\}$, the study of these $\mathcal{L}(R)$ is closely related to that of congruence varieties.

We will consider rings with prime power characteristic. All rings in this work, unless otherwise stated, will be assumed to be of characteristic p^k ($k > 1$, p prime). Let $\mathbf{W}(p^k)$ denote the set(!) $\{\mathcal{L}(R) : \text{char}\, R = p^k\}$. This set consists of lattice quasivarieties, so it has at most continuously many elements. $\mathbf{W}(p^k)$ is a partially ordered set under set theoretic inclusion, and it not a hard task to show that it is a join subsemilattice of the complete lattice of all lattice quasivarieties. The main question here is that how large is $\mathbf{W}(p^k)$. Hutchinson [Ht1] shows that while $\mathbf{W}(p)$ is a singleton, $\mathbf{W}(p^k)$ has at least two elements. A related result (cf. [HC1] or Theorems 2.A and 2.B in Chapter II) asserts that the *variety* $H\mathcal{L}(R)$ depends only on p^k, the characteristic of R. The main result of Chapter V, which is one of the main results in this dissertation, states that $\mathbf{W}(p^k)$ consists of continuously many quasivarieties $\mathcal{L}(R)$. Moreover, the "height" and "width" of the partially ordered set $\mathbf{W}(p^k)$ are as large as possible, for $\mathbf{W}(p^k)$ has ascending and descending chains and antichains with continuously many elements. The proof of this result is based on Hutchinson's theorem (Theorem 5.B) which gives a sufficient condition for $\mathcal{L}(R) \neq \mathcal{L}(R')$.

Now two problems arise naturally. Firstly, is Hutchinson's sufficient condition for $\mathcal{L}(R) \neq \mathcal{L}(R')$ also a necessary condition? Secondly, is $\mathbf{W}(p^k)$ closed with respect to arbitrary joins (taken in the lattice of all lattice quasivarieties)? Note that if either of these two problems had an affirmative answer then we could show that $\mathbf{W}(p^k)$ is a complete lattice. Unfortunately, we are unable to solve these two problems in the moment, and we do not know if $\mathbf{W}(p^k) = \langle \mathbf{W}(p^k), \subseteq \rangle$ is a lattice or not. Yet, we will show that at least one of the two problems we mentioned has a negative solution. This means that the partially ordered set $\mathbf{W}(p^k)$, lattice or not, probably has a more complicated structure than previously expected. The long proof involves a construction of a lattice Horn sentence $\chi(m,p)$, which generalizes the "irregular" Horn sentence of [CH1], and uses the method of [Cz5] to show the appropriateness of $\chi(m,p)$ (and to find it).

Our $\chi(m,p)$ is a Horn sentence with four variables, and not a particularly simple one. This raises the question whether a simpler Horn sentence could also be used. Say a Horn sentence with fewer variables. To show that this is not the case we conclude Chapter V by showing that any Horn sentence on at most three variables is trivial in some sense for modular lattices.

Chapter VI is devoted to involution lattices. A quadruplet $L = \langle L; \vee, \wedge, {}^* \rangle$ is called an *involution lattice* if $L = \langle L; \vee, \wedge \rangle$ is a lattice and ${}^*\colon L \to L$ is a lattice automorphism such that $(x^*)^* = x$ holds for all $x \in L$. To present a natural example, let us consider $\mathrm{Quord}(A)$, the set of quasiorders (i.e. reflexive, transitive and compatible relations), of an algebra or a set A. (If A is a set then compatibility means no restriction on relations.) Defining $\rho^* = \{\langle x, y\rangle \colon \langle y, x\rangle \in \rho\}$, $\mathrm{Quord}(A)$ becomes an involution lattice under set theoretic inclusion, i.e., $\mathrm{Quord}(A) = \langle \mathrm{Quord}(A); \vee, \wedge, {}^* \rangle$, where $\wedge$ is the intersection and $\vee$ is the transitive closure of the union. Note that $\mathrm{Quord}(A)$ reflects the congruence lattice of A since $\mathrm{Con}(A)$ is just the set (in fact the sublattice) of the fixed points of the involution in $\mathrm{Quord}(A)$. Motivated by a problem of Chajda and Pinus [CP3], we will prove that if an involution lattice I is algebraic and either $x^* = x$ holds for all $x \in I$ or I is finite and distributive then $I \cong \mathrm{Quord}(A)$ for some algebra A. These results and two others from [CC1] lead to the question whether the well-known Grätzer – Schmidt theorem [GS1] generalizes to the involution lattices $\mathrm{Quord}(A)$ or not. The answer is negative, for the main result of Chapter VI, Theorem 6.3, presents a Horn sentence which holds in every $\mathrm{Quord}(A)$ but not in every involution lattice. The proof uses our algorithm [Cz11] for the word problem of lattices and its computer implementation.

A former result of László Szabó [Sz1] asserts that for any *lattice* A, the lattice $\mathrm{Quord}(A)$ is isomorphic to the direct square of $\mathrm{Con}(A)$. The original proof is quite long. Now it appears that the adequate way to achieve this result is to use involution lattices. Indeed, $\mathrm{Con}^2(A)$ becomes an involution lattice if we consider the map which transposes the components of its members. First we formulate an easy structure theorem for distributive involution lattices that have an element ρ such that ρ^* is the complement of ρ. Armed with this structure theorem we can resort to [CL2], and the above-mentioned result of Szabó follows quite easily.

Then we take a further step. If A is a lattice then the isomorphism between $\mathrm{Quord}(A)$ and $\mathrm{Con}^2(A)$ is shown to be a particular instance of a natural equivalence between the functors Quord and $\mathrm{Con}^2(A)$. We will try to determine all natural equivalences between these two functors. Even if we cannot solve the problem in full generality our results, Theorem 6.7 and Example 6.22, indicate that the answer heavily depends on the common domain, which is a prevariety of lattices, of the two functors.

In **Chapter VII** a new class of related lattices from [CP1] are introduced and investigated. A corollary of the main result, Corollary 7.11, states that these lattices satisfy nontrivial Horn sentences. For definition, let P be a finite partially ordered set. The set of all subsets, alias coalitions, of P is denoted by $\mathcal{L}(P)$. This notion originates from game theory and the theory of human decision making where the ordering (or quasiordering) of P is given by a valuating function from P to the field of real numbers, which measures the "strength" of the individuals belonging to P. For example, P could be the set of shareholders of a given corporation or the set of political parties in a given country. For $X, Y \in \mathcal{L}(P)$, a map φ: $X \to Y$ is called an *extensive map* if φ is injective and for every $x \in X$ we have $x \le \varphi(x)$. Let $X \le Y$ mean that there exists an extensive map $X \to Y$; this definition turns $\mathcal{L}(P)$ into a partially ordered set $\mathcal{L}(P) = \langle \mathcal{L}(P), \le \rangle$. This $\mathcal{L}(P)$ carries all information what can be said about coalitions based merely on pairwise comparisons among the elements of P without knowing the numerical values of their strength. First we describe all finite partially ordered sets P for which $\mathcal{L}(P)$ is a lattice. (In fact, we do this in a slightly more general framework for quasiordered sets P.) It appears that $\mathcal{L}(P)$ is a lattice iff (the Hasse diagram of) P is a forest. Although there are two distinct proofs, one will be given in Chapter VII while the other is due to Gy. Pollák [CP1], in contrast with many other related lattices it is not so evident that coalitions of a forest form a lattice.

Later in the chapter we investigate the structure of coalition lattices. A generalization of the Hall – Dilworth lattice gluing yields a structural characterization of $\mathcal{L}(P)$ in Theorem 7.9. From this structure theorem it is easy to derive that coalition lattices satisfy nontrivial Horn sentences. On the other hand, we do not know in the moment if they satisfy nontrivial identities. But even if they do, we think that not too many identities hold in them. However, in spite of their poor equational theory, coalition lattices will be shown to satisfy the Jordan – Hölder chain condition. Finally, we show that a coalition lattice $\mathcal{L}(P)$ not only accompanies P, it determines P up to isomorphism.

Our terminology is the standard one of lattice theory and universal algebra, which follows Grätzer's books [Gr1] and [Gr2] with the exception that we speak of "terms" rather than "polynomials". Some really frequently needed notations, like $\mathrm{Con}(A)$, are given already here in the Introduction, but most of them will be defined again in the corresponding chapter.

This work is based on the author's previously written, in a few cases quite recently written, research papers. Each of the following chapters begins with listing the papers form which the chapter is composed. Although, according to the title, Horn sentences (including identities) in related lattices play the central role in all chapters, there are not too many cross-references among the following chapters. In particular, Chapters VI and VII are quite independent from each other and from the rest of this dissertation.

Since several of my papers were written with coauthors, a special method of referring to these papers is applied. (This happens in accordance with the coauthors' declarations, of course.) Sometimes, when a joint paper consists of more or less independently achieved results, only one name is mentioned in connection with a particular result. This is typical, e.g., in case of [CH2], where the larger part belongs to Hutchinson while the smaller part, which was put in Chapter V, belongs to me. But even if there is no such partition for a joint paper I do not include larger proportion of the joint results in this dissertation than my contribution to them. Beside space considerations this is why I have omitted [CC2] and [CC3] (where the single result of the paper cannot be divided) from this work. This is also one of the reasons that the assertions in this work are numbered in two ways. I have given two numbers, like "Theorem 7.1", to those which I consider an essential part of this work. They are always accompanied with proofs. On the other hand, results cited without proof are given a number plus a letter, like Corollary 6.C. This second category includes results from my coauthors in a joint paper, some joint achievements of me and coauthors, some of my former results (before or in [Cz16]), and other results cited from the literature.

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CHAPTER II

DIAMOND IDENTITIES IN CONGRUENCE VARIETIES AND GENERALIZED CONGRUENCE VARIETIES

Based on [Cz2], [Cz3] and [Cz9], the aim of this chapter is to contribute to the theory of the consequence relation among lattice identities in congruence varieties. Moreover, we will also consider some lattice varieties more general than congruence varieties. Since these generalizations do not make our proof unreasonably longer and, hopefully, shed more light on the topic, we will prove a part of the results immediately for the more general setting and deduce the case of congruence varieties as a particular case. Before indulging in generalizations let us overview the original notions.

For a class $\mathcal{K}$ of algebras let $\mathrm{Con}(\mathcal{K})$ denote $I\{\mathrm{Con}(A)\colon\ A \in \mathcal{K}\}$, i.e. the class of lattices isomorphic to congruence lattices of algebras in $\mathcal{K}$. If $\mathcal{K}$ is a variety then the lattice variety generated by $\mathrm{Con}(\mathcal{K})$ is called a *congruence variety*, cf. Jónsson [Jo1]. For a lattice identity λ and a set of lattice identities Σ, Σ is said to imply λ in congruence varieties, in notation $\Sigma \models_c \lambda$, if every congruence variety that satisfies (every member of) Σ also satisfies λ. If, in addition, Σ does not imply λ in all lattices, in notation $\Sigma \not\models \lambda$, then the consequence relation $\Sigma \models_c \lambda$ is called nontrivial. As we already mentioned in the Introduction, many nontrivial results of the form $\{\sigma\} \models_c \lambda$, also denoted by $\sigma \models_c \lambda$, have appeared so far. These results state that certain lattice identities are equivalent to the modular or distributive law in congruence varieties.

The aim of the present paper is threefold. We plan to give new nontrivial results of the form $\sigma \models_c \lambda$, we intend to broaden our knowledge on the compactness and effectiveness properties of $\models_c$, and we want to generalize these results for more general situations. To achieve the desired generality we will consider structures (i.e., nonempty sets equipped with operations and relations, cf. Weaver [We1] for an overview), not only algebras. The operators of forming subdirect squares, direct products and isomorphic copies will be denoted by Q^s, P and I, respectively. The relations on direct products are defined componentwise, while the relations for substructures (or subdirect products) are obtained via restriction to their base set. Another way of generalization is to consider Q^s-closed classes $\mathcal{K}$ instead of varieties. Let $\mathcal{K}$ be a class of similar structures and $A, B \in \mathcal{K}$. A map $\varphi\colon\ A \to B$ is called a *homomorphism* if it commutes with the fundamental operations and for any relation symbol ρ and arbitrary $a_1, \dots, a_n \in A$ if $\rho_A(a_1, \dots, a_n)$ then $\rho_B(\varphi(a_1), \dots, \varphi(a_n))$. Given $A \in \mathcal{K}$, the kernels of homomorphisms from A into other structures in $\mathcal{K}$ are called $\mathcal{K}$-congruences or *relative congruences* of A. Let $\mathrm{Con}^{\mathcal{K}}(A)$ be the set of $\mathcal{K}$-congruences of A. The proof of Theorem 3 in Weaver [We1] shows that $\mathrm{Con}^{\mathcal{K}}(A)$ is a lattice (with respect to inclusion) provided $\mathcal{K}$ is closed under direct products. Therefore $\mathrm{Con}^r(\mathcal{K}) := I\{\mathrm{Con}^{\mathcal{K}}(A)\colon\ A \in \mathcal{K}\}$ is a class

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of lattices when $\mathcal{K}$ is P-closed. Considering $\mathrm{Con}^r(\mathcal{K})$ instead of $\mathrm{Con}(\mathcal{K})$ offers us a way of generalization. For a structure A an equivalence relation Θ of A is called a $*$-congruence of A if Θ is a congruence in the algebraic sense and for any k-ary relation symbol ρ and any $\langle a_1, b_1\rangle, \dots, \langle a_k, b_k\rangle \in \Theta$ we have

$$\rho_A(a_1, \dots, a_k) \iff \rho_A(b_1, \dots, b_k),$$

cf. Weaver [We1]. The $*$-congruences of A form a sublattice of the equivalence lattice of A; this lattice will be denoted by $\mathrm{Con}^*(A)$. For an *algebra* A we have $\mathrm{Con}^*(A) = \mathrm{Con}(A)$.

A triple $\langle A; F; \leq\rangle$ is called an *ordered algebra* if $\langle A; F\rangle$ is an algebra, $\langle A; \leq\rangle$ is a partially ordered set, and every $f \in F$ is monotone with respect to $\leq$. Varieties of ordered algebras were studied e.g. in Bloom [Bl1]. In case of ordered algebras, the monotone and operation-preserving maps are called *homomorphisms*, and their kernels are called *order congruences*. Given an ordered algebra A, the set $\mathrm{Con}^<(A)$ of order congruences of A is an algebraic lattice. (This was proved in [CL2], where an inner definition of order congruences and a description of their join is also given.) For a class $\mathcal{K}$ of ordered algebras let $\mathrm{Con}^<(\mathcal{K}) := I\{\mathrm{Con}^<(A)\colon\ A \in \mathcal{K}\}$. For a class $\mathcal{K}$ of ordered algebras and $B \in \mathcal{K}$ the lattices $\mathrm{Con}^<(B)$, $\mathrm{Con}^*(B)$ and $\mathrm{Con}^{\mathcal{K}}(B)$ are pairwise different in general, even if $\mathcal{K}$ is closed under P and Q^s.

We will investigate three further consequence relations among lattice identities. Let λ be a lattice identity and let Σ be a set of lattice identities. Let $\Sigma \models_c \lambda\ (r; Q^s, P)$ resp. $\Sigma \models_c \lambda\ (*; Q^s)$ resp. $\Sigma \models_c \lambda\ (\leq; Q^s)$ denote that for every class $\mathcal{K}$ of structures which is closed under Q^s and P resp. every Q^s-closed class $\mathcal{K}$ of structures resp. every Q^s-closed class $\mathcal{K}$ of ordered algebras if Σ holds is $\mathrm{Con}^r(\mathcal{K})$ resp. $\mathrm{Con}^*(\mathcal{K})$ resp. $\mathrm{Con}^<(\mathcal{K})$ then so does λ. According to the notations above, $\models_c$ could be denoted by $\models_c\ (H, S, P)$. The reader will certainly notice by the end of the chapter that the Q^s-closedness of $\mathcal{K}$ could be replaced by the following weaker assumption: "if $A \in \mathcal{K}$ and α is a congruence (of the respective type) of A then α, as a subalgebra of A^2, belongs to $\mathcal{K}$.

Clearly, $\Sigma \models_c \lambda$ follows from any of the above-defined three consequence relations. Besides finding some new $\Sigma \models_c \lambda$ results, our goal is also to prove the converse under reasonable restrictions. I.e., we want to turn a lot of $\Sigma \models_c \lambda$ results into $\Sigma \models_c \lambda\ (r; Q^s, P)$, $\Sigma \models_c \lambda\ (*; Q^s)$ and $\Sigma \models_c \lambda\ (\leq; Q^s)$ statements.

The proofs of the classical $\Sigma \models_c \lambda$ results often involve particular tools. For example, free algebras are used in Day and Freese [DF1, p. 1156] or Jónsson [Jo1, p. 379]; Mal'cev conditions are used in Day [Da1] and Mederly [Me1], and even commutator theory is required in [KM1]. The scope of these tools is often extended far beyond varieties of algebras. There are free structures and there are Mal'cev conditions for $*$-congruences, cf. Weaver [We1]. Free ordered algebras and some Mal'cev conditions are available for ordered algebras (cf. Bloom [Bl1] and [CL1]). The methods used in [Cz1] and [Cz4] also indicate that certain $\models_c$ results can be generalized. Even commutator theory has been developed for relative congruences of quasivarieties of algebras and some Mal'cev-like conditions are also available, cf. Kearnes and McKenzie [KM1], Dziobiak [Dz1] and Nurakunov [Nu1]. However, all these recent developments are insufficient for our purposes as they require much stronger closedness assumption on $\mathcal{K}$.

Fortunately, some of the known $\Sigma \models_c \lambda$ results, namely those in Freese and Jónsson [FJ1] and Freese, Herrmann and Huhn [FHH1], are in fact $\Sigma \models_c \lambda\ (Q^s)$

results (for algebras and usual congruences), and we will not have much difficulty in generalizing them. In presence of modularity, the rest of the known $\Sigma \models_c \lambda$ results can, at least in principle, be deduced from Theorem 2.4 (cf. later or [Cz2]). Since [Cz2] relied on commutator theory, we had to find another approach which avoids commutator theory.

Let dist resp. mod stand for the distributive resp. modular law. Although the usage of "known" hurts mathematical rigour below, it is time to indicate that our aim is to prove the following

PROPOSITION 2.1. *([Cz9]) Suppose $\Sigma \models_c \lambda$ is a known result in the theory of congruence varieties and $\Sigma \models$ mod . Then $\Sigma \models_c \lambda \ (r; Q^s, P)$, $\Sigma \models_c \lambda \ (*; Q^s)$ and $\Sigma \models_c \lambda \ (\le; Q^s)$.*

Note that "known" includes the results in [Cz3], cf. Theorem 2.7 and its corollaries later in this chapter. We do not know if $\Sigma \models$ mod can be omitted or "known result" can be replaced by "true statement" in Proposition 2.1.

For structures A and B a homomorphism $\varphi\colon\ A \to B$ is called a *$*$-homomorphism* if for every relation symbol ρ and $a_1, \dots, a_k \in A$ we have

$$\rho_A(a_1, \dots, a_k) \iff \rho_B(\varphi(a_1), \dots, \varphi(a_k)).$$

It is easy to see, cf. Weaver [We1], that $*$-congruences are precisely the kernels of $*$-homomorphisms. A homomorphism resp. $*$-homomorphism $\varphi\colon\ A \to A$ is called a *retraction* resp. *$*$-retraction* if $\varphi \circ \varphi = \varphi$. The retraction of an ordered algebra is defined analogously; then φ must be monotone, of course. If $\varphi\colon\ A \to A$ is a retraction then $B := \varphi(A)$ is called a retract of A. (The relations on B are defined as the restrictions of the relations on A.) Associated with this φ we have a map $\hat{\varphi}$ from the set of equivalences of B into the set of equivalences of A defined by $\hat{\varphi}(\Theta) = \{\langle a, b\rangle \in A^2\colon\ \langle \varphi(a), \varphi(b)\rangle \in \Theta\}$. In the sequel, the restriction of $\hat{\varphi}$ to $\mathrm{Con}^*(B)$, $\mathrm{Con}^r(B)$ or $\mathrm{Con}^<(B)$ will also be denoted by $\hat{\varphi}$.

LEMMA 2.1. *([Cz9]) Suppose $\varphi\colon\ A \to A$ is a retraction, $A \in \mathcal{K}$, and $B = \varphi(A)$.*
(A) *If φ is a $*$-retraction then $\hat{\varphi}\colon\ \mathrm{Con}^*(B) \to \mathrm{Con}^*(A)$*
(B) *If $B \in \mathcal{K}$ and $\mathcal{K}$ is P-closed then $\hat{\varphi}\colon\ \mathrm{Con}^{\mathcal{K}}(B) \to \mathrm{Con}^{\mathcal{K}}(A)$*
(C) *If A is an ordered algebra and φ is monotone then $\hat{\varphi}\colon\ \mathrm{Con}^<(B) \to \mathrm{Con}^<(A)$*

is a lattice embedding.

PROOF. Since the meet coincides with the intersection, it is evident that $\hat{\varphi}$ is a meet-homomorphism in all the three cases. If Θ is an equivalence on B and $a, b \in B$ then $\langle a, b\rangle \in \hat{\varphi}(\Theta) \iff \langle \varphi(a), \varphi(b)\rangle \in \Theta \iff \langle a, b\rangle = \langle \varphi(a), \varphi(b)\rangle \in \Theta$, thus $\hat{\varphi}$ is injective. The treatment for joins is more or less the same for all the three cases, thus we detail (B) only. Assume that for $C, D, E \in \mathcal{K}$ and homomorphisms $\alpha\colon\ B \to C$, $\beta\colon\ B \to D$ and $\gamma\colon\ B \to E$ we have $\mathrm{Ker}\alpha \vee \mathrm{Ker}\beta = \mathrm{Ker}\gamma$ in $\mathrm{Con}^{\mathcal{K}}(B)$. Then $\hat{\varphi}(\mathrm{Ker}\alpha) = \mathrm{Ker}(\alpha \circ \varphi)$, $\hat{\varphi}(\mathrm{Ker}\beta) = \mathrm{Ker}(\beta \circ \varphi)$ and $\hat{\varphi}(\mathrm{Ker}\gamma) = \mathrm{Ker}(\gamma \circ \varphi)$. Since $\hat{\varphi}$ is monotone, $\hat{\varphi}(\mathrm{Ker}\alpha) \le \hat{\varphi}(\mathrm{Ker}\gamma)$ and $\hat{\varphi}(\mathrm{Ker}\beta) \le \hat{\varphi}(\mathrm{Ker}\gamma)$. Now let $\delta\colon\ A \to F$ be an arbitrary homomorphism such that $F \in \mathcal{K}$, $\mathrm{Ker}\delta \supseteq \hat{\varphi}(\mathrm{Ker}\alpha) = \mathrm{Ker}(\alpha \circ \varphi)$ and $\mathrm{Ker}\delta \supseteq \hat{\varphi}(\mathrm{Ker}\beta) = \mathrm{Ker}(\beta \circ \varphi)$; we have to show that $\mathrm{Ker}\delta \supseteq \mathrm{Ker}(\gamma \circ \varphi)$. Suppose $\langle a, b\rangle \in \mathrm{Ker}(\gamma \circ \varphi)$ for some $a, b \in A$. Since $\langle \varphi(\varphi(a)), \varphi(a)\rangle = \langle \varphi(a), \varphi(a)\rangle \in \mathrm{Ker}\alpha$, we have $\langle \varphi(a), a\rangle \in \mathrm{Ker}(\alpha \circ \varphi) \subseteq \mathrm{Ker}\delta$. Similarly, $\langle \varphi(b), b\rangle \in \mathrm{Ker}\delta$. Now consider the restriction $\delta|_B\colon\ B \to F$, which is a homomorphism. If $c, d \in B$ and $\langle c, d\rangle \in$

Kerα then $\langle c,d\rangle = \langle\varphi(c),\varphi(d)\rangle \in \mathrm{Ker}(\alpha\circ\varphi) \subseteq \mathrm{Ker}\delta$. Thus $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}(\delta|_B)$, and $\mathrm{Ker}\beta \subseteq \mathrm{Ker}(\delta|_B)$ comes similarly. Therefore $\mathrm{Ker}\gamma \subseteq \mathrm{Ker}(\delta|_B)$. From $\langle a,b\rangle \in \mathrm{Ker}(\gamma\circ\varphi)$ we infer $\langle\varphi(a),\varphi(b)\rangle \in \mathrm{Ker}\gamma \subseteq \mathrm{Ker}(\delta|_B) \subseteq \mathrm{Ker}\delta$, and $\langle a,b\rangle \in \mathrm{Ker}\delta$ follows by transitivity. Therefore $\hat\varphi$ is a $\vee$-homomorphism, and (B) is proved.

The arguments for (A) resp. (C) are quite analogous: we have to use $*$-homomorphisms resp. monotone homomorphisms, and A, B, C, D, E, F will be arbitrary structures resp. arbitrary ordered algebras, not necessarily in $\mathcal{K}$. $\square$

The amalgamation property we are going to consider first appeared in Freese and Jónsson [FJ1], and played a central role in Freese, Herrmann and Huhn [FHH1].

Definition 2.1. A class $\mathcal{C}$ of lattices is said to satisfy the Freese—Jónsson amalgamation property, in short FJAP, if for each $L \in \mathcal{C}$ and $a \in L$ there exists an $M \in \mathcal{C}$ and embeddings φ_1, φ_2 of L in M such that

(a) $\varphi_1(x) = \varphi_2(x)$ for all $x \geq a$ in L,
(b) $\varphi_1(x) \vee \varphi_2(x) = \varphi_1(a)$ for all $x \leq a$ in L, and
(c) $\varphi_i(y) \vee (\varphi_1(x) \wedge \varphi_2(x)) = \varphi_i(x)$ for all $y \leq x$ in L and $i = 1, 2$.

Lemma 2.2. *([Cz9]) Let $\mathcal{C}$ be one of the following classes:*

(A) $\mathrm{Con}^*(\mathcal{K})$ *where $\mathcal{K}$ is a Q^s-closed class of structures;*
(B) $\mathrm{Con}^r(\mathcal{K})$ *where $\mathcal{K}$ is a class of structures closed under P and Q^s;*
(C) $\mathrm{Con}^<(\mathcal{K})$ *where $\mathcal{K}$ is a Q^s-closed class of ordered algebras.*

Then $\mathcal{C}$ satisfies FJAP.

Proof. The construction needed by the proof of this lemma is the same as that for a Q^s-closed class of algebras (cf. Freese and Jónsson [FJ1] or Hagemann and Herrmann [HH1]). We give the details in case (B) only. Suppose $C \in \mathcal{K}$ and $\alpha \in \mathrm{Con}^{\mathcal{K}}(C)$. Let $A := \{\langle x,y\rangle \in C^2\colon\ x\alpha y\}$. Since A is a subdirect square of C, it belongs to $\mathcal{K}$. Let ι denote the embedding $C \to A$, $x \mapsto \langle x,x\rangle$, and denote $\iota(C)$ by B. Then $\iota\colon\ C \to B$ is an isomorphism, which induces an isomorphism, also denoted by ι, from $\mathrm{Con}^r(C)$ to $\mathrm{Con}^r(B)$. Let ψ_i be the retraction $A \to B$, $\langle x_1,x_2\rangle \mapsto \langle x_i,x_i\rangle$. Then $\hat{\psi_i}\colon\ \mathrm{Con}^{\mathcal{K}}(B) \to \mathrm{Con}^{\mathcal{K}}(A)$ is an embedding by Lemma 2.1. Therefore $\hat{\psi_i}\circ\iota\colon\ \mathrm{Con}^{\mathcal{K}}(C) \to \mathrm{Con}^{\mathcal{K}}(A)$, $\Theta \mapsto \Theta_i := \{\langle\langle x_1,x_2\rangle,\langle y_1,y_2\rangle\rangle \in A^2\colon\ x_i\Theta y_i\}$ is a lattice embedding for $i = 1, 2$. For $\Theta \geq \alpha$, $\Theta_1 = \Theta_2$ is obvious. For $\Theta \leq \alpha$ it is easy to see that $\Theta_1 \circ \Theta_2 \supseteq \alpha_1 = \alpha_2$, thus we obtain that $\alpha_1 \subseteq \Theta_1 \circ \Theta_2 \subseteq \Theta_1 \vee_{\mathcal{K}} \Theta_2 \subseteq \alpha_1 \vee_{\mathcal{K}} \alpha_1 = \alpha_1$, showing (b) in the definition of FJAP. (Here $\vee_{\mathcal{K}}$ stands for the join taken in $\mathrm{Con}^{\mathcal{K}}(A)$.) Now let $i \in \{1,2\}$ and $\Theta \subseteq \Psi \in \mathrm{Con}^{\mathcal{K}}(C)$. Then $\Psi_i \subseteq 0_i \circ (\Psi_1 \cap \Psi_2) \circ 0_i$ where 0 denotes the smallest (relative) congruence of C. Indeed, e.g. for $i = 1$, if $\langle x_1,x_2\rangle\Psi_1\langle y_1,y_2\rangle$ then $\langle x_1,x_2\rangle 0_1 \langle x_1,x_1\rangle \Psi_1 \cap \Psi_2 \langle y_1,y_1\rangle 0_1 \langle y_1,y_2\rangle$. Therefore $\Psi_i \subseteq 0_i \circ (\Psi_1 \cap \Psi_2) \circ 0_i \subseteq 0_i \vee_{\mathcal{K}} (\Psi_1 \wedge \Psi_2) \subseteq \Theta_i \vee_{\mathcal{K}} (\Psi_1 \wedge \Psi_2) \subseteq \Psi_i \vee_{\mathcal{K}} (\Psi_i \wedge \Psi_i) = \Psi_i$, proving (c) in the definition of FJAP. This completes the proof of (B). The arguments for (A) resp. (C) are analogous, for Ψ_i becomes a $*$-retraction resp. monotone retraction. $\square$

Given a ring R with 1, let $H\mathcal{L}(R)$ denote the class of homomorphic images of lattices embeddable in the submodule lattice of (unitary left) R-modules. $H\mathcal{L}(R)$ is just the congruence variety $HSP(\mathrm{Con}(R\text{-}\mathbf{Mod}))$. For integers $m \geq 0$ and $n \geq 1$ let $D(m,n)$ denote the ring sentence $(\exists r)(m\cdot r = n\cdot 1)$. (Here 1 is the ring unit and $k\cdot x = x + x + \ldots + x$, k times.) $D(m,n)$ is called a divisibility condition. In

[HC1] an algorithm is given which associates a pair $\langle m_\varepsilon, n_\varepsilon\rangle$ of integers, $m_\varepsilon \geq 0$, $n_\varepsilon \geq 1$, with an arbitrary lattice identity ε such that for any R we have

THEOREM 2.A. *([HC1]) ε holds in $H\mathcal{L}(R)$ iff $D(m_\varepsilon, n_\varepsilon)$ holds in R.*

Let $V(0) := H\mathcal{L}(\mathbf{Q})$, i.e., the lattice variety generated by the rational projective geometries. For $k > 0$ let $V(k) := H\mathcal{L}(\mathbf{Z}_k)$ where $\mathbf{Z}_k$ is the factor ring of integers modulo k. For a nonnegative integer k and a prime p let $\operatorname{expt}(k,p)$ denote the largest integer $i \geq 0$ for which $p^i \mid k$; by $\operatorname{expt}(0,p)$ we mean the smallest infinite ordinal ∞. From [HC1, Proposition 1] we invoke

THEOREM 2.B. *$D(m,n)$ holds in a ring R iff for any prime p with* $\operatorname{expt}(m,p) > \exp(n,p)$ *R satisfies $D(p^{\operatorname{expt}(n,p)+1}, p^{\operatorname{expt}(n,p)})$ and, in addition, $m = 0$ implies that the characteristic of R is not 0. In case the characteristic of R is $k > 0$ then $D(m,n)$ holds in R iff $(m,k) \mid n$.*

For technical reasons, in connection with Theorem 2.B, we define $G(m,n) := \{p^{i+1}\colon\ p \text{ prime},\ i = \operatorname{expt}(n,p) < \operatorname{expt}(m,p)\} \cup \{i\colon\ i = 0 = m\}, \quad m \geq 0,\ n \geq 1$. Note that $\{i\colon\ i = 0 = m\}$ is $\{0\}$ or $\emptyset$, and $G(m,n) = \emptyset$ if m divides n.

For $n \geq 2$, an n-diamond in a modular lattice L is defined to be an $(n+1)$-tuple $\vec{a} = \langle a_0, a_1, \ldots, a_n\rangle \in L^{n+1}$ satisfying $\bigvee_{i\neq j}^{0,n} a_i = 1_{\vec{a}}$ and $a_\ell \wedge \bigvee_{i\neq k,\ell}^{0,n} a_i = 0_{\vec{a}}$ for all j and all $k \neq \ell$, where $1_{\vec{a}} = \bigvee_i^{0,n} a_i$ and $0_{\vec{a}} = \bigwedge_i^{0,n} a_i$. This concept is due to András Huhn [Hn1], [HH2] (who calls it an $(n-1)$-diamond.) Huhn has also shown that, for any n-diamond $\vec{a}$ in a modular lattice either $a_0 = a_1 = \ldots = a_n$ or $|\{a_0, a_1, \ldots, a_n\}| = n+1$. In the former case, when all components of $\vec{a}$ are equal, the diamond is called trivial.

DEFINITION 2.2. ([Cz2], [Cz9]) Let λ: $p(x_1, \ldots, x_t) = q(x_1, \ldots, x_t)$ be a lattice identity. The conjunction of λ and the modular law is called a *diamond identity*, cf. [Cz2], if there are $(n+1)$-ary lattice terms $c_1(y_0, y_1, \ldots, y_n), \ldots, c_t(y_0, y_1, \ldots, y_n)$ for some $n \geq 2$ such that for an arbitrary modular lattice L if $p(c_1(\vec{a}), \ldots, c_t(\vec{a})) = q(c_1(\vec{a}), \ldots, c_t(\vec{a}))$ for every n-diamond $\vec{a}$ in L then λ holds in L.

The distributive law dist: $x_1 \wedge (x_2 \vee x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ is the most natural example of a diamond identity. Indeed, dist holds in a modular lattice L iff L includes no 2-diamond iff dist holds for the elements of every 2-diamond of L; thus we can put $n = 2$, $t = 3$ and c_i the $(i-1)$-th projection for $i = 1, 2, 3$. The modular law is clearly another example. Thanks to András Huhn's connection between n-diamonds and the so-called n-distributive identities, cf. Huhn [Hn1, HH2], the conjunction of the n-distributive identity with modularity is also a diamond identity. At this point it is worth mentioning the well-known folklore fact that the conjunction of two (or finitely many) lattice identities is always equivalent to an appropriate single lattice identity, modulo lattice theory. Indeed, the conjunction $p_1(x_1, \ldots, x_k) \leq q_1(x_1, \ldots, x_k)$ and $p_2(y_1, \ldots, y_\ell) \leq q_2(y_1, \ldots, y_\ell)$ is clearly equivalent to

$$p_1(x_1, \ldots, x_k) \vee p_2(y_1, \ldots, y_\ell) \leq q_1(x_1, \ldots, x_k) \vee q_2(y_1, \ldots, y_\ell).$$

Huhn [Hn1] has shown that n-diamonds are projective in the variety of modular lattices. This means that there are lattice terms $g_0(x_0, \ldots, x_n), \ldots, g_n(x_0, \ldots, x_n)$ such that for any modular lattice L and any $\vec{a} \in L^{n+1}$, $\langle g_0(\vec{a}), \ldots, g_n(\vec{a})\rangle$ is an n-diamond, and if $\vec{b} \in L^{n+1}$ is an n-diamond then $g_i(\vec{b}) = b_i$ for all i. (He has also given concrete lattice terms $g_0(x_0,\dots,x_n)$, …, $g_n(x_0,\dots,x_n)$ for this purpose, but we will need only the existence of these terms in the sequel.) Therefore with any lattice identity μ: $p(x_1,\dots,x_t)=q(x_1,\dots,x_t)$ and any choice of $n\ge 2$ and $(n+1)$-ary lattice terms c_1, …, c_t we can associate a diamond identity λ in a very natural way. This λ will hold only in modular lattices and it will hold in a modular lattice L if and only if for any n-diamond $\vec a\in L^{n+1}$ we have $p(c_1(\vec a),\dots,c_t(\vec a))=q(c_1(\vec a),\dots,c_t(\vec a))$. This condition, i.e. requiring the satisfaction of μ for certain elements of n-diamonds, looks just a Horn sentence at first sight. Yet, thank to the projectivity of n-diamonds, it is equivalent to a lattice identity, namely the conjunction of the modular law and

$$p(c_1(g_0(\vec x),\dots,g_n(\vec x)),\dots,c_t(g_0(\vec x),\dots,g_n(\vec x)))=\\ q(c_1(g_0(\vec x),\dots,g_n(\vec x)),\dots,c_t(g_0(\vec x),\dots,g_n(\vec x)))$$

where $\vec x$ stands for $x_0,x_1,\dots,x_n$. The conjunction of the modular law with any of the identities in [HC1], [Cz3], Herrmann and Huhn [HH3], and Freese and McKenzie [FM1, Ch. XIII] is also an interesting example of diamond identities.

One of our main results is the following

Theorem 2.1. *([Cz9]) Let Σ be a set of lattice identities with $\Sigma\models$ modularity and let λ be a diamond identity. Then the following five conditions are equivalent*

(i) $\Sigma\models_c\lambda$,
(ii) $\Sigma\models_c\lambda\ (*;Q^s)$,
(iii) $\Sigma\models_c\lambda\ (r;Q^s,P)$,
(iv) $\Sigma\models_c\lambda\ (\le;Q^s)$,
(v) $\{0\}\cap\{m_\lambda\}\subseteq\{m_\varepsilon\colon\ \varepsilon\in\Sigma\}$, *and for any prime p if* $\mathrm{expt}(m_\lambda,p)>\mathrm{expt}(n_\lambda,p)$ *then* $\mathrm{expt}(n_\lambda,p)\ge\mathrm{expt}(n_\varepsilon,p)<\mathrm{expt}(m_\varepsilon,p)$ *holds for some* $\varepsilon\in\Sigma$.

After proving this theorem, we will give a simpler (commutator theoretic) approach to the equivalence of (i) and (v). To unify the treatment for several kinds of congruences, another consequence relation is worth introducing. Let T be a "set" of lattice varieties. We say that $\Sigma\models^T\lambda$ if for every $U\in T$ if Σ holds in U then so does λ. Now, in virtue of Lemma 2.2, Theorem 2.1 will clearly follow from

Theorem 2.2. *([Cz9]) Let Σ be a set of lattice identities with $\Sigma\models$ modularity and let λ be a diamond identity. Let T be a set of lattice varieties such that each U in T is generated by a class satisfying FJAP and $V(k)\in T$ for all $k\ge 0$. Then $\Sigma\models^T\lambda$ if and only if (v) of Theorem 2.1 holds.*

The key to this theorem is the following generalization of Freese [Fr3] (when λ is the distributive law, cf. also Freese, Herrmann and Huhn [FHH, Cor. 14]) and [Cz2, Thm. 1].

Theorem 2.3. *([Cz9]) Let T be as in Theorem 2.2, and let $U\in T$. Suppose that a diamond identity λ does not hold in U and U consists of modular lattices. Then there is an h in $G(m_\lambda,n_\lambda)$ such that $V(h)$ is a subvariety of U.*

Before starting our proof we need some further tools. For a prime power p^k let $R(p,k)$ denote $\mathbf{Z}_{p^k}$, the factor ring of integers modulo p^k. Let $R(p,\infty)$ denote the ring of rational numbers whose denominator is not divisible by p, and let

$R(0,1) := \mathbf{Q}$, the ring of rational numbers. For any of these rings $R(u,v)$, let $L(u,v,n)$ be the lattice of submodules of ${}_{R(u,v)}R(u,v)^n$. One of the main tools we need is taken from Herrmann [He1]:

Theorem 2.C. *([He1]) Every subdirectly irreducible modular lattice which is generated by an n-diamond is isomorphic or dually isomorphic to one of the following lattices: $L(p,k,n)$ for a prime power p^k, $L(p,\infty,n)$ for a prime p, or $L(0,1,n)$.*

Note that an important particular case of this theorem was proved in Herrmann and Huhn [HH2], which could also be used for our purposes in virtue of Freese, Herrmann and Huhn [FHH1, Prop. 12].

Proof of Theorem 2.3. Suppose the assumptions of the theorem hold, and let U_0 be a class of lattices which satisfies FJAP and generates the variety U. For a lattice identity ε let ε^{d} denote the dual of ε. For a prime p let $V(p^\infty) := H\mathcal{L}(R(p,\infty))$. Then $V(p^k) = H\mathcal{L}(R(p,k))$ for every $p \in \{0\} \cup \{\text{primes}\}$ and $1 \le k \le \infty$. Since λ fails in U, there is an integer $f > 1$, an $M = M_f \in U_0$, and an f-diamond $\vec{a}$ in M such that λ fails in the sublattice $L = L_f$ generated by (the elements $a_0, a_1, \dots, a_f$ of) $\vec{a}$. By Freese, Herrmann and Huhn [FHH1, Lemma 11], by the equivalence of n-diamonds with dual n-diamonds (cf. Huhn [Hn2]) and by the equivalence of von Neumann n-frames with n-diamonds (cf. Herrmann and Huhn [HH2, (1.7)]) we obtain that for any integer $g \ge f$ there is a lattice $M_g \in U_0$, a sublattice L_g generated by a g-diamond in M_g and an embedding $\varphi\colon M \to M_g$ such that the restriction $\varphi|_L$ of φ is an $L \to L_g$ embedding. Clearly, for every $g \ge f$, λ fails in L_g and $L_g \in U$. Decomposing L_g as a subdirect product of subdirectly irreducible lattices, every factor will be generated by a g-diamond, namely by the image of the original diamond under the natural projection. These subdirect factors belong to U and at least one them fails λ. Therefore (up to notational changes) we may assume that the $L_g \in U$ are subdirectly irreducible.

By Hutchinson's duality result [HC1, Thm. 7] the congruence varieties $H\mathcal{L}(R)$ are selfdual lattice varieties. Therefore, thanks to congruence permutability and strong Mal'cev conditions associated with an arbitrary lattice identity ε and its dual (cf. Wille [Wi1] or Pixley [Pi1], or for a more explicit form [HC1, Thm. 1]),

(1) there is an integer $r(\varepsilon)$ such that, for any ring R, ε holds in $H\mathcal{L}(R)$ iff ε holds in $\mathrm{Con}({}_RR^n)$ for some $n \ge r(\varepsilon)$ iff ε^{d} holds in $\mathrm{Con}({}_RR^n)$ for some $n \ge r(\varepsilon)$.

For $b \in \{0,1,2,\dots\} \cup \{p^\infty\colon\ p \text{ prime}\}$ and $a \in \{0,1,2,\dots\}$ we define the "generalized greatest common divisor" as follows:

$$(a,b)' := \begin{cases} 0, & \text{if } b = 0 \text{ and } a = 0 \\ 1, & \text{if } b = 0 \text{ and } a > 0 \\ (a,b), & \text{if } b \in \{1,2,3,\dots\} \\ p^{\mathrm{expt}(a,p)}, & \text{if } b = p^\infty \text{ and } a > 0 \\ p^\infty, & \text{if } b = p^\infty \text{ and } a = 0. \end{cases}$$

Note that $(-,-)'$ is not a commutative operation, and p^∞ divides no positive integer. Combining (1) and Theorems 2.A and 2.B we obtain for any $p \in \{0\} \cup \{\text{primes}\}$ and any $1 \le k \le \infty$:

(2) Suppose $n \ge r(\varepsilon)$. Then ε holds in $V(p^k)$ iff ε holds in $L(p,k,n)$ iff ε holds in the dual of $L(p,k,n)$ iff $D(m_\varepsilon, n_\varepsilon)$ holds in $R(p,k)$ iff $(m_\varepsilon, p^k)' \mid n_\varepsilon$.

By Theorem 2.C, each of the $L_g \in U \quad (g \geq f)$ is of the form $L(p_g, k_g, g)^{u_g}$ where $p_g \in \{0\} \cup \{\text{primes}\}$, $1 \leq k_g \leq \infty$, $u_g \in \{0,1\}$, and $k_g = 1$ when $p_g = 0$. Here $L(p_g, k_g, g)^1 := L(p_g, k_g, g)$ and $L(p_g, k_g, g)^0 := L(p_g, k_g, g)^{\mathrm{d}}$, the dual of $L(p_g, k_g, g)$. Since λ fails in L_g, we conclude from (2) that

(3) For $g \geq f$ we have that $(m_\lambda, p_g^{k_g})'$ does not divide n_λ.

For $q \in \{0\} \cup \{\text{primes}\}$ let $J_q := \{g\colon\ g \geq f \text{ and } p_g = q\}$. Now the proof ramifies depending on m_λ.

Assume first that $m_\lambda = 0$. Suppose J_0 is infinite, and let ε be an identity which holds in U. Then ε holds in $L(0,1,g)^{u_g}$ for infinitely many g. (2) yields that $m_\varepsilon > 0$, whence ε holds in $V(0)$ by (2). Thus $V(0) \subseteq U$, and $0 \in G(m_\lambda, n_\lambda)$.

Suppose J_q is infinite for some $q > 0$ and let $i := \operatorname{expt}(n_\lambda, q)$. Then $k_g > i$ for $g \in J_q$ and $q^{i+1} \in G(m_\lambda, n_\lambda)$ by (3). Suppose an identity ε holds in U. Taking a sufficiently large $g \in J_q$ we conclude from (2) that ε holds in $V(q^{k_g})$. But $(m_\varepsilon, q^{k_g})' \mid n_\varepsilon$ implies $(m_\varepsilon, q^{i+1})' \mid n_\varepsilon$, whence ε holds in $V(q^{i+1})$ by (2). This shows that $V(q^{i+1}) \subseteq U$.

Suppose now that J_q is finite for every $q \in \{0\} \cup \{\text{primes}\}$. Then $\{p_g\colon\ g \geq f\}$ is an infinite set of primes. By (2), no divisibility condition of the form $D(0,t)$ can hold in each of the rings $R(p_g, k_g) \quad (g \geq f)$. Consequently, if $m_\varepsilon = 0$ for a lattice identity ε then ε does not hold in U. Thus $m_\varepsilon > 0$ for all ε that hold in U, and these ε hold in $V(0)$ by (2). We have obtained that $V(0) \subseteq U$ and, of course, $0 \in G(m_\lambda, n_\lambda)$.

Now let us assume that $m_\lambda > 0$. First observe by Theorem 2.B that for distinct primes p, q and any $0 \leq k \leq \infty$ the divisibility condition $D(q^{\ell+1}, q^\ell)$ holds in $R(p,k)$ for all $\ell \in \{0,1,2,\dots\}$.

Hence, by (3), (2) and Theorem 2.B, we conclude that, for every $g \geq f$, $\operatorname{expt}(m_\lambda, p_g) > \operatorname{expt}(n_\lambda, p_g)$ but $D(p_g^{\operatorname{expt}(n_\lambda,p_g)+1}, p_g^{\operatorname{expt}(n_\lambda,p_g)})$ fails in $R(p_g, k_g)$. Hence, by Theorem 2.B, we conclude $i := \operatorname{expt}(p_g, n_\lambda) < k_g$ for all $g \geq f$. On the other hand, $\operatorname{expt}(m_\lambda, p) > \operatorname{expt}(n_\lambda, p)$ can hold for finitely many primes p only, whence there is a prime q such that J_q is infinite. I.e., U contains $L(q, k_g, g)^{u_g}$ for infinitely many g. Suppose ε holds in U and choose a $g \in J_q$ with $g \geq r(\varepsilon)$. From (2) we obtain $(m_\varepsilon, q^{k_g})' \mid n_\varepsilon$, whence $(m_\varepsilon, q^{i+1})' \mid n_\varepsilon$, implying that ε holds in $V(q^{i+1})$. We have obtained $V(q^{i+1}) \subseteq U$, and evidently q^{i+1} belongs to $G(m_\lambda, , n_\lambda)$. $\square$

Proof of Theorem 2.2. Let us assume that $\Sigma \models^T \lambda$ and the conditions of the theorem are fulfilled. If $m_\lambda = 0$ but $m_\varepsilon > 0$ for all $\varepsilon \in \Sigma$ then, by Theorems 2.A and 2.B, Σ would hold but λ would fail in $V(0) \in T$. This is not the case and we conclude that $\{0\} \cap \{m_\lambda\} \subseteq \{m_\varepsilon\colon\ \varepsilon \in \Sigma\}$. If $\operatorname{expt}(m_\lambda, p) > \operatorname{expt}(n_\lambda, p) = i$ then, by Theorems 2.A and 2.B, λ and therefore Σ fails in $V(p^{i+1}) \in T$. Therefore, again by Theorems 2.A and 2.B, there exists an $\varepsilon \in \Sigma$ with $\operatorname{expt}(n_\lambda, p) = i \geq \operatorname{expt}(n_\varepsilon, p) < \operatorname{expt}(m_\varepsilon, p)$, proving (v).

Now assume that (v) holds but $\Sigma \models^T \lambda$ fails. Therefore there is a $U \in T$ such that λ fails in U but Σ holds in U. By Theorem 2.3, $V(h) \subseteq U$ for some $h \in G(m_\lambda, n_\lambda)$. Clearly, Σ holds in $V(h)$. If $h = 0 = m_\lambda$ then $m_\varepsilon = 0$ for some $\varepsilon \in \Sigma$ by (v). Hence, by Theorems 2.A and 2.B, ε cannot hold in $V(h)$. Therefore $h = p^{i+1}$ where $i = \operatorname{expt}(n_\lambda, p) < \operatorname{expt}(m_\lambda, p)$ for some p. By (v) there is an $\varepsilon \in \Sigma$ with $i \geq \operatorname{expt}(n_\varepsilon, p) < \operatorname{expt}(m_\varepsilon, p)$. Consequently, by Theorems 2.A and 2.B, ε cannot hold in $V(h)$; a contradiction again. $\square$

"Proof" of Proposition 2.1. Now we outline how the proposition can be shown. The quotation marks around the word proof indicate that the result was not formulated with the usual mathematical preciseness. Most of the $\Sigma \models_c \lambda$ statements in the scope of Proposition 2.1 are settled by Theorem 2.1; there are only two exceptions, up to the author's present knowledge. It is shown in Freese and Jónsson [FJ1] that mod $\models_c$ Arguesian law. In Freese, Herrmann and Huhn [FHH1], some identities $\gamma_{n,m}(w_k)$ (n odd, $n > 1$, $k > 1$), even stronger than the Arguesian law, are constructed and it is shown that mod $\models_c \gamma_{n,m}(w_k)$. Fortunately, the proof of these results is based on FJAP. Therefore Proposition 2.1 holds for these cases, too. $\square$

Now we return to the consequence relation considered only in congruence varieties. Of course, in virtue of Proposition 2.1, with the additional assumption of modularity the forthcoming results of this chapter apply to the consequence relations $\models_c \ (r; Q^s, P)$, $\models_c \ (*; Q^s)$ and $\models_c \ (\leq; Q^s)$ as well.

Theorem 2.4. *([Cz2]) Let Σ be an arbitrary set of lattice identities. Then the following two conditions are equivalent*

(i) $\Sigma \models_c \lambda$,
(ii) $\{0\} \cap \{m_\lambda\} \subseteq \{m_\varepsilon\colon \ \varepsilon \in \Sigma\}$, *and for any prime p if* $\mathrm{expt}(m_\lambda, p) > \mathrm{expt}(n_\lambda, p)$ *then* $\mathrm{expt}(n_\lambda, p) \geq \mathrm{expt}(n_\varepsilon, p) < \mathrm{expt}(m_\varepsilon, p)$ *holds for some* $\varepsilon \in \Sigma$, *and* $\Sigma \models_c$ *mod.*

Proof. Although this result clearly follows from Theorem 2.1, we are going to present a simpler proof. In fact, we are going to prove the following special case of Theorem 2.3:

(4) Let U be a modular congruence variety such that a diamond identity λ does not hold in U. Then there is an h in $G(m_\lambda, n_\lambda)$ such that $V(h)$ is a subvariety of U.

This will be sufficient, for (4) yields Theorem 2.4 the same easy way as Theorem 2.3 implies Theorem 2.1.

Let W be a variety of algebras such that U is the congruence variety of W. Then there is an n-diamond $\vec{a}$ in the congruence lattice $\mathrm{Con}(A)$ of some algebra A in W such that λ fails in the interval $L = [0_{\vec{a}}, 1_{\vec{a}}]$ of $\mathrm{Con}(A)$. Here $0_{\vec{a}}$ and $1_{\vec{a}}$ denotes $\bigwedge_{i=0}^{n} a_i$ and $\bigvee_{i=0}^{n} a_i$, respectively. We can assume that $0_{\vec{a}}$ is the least element of $\mathrm{Con}(A)$ as otherwise A could be replaced by $A/0_{\vec{a}} \in W$. By Day and Kiss [DK1, Lemma 3.1], $1_L = 1_{\vec{a}}$ is an Abelian congruence of A. Therefore Lemma 7.1 and Theorem 7.2 of Day and Kiss [DK1] yield the existence of a ring S such that $L \in HSP(\mathrm{Con}(S\text{-}\mathbf{Mod})) \subseteq HSP(\mathrm{Con}(W)) = U$. Since λ fails in $H\mathcal{L}(R) = HSP(\mathrm{Con}(S\text{-}\mathbf{Mod})$, a routine calculation based on Theorems 2.A and 2.B and the description of the inclusion relation amongst all $HSP(\mathrm{Con}(R\text{-}\mathbf{Mod}))$, cf. [HC1, Theorem 5] yields (4). $\square$

Armed with Theorem 2.4 now we can say something about the compactness and effectiveness of the consequence relation in congruence varieties.

Theorem 2.5. *([Cz2]) The consequence relation in congruence varieties is compact at any diamond identity. I.e., if Σ is a set of lattice identities, λ is a* *diamond identity and $\Sigma \models_c \lambda$ then there exists a finite subset Σ' of Σ such that $\Sigma' \models_c \lambda$.*

PROOF. Assume that $\Sigma \models_c \lambda$. The condition, in fact a part of Theorem 2.4.(ii),

> "for any prime p if $\mathrm{expt}(m_\lambda, p) > \mathrm{expt}(n_\lambda, p)$ then $\mathrm{expt}(n_\lambda, p) \geq \mathrm{expt}(n_\varepsilon, p) < \mathrm{expt}(m_\varepsilon, p)$ holds for some $\varepsilon \in \Sigma$"

will be denoted by (5). By a very deep result of Day and Freese [DF1, Theorem 6,4] there is a $\kappa \in \Sigma$ such that $\kappa \models_c$ modularity. If $m_\lambda = 0$ then, by Theorem 2.4, there is an $\eta \in \Sigma$ with $m_\eta = 0$. This η can serve (5) for all primes not dividing n_η. Hence there is a finite set Σ_1 such that $\eta \in \Sigma_1 \subseteq \Sigma$ and (5) is fulfilled by Σ_1. Clearly, $\{\kappa\} \cup \Sigma_1 \models_c \lambda$. If $m_\lambda \neq 0$ then (5) requires the existence of an $\varepsilon = \varepsilon_p$ for finitely many p only. These ε_p constitute a finite set Σ_2 and $\{\kappa\} \cup \Sigma_2 \models_c \lambda$. $\square$

After having generalized [Cz12] and Day and Freese [DF1] to diamond identities in the previous theorem, we are going to do the same with [CF1] in the following

THEOREM 2.6. *([Cz2]) There is an algorithm which for any diamond identity λ and any finite set Σ of lattice identities decides if $\Sigma \models_c \lambda$.*

PROOF. Although the present consideration seems quite short, it is just because our algorithm invokes two other ones. If we gave detailed descriptions of these two algorithms (even without proofs) here then the current proof would be several pages long. To test whether $\Sigma \models_c \lambda$ we first test if $\Sigma \models_c$ mod according to Freese [CF1]. If not then $\Sigma \models_c \lambda$ does not hold. If λ has successfully passed the first test then we calculate the integers m_λ and n_λ, and m_ε and n_ε for each $\varepsilon \in \Sigma$, using the algorithm given in [HC1], which we have already mentioned in connection with Theorem 2.A. Having access to these numbers now it is easy to check if Theorem 2.4(ii) holds, for we have to consider finitely many primes p only. If we find that 2.4(ii) holds then $\Sigma \models_c \lambda$, otherwise not. $\square$

We do not know if Theorems 2.5 and 2.6 hold for some lattice identities λ which are not diamond identities or if they hold for all λ. It is interesting to draw a parallel between the $\models_c$ relation studied at λ and the existence of a Mal'cev condition characterizing the satisfaction of λ in the congruence variety of a given variety. It is shown in Freese and McKenzie [FM1, Chapter XIII] that diamond identities are characterizable by Mal'cev conditions. (Note that Freese and Mckenzie [FM1] consider some identities seemingly different from diamond identities but, using the equivalence of Huhn diamonds with von Neumann's frames, it is not hard to see that their identities are just diamond identities modulo lattice theory.) Moreover, all lattice identities that are known to be characterizable by Mal'cev conditions are equivalent to diamond identities in congruence variety sense. (I.e., if μ is known to be characterizable by a Mal'cev condition then $\mu \models_c \lambda$ and $\lambda \models_c \mu$ holds for an appropriate diamond identity λ.) The question whether every diamond identity can be characterized by Mal'cev condition is a longstanding open problem. In principle, there might be a connection between lattices characterizable by Mal'cev conditions and those for which Theorem 2.5 and/or 2.6 hold, perhaps these lattice identities are the same, but our knowledge is zero in this respect.

Now we set out to enlarge the list of previously known nontrivial $\mu \models_c \lambda$ results. (The results we are going to present were, of course, taken into account when formulating Proposition 2.1.)

For an integer $n > 2$ and a modular lattice L, a system

$$\vec{f} = (a_i, c_{ij}\colon\ 1 \le i \le n,\ \ 1 \le j \le n,\ \ i \ne j)$$

of elements of L is called a (von Neumann) n-frame in L if $a_j \sum_{i \ne j} a_i = 0_{\vec{f}}$, $c_{jk} = c_{kj}$, $a_j c_{jk} = 0_{\vec{f}}$, $a_j + c_{jk} = a_j + a_k$ and $c_{jk} = (a_j + a_k)(c_{j\ell} + c_{\ell k})$ for all distinct $j, k, \ell \in \{1, 2, \dots, n\}$ where $0_{\vec{f}}$ resp. $1_{\vec{f}}$ are the meet resp. the join of all elements of $\vec{f}$, cf. von Neumann [Ne1]. Here and often in the sequel we write $x + y$ resp. xy for the join resp. meet of x and y. On the set $\{x_1, \dots, x_4\}$ of variables we define lattice terms e_k and f_k for $k \ge 0$ by the following recursion equations:

$$e_0 = x_1, \qquad e_{k+1} = (f_{k+1} + (x_1 + x_2)(x_3 + x_4))(x_1 + x_3),$$
$$f_0 = x_2, \qquad f_{k+1} = (e_k + x_4)(x_2 + x_3).$$

(By substitution, we can obtain recursion relations expressing e_{k+1} from e_k, x_1, x_2, x_3 and x_4, and the same is true for f_{k+1}.) Denoting $x_2 + e_n + f_m$ by $q_{mn}(x_1, \dots, x_4)$ let $\Delta(m, n)$ stand for the lattice identity

$$(x_1 + x_2)(x_3 + x_4) \le q_{mn}(x_1, x_2, x_3, x_4).$$

This is just Hutchinson's identity defined in [HC1, page 289]. It is shown in [HC1, Proposition 6] that $m_{\Delta(m,n)} = m$ and $n_{\Delta(m,n)} = n$, cf. Theorem 2.A of the present chapter to appreciate this fact.

Frames are projective in the variety of modular lattices. This was proved in two steps; first for n-diamonds in Huhn [Hn1] (for a more explicit statement cf. the remarks after the definition of diamond identities in the present chapter or Freese [Fr4]), and then frames and diamonds turned out to be equivalent in Herrmann and Huhn [HH2, page 104]. (The equivalence of these two notions means that the finitely presented modular lattices determined by them are isomorphic.) Therefore there are lattice terms $b_i(\vec{x})$ and $d_{ij}(\vec{x})$ in variables $\vec{x} = (x_i, x_{ij}\colon\ 1 \le i \le k,\ 1 \le j \le k,\ i \ne j)$ such that these terms produce a k-frame $(b_i(\vec{y}), d_{ij}(\vec{y})\colon\ 1 \le i \le k,\ 1 \le j \le k,\ i \ne j)$ from any system $\vec{y}$ of elements in a modular lattice L and, in addition, if $\vec{f} = (a_i, c_{ij}\colon\ 1 \le i \le k,\ 1 \le j \le k,\ i \ne j)$ is a k-frame in L then $b_i(\vec{f}) = a_i$ and $d_{ij}(\vec{f}) = c_{ij}$ for every i and $j \ne i$.

For $k \ge 4$ the conjunction of the modular law and the identity

$$(d_{13}(\vec{x}) + d_{23}(\vec{x}))(d_{14}(\vec{x}) + d_{24}(\vec{x})) \le q_{mn}(d_{13}(\vec{x}), d_{23}(\vec{x}), d_{14}(\vec{x}), d_{24}(\vec{x})),$$

where $\vec{x} = (x_i, x_{ij}\colon\ 1 \le i \le k,\ 1 \le j \le k,\ i \ne j)$, will be denoted by $\Delta(m, n, k)$. Clearly, $\Delta(m, n, k)$ is equivalent to a single identity modulo lattice theory. Note that $\Delta(m, n, k)$ holds in a modular lattice L iff for any k-diamond $\vec{f} = (a_i, c_{ij}\colon\ 1 \le i \le k,\ 1 \le j \le k,\ i \ne j)$ in L the identity $\Delta(m, n)$ holds when c_{13}, c_{23}, c_{14} and c_{24} are substituted for its variables x_1, x_2, x_3 and x_4, respectively.

THEOREM 2.7. *([Cz3]) Consider arbitrary integers $m', m_i \ge 0$, $n', n_i \ge 1$, and $k', k_i \ge 4$ $(i \in I)$ where I is an index set. Then $\{\Delta(m_i, n_i, k_i\colon\ i \in I\} \models_c \Delta(m', n', k')$ if and only if $\{D(m_i, n_i)\colon\ i \in I\}$ implies $D(m', n')$ in the class of rings with 1.*

It is evident from this theorem that we have

COROLLARY 2.1. *([Cz3]) If m does not divide n and $k \geq 5$ then*
$$\Delta(m,n,k) \models_c \Delta(m,n,k-1).$$

To show that the $\models_c$ in Corollary 2.1 is nontrivial we formulate

PROPOSITION 2.2. *([Cz3]) If m does not divide n and $k \geq 5$ then $\Delta(m,n,k) \not\models \Delta(m,n,k-1)$.*

To point out that (some of) the identities occurring in Corollary 2.1 (and Proposition 2.2) are distinct in a very strong sense we present the following

PROPOSITION 2.3. *([Cz3]) For any $k \geq 4$ the set $\{\Delta(p,1,k)\colon\ p$ is prime $\}$ is independent in congruence varieties in the sense that for every prime q*

$$\{\Delta(p,1,k)\colon\ p \text{ prime, } p \neq q\} \not\models_c \Delta(q,1,k).$$

PROOF OF THEOREM 2.7. As we have already mentioned, n-diamonds and n-frames are equivalent (Herrmann and Huhn [HH2, page 104]). Thus, we can conclude that our lattice identities $\Delta(m,n,k)$ are diamond identities. What Theorem 2.4 asserts can be formulated less technically as follows:

(6) For any diamond identity λ, $\Sigma \models_c \lambda$ iff for any ring R with 1 if Σ holds in $H\mathcal{L}(R) = HSP(\mathrm{Con}(R\text{-}\mathbf{Mod}))$ then so does λ.

Note that (6) is implicit in the proof of Theorem 2.4. Therefore, in virtue of Theorem 2.A, it suffices to show that $D(n_{\Delta(m,n,k)}, m_{\Delta(m,n,k)})$ is equivalent to $D(m,n)$ for any meaningful m,n,k. Although there is a method in [HC1] for calculating $n_{\Delta(m,n,k)}$ and $m_{\Delta(m,n,k)}$, this would be hopelessly too complicated in the present situation. But fortunately $n_{\Delta(m,n)} = n$ and $m_{\Delta(m,n)} = m$, so by Theorem 2.A it suffices to show that, for any R, $\Delta(m,n)$ holds in $H\mathcal{L}(R)$ iff $\Delta(m,n,k)$ holds in $H\mathcal{L}(R)$. Clearly, $\Delta(m,n)$ implies $\Delta(m,n,k)$ in $H\mathcal{L}(R)$, in fact in any lattice.

Conversely, assume that $\Delta(m,n,k)$ holds in $H\mathcal{L}(R)$. Let $M = M(u_1, \dots, u_k)$ denote the R-module freely generated by $\{u_1, \dots, u_k\}$. Then $\Delta(m,n,k)$ holds in $\mathrm{Sub}(M)$, the submodule lattice of M, for $\mathrm{Sub}(M) \cong \mathrm{Con}(M)$. It is easy to see (or cf. von Neumann [Ne1]) that the cyclic submodules $(Ru_i, R(u_i - u_j)\colon\ 1 \leq i \leq k,\ 1 \leq j \leq k,\ i \neq j)$ constitute a k-frame in $\mathrm{Sub}(M)$. (In fact, this is the most typical example of a k-frame.) Therefore

$$\begin{aligned}(R(u_1-u_3)+R(u_2-u_3))(R(u_1-u_4)+R(u_2-u_4)) \leq \\ q_{mn}(R(u_1-u_3), R(u_2-u_3), R(u_1-u_4), R(u_2-u_4))\end{aligned} \tag{7}$$

holds in $\mathrm{Sub}(M)$ and even in $\mathrm{Sub}(M(u_1,u_2,u_3,u_4))$. Now the theory of Mal'cev conditions (cf. Wille [Wi1] or Pixley [Pi1]) together with the canonical isomorphism between $\mathrm{Sub}(M(u_1,u_2,u_3,u_4))$ and the congruence lattice of $M(u_1,\, u_2,\, u_3,\, u_4)$ easily yield that $\Delta(m,n)$ holds in $H\mathcal{L}(R)$. (Indeed, in congruence permutable case the strong Mal'cev condition characterizing $\Delta(m,n)$ is derived from the satisfaction of $\Delta(m,n)$ in $\mathrm{Con}(M(u_1,u_2,u_3,u_4))$ when the four principal congruences generated by the respective pairs $\langle u_1,u_3\rangle$, $\langle u_2,u_3\rangle$, $\langle u_1,u_4\rangle$ and $\langle u_2,u_4\rangle$ are substituted for the variables of $\Delta(m,n)$. For further details the reader may turn to the first nine rows in the proof of [HC1, Proposition 6] where it is shown that (7) implies the satisfaction of $\Delta(m,n)$ in $H\mathcal{L}(R)$.) $\square$

Proof of Proposition 2.2. Let $\mathbf{Z}$ be the ring of integers. Then $\Delta(m,n)$ does not hold in $H\mathcal{L}(\mathbf{Z})$ by Theorem 2.A. We have shown in the previous proof that, with k instead of $k-1$, $\Delta(m,n,k-1)$ implies $\Delta(m,n)$ in any $H\mathcal{L}(R)$. This applies for $R=\mathbf{Z}$ and we conclude that $\Delta(m,n,k-1)$ fails in $H\mathcal{L}(\mathbf{Z})$. It is shown in Herrmann and Huhn [HH3, Satz 7] that the variety $H\mathcal{L}(\mathbf{Z})$ is generated by its finite members. Therefore there is a finite (and necessarily modular) lattice $L \in H\mathcal{L}(\mathbf{Z})$ with minimal number of elements such that $\Delta(m,n,k-1)$ fails in L. We intend to show that $\Delta(m,n,k)$ holds in L; the assertion then will follow. Assume the contrary. Then there is a k-frame $\vec{f}=(a_i, c_{ij}\colon\ 1\le i\le k,\ \ 1\le j\le k,\ \ i\ne j)$ in L such that $\Delta(m,n)$ fails when c_{13}, c_{23}, c_{14} and c_{24} are substituted for its variables. It is known that either all elements of a frame are equal or a_1, a_2, $\dots$, a_k are distinct atoms of a Boolean sublattice of length k, cf. e.g., Herrmann and Huhn [HH2, (iii) on page 101 and page 104]. Now only the latter is possible since the one element lattice satisfies any identity. Hence $a_1+\dots+a_{k-1}<a_1+\dots+a_k$. Consider the subframe $\vec{g}=(a_i, c_{ij}\colon\ 1\le i\le k-1,\ \ 1\le j\le k-1,\ \ i\ne j)$ and let L' be the interval $[0_{\vec{g}}, 1_{\vec{g}}]$. Since, by $k\ge 5$, the elements c_{13}, c_{23}, c_{14} and c_{24} belong to L', $\Delta(m,n,k-1)$ fails L'. But this contradicts the choice of L, for $1_{\vec{g}}=a_1+\dots+a_{k-1}<a_1+\dots+a_k$ implies $|L'|<|L|$. $\square$

Proof of Proposition 2.3. In virtue of Theorem 2.7 it suffices to show that $\{D(p,1)\colon\ p\ne q\}$ does not imply $D(q,1)$. Indeed, this is witnessed by the ring of those rational numbers whose denominator is not divisible by q. $\square$

As we have mentioned in the Introduction, all $\mu\models_c\lambda$ type results before Corollary 2.1 were located at modularity or distributivity. Corollary 2.1, taken from [Cz3], was the first example for nontrivial $\models_c$ results of a different kind. Later Freese [Fr2], which is still to appear, showed that if a modular congruence variety U contains a nondistributive lattice then U cannot be defined by a finite set of lattice identities. This and the existence of infinitely many (in fact continuously many by [HC1]) modular congruence varieties easily yield the *existence* of infinitely many nontrivial $\lambda\models_c\mu$ results which are pairwise distinct in the (stronger) $\models_c$ sense. Moreover, for each λ if $\lambda\not\models_c$ distributivity then there is a μ such that $\lambda\models_c\mu$ and but $\lambda\not\models\mu$. From one aspect, this is more than Corollary 2.1. On the other hand, Corollary 2.1 exhibits concrete lattice identities, which is more than proving their existence only.

CHAPTER III

A NON-SELFDUAL MODULAR CONGRUENCE VARIETY

Based on [Cz8], this chapter is to present an example of a non-selfdual modular congruence variety. For a ring R, always with with unit element, let $\mathcal{L}(R)$ denote the class of lattices embeddable in submodule lattices of R-modules. Then $\mathbf{H}\mathcal{L}(R)$, the variety generated by $\mathcal{L}(R)$, is a self-dual congruence variety by Hutchinson [HC1, Thm. 7] and [Ht4]. On the other hand, non-modular congruence varieties need not be self-dual by Day and Freese [DF1]. The $\mathbf{H}\mathcal{L}(R)$ had been the only known congruence varieties for a long time, which led to the impression that the congruence variety of Abelian groups, alias $\mathbf{H}\mathcal{L}(\mathbf{Z})$, was the largest modular congruence variety. This picture was refuted in two steps. First, an unpublished work of Kiss and Pálfy [KP1] showed that the congruence lattice of a certain metaabelian group cannot be embedded in the congruence lattice of any Abelian group.

I am indebted to Emil W. Kiss [Ki1] for drawing my attention to $\mathbf{Con}(\mathbf{M}_4)$ and for the very interesting conversations. In 1987, one of these conversations led to the idea that there is an algorithm to check if a given identity holds in the congruence variety of the variety generated by the quaternion group, and "only" the problem of finding appropriate identities remained. (This algorithm will be used to prove part (B) of the forthcoming theorem.)

Developing the ideas of [KP1] further, Pálfy and Szabó [PS1] and [PS2] have recently shown that the congruence varieties of certain group varieties are not subvarieties of $\mathbf{H}\mathcal{L}(\mathbf{Z})$. This leads to the problem whether every modular congruence variety is self-dual, cf. Pálfy and Szabó [PS2, Problem 4.2] for a slightly different formulation. The aim of the present chapter is to give a negative solution.

For a variety V let $\mathbf{Con}(V)$ denote the congruence variety of V, i.e., the lattice variety generated by the congruence lattices of all algebras in V. Let $\mathbf{M}$ be the variety of metaabelian groups. $\mathbf{M}$ is defined by the identity $[x,y]z = z[x,y]$ where $[x,y] = x^{-1}y^{-1}xy$. By the elementary properties of the commutator (cf., e.g., Gorenstein [Go1, Ch. 2.2]) it is easy to see that $\mathbf{M}$ satisfies the identities

$$
\begin{aligned}
[a,b]^{-1} &= [b,a] = [a^{-1},b] = [a,b^{-1}] \\
ba &= ab[a,b]^{-1} \\
[ab,c] &= [a,c][b,c], \quad [a,bc] = [a,b][a,c] \\
b^l a^k &= a^k b^l [a,b]^{-kl} \quad (k,l \in \mathbf{Z}).
\end{aligned}
\tag{3.1}
$$

Let $\mathbf{A}$ be the variety of Abelian groups and let $\mathbf{M}_4$ be the variety generated by the quaternion group. Then $\mathbf{M}_4$ is a subvariety of $\mathbf{M}$, and it is defined by the identities $[x,y]z = z[x,y]$, $x^4 = 1$ and $[x,y]^2 = 1$. Pálfy and Szabó [PS1] and [PS2] gave identities satisfied in $\mathbf{Con}(\mathbf{A})$ but not in $\mathbf{Con}(\mathbf{M}_4)$. However, the duals of their identities do the same, so we have to consider another identity. In the variables $\alpha_1, \alpha_2, \ldots, \alpha_{13}$ let us consider the following lattice terms:

$$p = (\alpha_{12} + \alpha_{13})\big(\alpha_4 + \alpha_5 + (\alpha_1 + \alpha_6 + \alpha_7)(\alpha_2 + \alpha_8 + \alpha_9)(\alpha_3 + \alpha_{10} + \alpha_{11})\big),$$

$$\begin{aligned}
q_1 &= \alpha_1 + \alpha_2 + \alpha_3, & q_2 &= \alpha_6 + \alpha_7 + \alpha_{12} + \alpha_{13},\\
q_3 &= \alpha_1 + \alpha_4 + \alpha_5 + \alpha_{10}, & q_4 &= \alpha_3 + \alpha_8 + \alpha_9,\\
q_5 &= \alpha_2 + \alpha_4 + \alpha_{10} + \alpha_{11} + \alpha_{12}, & q_6 &= \alpha_2 + \alpha_{11} + \alpha_{12} + \alpha_{13},\\
q_7 &= \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_9, & q_8 &= \alpha_1 + \alpha_3 + \alpha_5,\\
q_9 &= \alpha_6 + \alpha_7 + \alpha_8 + \alpha_{10} + \alpha_{11}, & q_{10} &= \alpha_3 + \alpha_6 + \alpha_9 + \alpha_{12} + \alpha_{13},\\
q_{11} &= \alpha_4 + \alpha_5 + \alpha_{10} + \alpha_{11}, & q_{12} &= \alpha_1 + \alpha_2 + \alpha_{13},\\
q_{13} &= \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9
\end{aligned}$$

and

$$q = q_1 + (q_2q_3 + q_4q_5)(q_6q_7 + q_8q_9)(q_{10}q_{11} + q_{12}q_{13}).$$

Let μ_{13} denote the identity

$$p \le q,$$

and let μ_{13}^d denote the dual of μ_{13} . Note that μ_{13} was found by modifying, in fact weakening, the dual of the identity in Pálfy and Szabó [PS1]. Therefore any lattice satisfying the identity of [PS1] also satisfies μ_{13}^d .

Theorem 3.1.

(A) μ_{13} *holds in* $\mathbf{Con}(\mathbf{M})$.
(B) μ_{13}^d *fails in* $\mathbf{Con}(\mathbf{M})$.

We will actually show that μ_{13}^d fails even in $\mathbf{Con}(\mathbf{M}_4)$. Therefore the modular congruence varieties $\mathbf{Con}(\mathbf{M})$ and $\mathbf{Con}(\mathbf{M}_4)$ are not self-dual.

Proof. (B) The rather long calculations required by this part of the proof were done by a personal computer; here we outline the algorithm only. The author has developed a Pascal program, Borland's Turbo Pascal 6.0. On the floppy disk attached to this dissertation, the program can be found in the SELFDUAL directory. It contains a distribution file METAP.DST. Having a Turbo Pascal compiler (DOS version, 6.0 or 7.0) and following the easy instructions written in the README.1ST file, the reader can easily make an executable program plus some data files from the distribution file. But even this is not necessary because, for convenience, the directory already contains this executable file, called METAP.EXE, plus four datafiles called MU13.PRI, MU13DUAL.PRI, DESARG.PRI and P3SZCS.PRI (the last named is the identity of Pálfy and Szabó).

As it is clear from the above list of data files, there is another application of this program: it offers a very short proof of Pálfy and Szabó's main result in [PS1] and [PS2]. Indeed, proving their result takes about a second on an AT286 personal computer.

The Wille — Pixley algorithm [Wi1, Pi1] offers a standard way to check if a lattice identity holds in the congruence variety of a variety with permuting congruences. Like in [HC1], we can construct a strong Mal'cev condition (MC) such that (MC) holds in $\mathbf{M}_4$ iff μ_{13}^d holds in $\mathbf{Con}(\mathbf{M}_4)$. (Cf. [HC1] for details on this construction.) This Mal'cev condition is a finite collection of n-ary term symbols f_k and equations of the form

$$f_l(x_{1C}, x_{2C}, \dots, x_{nC}) = f_r(x_{1C}, x_{2C}, \dots, x_{nC}) \tag{3.2}$$

or

$$f_l(x_{1C}, x_{2C}, \dots, x_{nC}) = x_j \tag{3.3}$$

where C is a partition on the set $\{1, 2, \dots, n\}$ and iC denotes the smallest element of the C-block containing i. Suppose μ_{13}^d holds in $\mathbf{Con}(\mathbf{M}_4)$, then there exist group terms f_k such that all the equations (3.2) and (3.3) of (MC) are valid identities in $\mathbf{M}_4$. By Pálfy and Szabó [PS2] or the identities (3.1) each n-ary group term $g(x_1, \dots, x_n)$ in $\mathbf{M}_4$ can uniquely be represented in the form

$$\prod_{i=1}^{n} x_i^{a_i} \prod_{i<j} [x_i, x_j]^{b_{ij}} \tag{3.4}$$

where $a_i \in \mathbf{Z}_4 = \{0, 1, 2, 3\}$ and $b_{ij} \in \mathbf{Z}_2 = \{0, 1\}$. Here

$$\prod_{i=1}^{n} x_i^{a_i} \quad \text{and} \quad \prod_{i<j} [x_i, x_j]^{b_{ij}}$$

are called the Abelian part and the commutator part of g, respectively.

The variety of Abelian groups of exponent four is a subvariety of $\mathbf{M}_4$, whence (MC) holds in it. Since $\mathbf{M}_4$ is term equivalent to the variety of modules over $\mathbf{Z}_4$, we can use the algorithm described in [HC1] to determine the $a_i^{(k)}$, the exponents occurring in the Abelian part of f_k according to (3.4). Luckily enough, these $a_i^{(k)}$ are uniquely determined by (MC).

Now let $C_1, \dots, C_w$ be the blocks of a partition C such that the minimal representatives $c_i \in C_i$ satisfy $c_1 < c_2 < \dots < c_w$. For a term g of the form (3.4) the term $g(x_{1C}, \dots, x_{nC})$ can be written in the (unique) form

$$\prod_{i=1}^{w} x_{c_i}^{d_i} \prod_{i<j} [x_{c_i}, x_{c_j}]^{t_{ij}}$$

Here $d_i = \sum_{j \in C_i} a_j$. To determine the t_{ij} for $i < j$ let us consider an $u \in C_i$ and a $v \in C_j$. If $u < v$ then $[x_u, x_v]^{b_{uv}}$ turns into $[x_{c_i}, x_{c_j}]^{b_{uv}}$. If $u > v$ then $[x_v, x_u]^{b_{vu}}$ turns into $[x_{c_j}, x_{c_i}]^{b_{vu}} = [x_{c_i}, x_{c_j}]^{-b_{vu}}$ and exchanging the places of $x_{c_j}^{a_v}$ and $x_{c_i}^{a_u}$ in the Abelian part enters $[x_{c_i}, x_{c_j}]^{-a_u a_v}$ as well. Combining all these effects we obtain that

$$t_{ij} = \sum_{\substack{u<v \\ u \in C_i,\ v \in C_j}} b_{uv} - \sum_{\substack{u>v \\ u \in C_i,\ v \in C_j}} (b_{vu} + a_u a_v). \tag{3.5}$$

Therefore, if the a_i and b_{ij} for f_k are denoted by $a_i^{(k)}$ and $b_{ij}^{(k)}$, (3.2) implies

$$\begin{aligned} &\sum_{\substack{u<v \\ u \in C_i,\ v \in C_j}} b_{uv}^{(l)} - \sum_{\substack{u>v \\ u \in C_i,\ v \in C_j}} (b_{vu}^{(l)} + a_u^{(l)} a_v^{(l)}) = \\ &\qquad \sum_{\substack{u<v \\ u \in C_i,\ v \in C_j}} b_{uv}^{(r)} - \sum_{\substack{u>v \\ u \in C_i,\ v \in C_j}} (b_{vu}^{(r)} + a_u^{(r)} a_v^{(r)}) \end{aligned} \tag{3.6}$$

for all meaningful $i<j$. The equations (3.6) and the analogous equations derived from (3.3) constitute a system of linear equations over the two-element field with the $b_{uv}^{(k)}$ being the unknowns. Using some reductions, including the one offered by [Cz1, Prop. 2] (to be mentioned also in the next chapter after Proposition 4.2) or its special case for groups [PS2, Lemma 1.1], the system eventually considered consists of 130 equations for 108 unknowns. Since this system proved to be unsolvable, μ_{13}^d fails in $\mathbf{Con}(\mathbf{M})$.

(A) Assume that $\alpha_1, \alpha_2, \ldots, \alpha_{13}$ are congruences of a metaabelian group $G \in \mathbf{M}$ and y_1 is an element of $[1]p$, the $p(\alpha_1, \alpha_2, \ldots, \alpha_{13})$-block of the group unit 1. From the permutability of group congruences and $(1, y_1) \in p$ we infer that there exists an element $y_2 \in G$ such that $(1, y_2) \in \alpha_{12}$ and $(y_2, y_1) \in \alpha_{13}$. Parsing the lattice term p further we obtain elements $y_3, y_4, \ldots, y_{13} \in G$ such that

$$\begin{gathered}(1, y_4) \in \alpha_4, \quad (y_4, y_3) \in \alpha_5, \quad (y_3, y_5) \in \alpha_1, \quad (y_5, y_7) \in \alpha_6,\\ (y_7, y_1) \in \alpha_7, \quad (y_3, y_6) \in \alpha_2, \quad (y_6, y_9) \in \alpha_8, \quad (y_9, y_1) \in \alpha_9,\\ (y_3, y_8) \in \alpha_3, \quad (y_8, y_{10}) \in \alpha_{10}, \quad (y_{10}, y_1) \in \alpha_{11}.\end{gathered}$$

Consider the group elements

$$\begin{aligned}
f_1 &= y_1 y_5^{-1} y_6 [y_1, y_2][y_1, y_6]^{-1}[y_2, y_5][y_3, y_6]^{-1}[y_3, y_9][y_6, y_9]^{-1},\\
f_2 &= y_3^{-1} y_5^{-1} y_6 y_8 [y_1, y_3][y_1, y_5]^{-1}[y_2, y_5][y_2, y_8]^{-1}[y_3, y_5]^{-1}[y_3, y_9][y_6, y_9]^{-1},\\
f_3 &= y_3^{-1} y_6 y_8 [y_2, y_3][y_2, y_8]^{-1}[y_3, y_9][y_6, y_9]^{-1},\\
f_4 &= y_1 y_5^{-1} y_8 [y_2, y_5][y_2, y_8]^{-1}.
\end{aligned}$$

We claim that

$$\begin{aligned}
&(1, f_2) \in q_1, \quad (f_2, f_1) \in q_{11}, \quad (f_2, f_1) \in q_{10},\\
&(f_1, y_1) \in q_{12}, \quad (f_1, y_1) \in q_{13}, \quad (f_2, f_3) \in q_3,\\
&(f_2, f_3) \in q_2, \quad (f_3, y_1) \in q_4, \quad (f_3, y_1) \in q_5,\\
&(f_2, f_4) \in q_6, \quad (f_2, f_4) \in q_7, \quad (f_4, y_1) \in q_8,\\
&(f_4, y_1) \in q_9.
\end{aligned} \tag{3.7}$$

Each of the relations of (3.7) follows easily from (3.1) and the definitions. E.g., to verify $(f_2, f_1) \in q_{11}$ we can compute as follows. Since 1, y_4 and y_3 are pairwise congruent modulo q_{11} and so are y_1 and y_8 we obtain

$$\begin{aligned}
f_2 \; q_{11} \; & 1^{-1} y_5^{-1} y_6 y_1 [y_1, 1][y_1, y_5]^{-1}[y_2, y_5][y_2, y_1]^{-1}[1, y_5]^{-1}[1, y_9][y_6, y_9]^{-1} =\\
& y_5^{-1} y_6 y_1 [y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} =\\
& y_5^{-1} y_1 y_6 [y_1, y_6]^{-1}[y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} =\\
& y_1 y_5^{-1}[y_1, y_5^{-1}]^{-1} y_6 [y_1, y_6]^{-1}[y_1, y_5^{-1}][y_2, y_5][y_1, y_2][y_6, y_9]^{-1} =\\
& y_1 y_5^{-1} y_6 [y_1, y_6]^{-1}[y_2, y_5][y_1, y_2][y_6, y_9]^{-1} \qquad \text{and}\\
f_1 \; q_{11} \; & y_1 y_5^{-1} y_6 [y_1, y_2][y_1, y_6]^{-1}[y_2, y_5][1, y_6]^{-1}[1, y_9][y_6, y_9]^{-1} =\\
& y_1 y_5^{-1} y_6 [y_1, y_6]^{-1}[y_2, y_5][y_1, y_2][y_6, y_9]^{-1},
\end{aligned}$$

showing $(f_2, f_1) \in q_{11}$. From (3.7) it follows that $(1, y_1) \in q$. Therefore the p-class of 1 is included in the q-class of 1. By the canonical bijection between group congruences and normal subgroups we conclude that μ_{13} holds in $\mathbf{Con}(\mathbf{M})$.

Problem. Note that, in spite of some particular positive results of Haiman [Ha1], it is still an open question if the variety generated by all linear lattices is self-dual. Thus it would be interesting to know if μ_{13} holds in every linear lattice, but we do not know even if it holds in the normal subgroup lattice of any group.

CHAPTER IV

LATTICE IDENTITIES FOR CONGRUENCES AT CONSTANTS

The several consequence relations among lattice identities that we considered in Chapter II were all stronger than the usual $\models$ relation in the class of all lattices. Based on [Cz1], this chapter is devoted to a different consequence relation.

Let A be an algebra with a distinguished nullary operation symbol e in its type, and let

$$\lambda\colon \quad p(x_1, \dots , x_n) \leq q(x_1, \dots , x_n)$$

be a lattice identity. Following Chajda [Ch1] we say that the congruences of A satisfy the identity λ at e if

$$[e]p(\alpha_1, \dots , \alpha_n) \subseteq [e]q(\alpha_1, \dots , \alpha_n)$$

for any congruences $\alpha_1, \dots , \alpha_n$ of A. Here, for $\beta \in \mathrm{Con}(A)$, $[e]\beta = \{a \in A\colon \langle e, a\rangle \in \beta\}$ denotes the congruence class of β containing e. If V is a class of similar algebras and e is a constant operation symbol in its type then we say that λ holds for the congruences of V at e if the congruences of A satisfy the identity λ at e for every $A \in V$.

Let Q_{fin} and S denote the operators of forming finite direct powers and subalgebras, respectively. (By convention, our classes will automatically be closed under taking isomorphic copies, so we do not need an extra operator for this.) For a lattice identity μ and a set of lattice identities Σ let $\Sigma \models_c^{\mathrm{const}} \mu$ stand for the fact that for any SQ_{fin}-closed class V with a constant operation symbol e in its type if every $\sigma \in \Sigma$ holds for the congruences of V at e then μ also holds for the congruences of V at e. As usual, we will write $\sigma \models_c^{\mathrm{const}} \mu$ rather than $\{\sigma\} \models_c^{\mathrm{const}} \mu$. It is almost trivial that

$$\text{(4.1)} \qquad \text{if} \quad \Sigma \models_c^{\mathrm{const}} \mu \quad \text{then} \quad \Sigma \models_c \mu.$$

Indeed, suppose $\Sigma \models_c^{\mathrm{const}} \mu$ and let V be a variety of type τ such that (every member of) Σ holds in $\mathrm{Con}(V)$. By adding a new constant symbol e to τ we define a new type τ'. Make every $A \in V$ into a τ'-algebra by assigning an element $e_A \in A$ in all possible ways, and let U be the class of all τ'-algebras obtained this way. Obviously, U is closed under S and Q_{fin}, and Σ holds for congruences of U at e. Hence μ also holds for congruences of U at e. In the language of V this means that the $p(\alpha_1, \dots , \alpha_n)$-class of every $e \in A$ is included in the $q(\alpha_1, \dots , \alpha_n)$-class of e, where $A \in V$ and λ is of the form $p(x_1, \dots , x_n) \leq q(x_1, \dots , x_n)$. Therefore λ holds in $\mathrm{Con}(V)$, proving (4.1)

This easy assertion indicates that any $\models_c^{\mathrm{const}}$ result is a generalization of the corresponding $\models_c$ result. One of the standard ways to prove $\models_c$ results is to use

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Mal'cev conditions. Chajda [Ch1] was the first to observe that the satisfaction of distributivity for congruences of a variety V at a constant e can be characterized by a Mal'cev condition. His Mal'cev condition is very similar to Jónsson's one given in [Jo2], the only difference is that one has to replace the first variable by e in each Jónsson term to obtain Chajda's Mal'cev condition.

In general, only a weak Mal'cev condition is known to characterize whether a given lattice identity λ holds in the congruence variety of a variety V. This weak Mal'cev condition is of the form $(\forall m)(\exists n)$ (U_{mn}) where the U_{mn} are the strong Mal'cev conditions given by the Wille – Pixley algorithm, cf. [Wi1] or [Pi1] (or cf. [HC1] where this algorithm is used). We do not detail this algorithm here, so the following few lines, up to the end of Proposition 4.2, will be informative only for those who know something about the Wille – Pixley algorithm. In this algorithm, two of the variables of the "term variable symbols" f_i occurring in U_{mn}, say the first and last variables, have distinguished role. Let U'_{mn} denote the strong Mal'cev condition obtained from U_{mn} via substituting e for the first variable everywhere. (For example, we have to write $t_3(x_2,\dots,x_k)$ resp. $e = t_2(x_2,e,x_2,e)$ instead of $t_3(x_1,x_2,\dots,x_k)$ resp. $x_1 = t_2(x_1,x_2,x_1,x_2,x_1)$.) Now we can formulate

PROPOSITION 4.2. *([Cz1]) Let V be a variety with a nullary operation symbol e in its type. Then λ holds for the congruences of V at e iff for each $m \geq 2$ there is an $n \geq 2$ such that U'_{mn} holds in V.*

We omit the long but straightforward technicalities which merge the Wille – Pixley's algorithm and Chajda's approach to a proof of Proposition 4.2. Instead, we point out that this Proposition is useful in many cases. Suppose V is a congruence permutable variety. Then λ holds in $\mathrm{Con}(V)$ iff U_{22} holds in V. If, in addition, e is a constant in V then λ holds for the congruences of V at e iff U'_{22} holds in V. Now let V be a variety of groups or a variety of modules, and let e stand for 1 or 0, respectively. By the well-known canonical correspondence between congruences and congruence classes containing e, λ holds in $\mathrm{Con}(V)$ iff λ holds for congruences of V at e. Thus,

$$\lambda \text{ holds in } \mathrm{Con}(V) \text{ iff } U'_{22} \text{ holds in } V. \tag{4.3}$$

The advantage of U'_{mn} over U_{mn} is that U'_{mn} has fewer variables. Note that (4.3) was used in the computer program mentioned in the previous chapter (that is why we had 130 equations for 108 unknowns "only"), and it could have been used (if it had been known that time) in [HC1] to simplify some calculations. The group theoretic version of (4.3) was used in Pálfy and Szabó [PS1] and [PS2].

Now we intend to generalize one of the first $\models_c$ results to an $\models_c^{\mathrm{const}}$ result. Let dist denote the distributive law

$$x(y+z) \leq xy + xz.$$

Calling a lattice identity nontrivial if it does not hold in all lattices we have

THEOREM 4.4. *([Cz1]) Suppose ε is a nontrivial lattice identity of the form*

$$\sigma_0 w \leq \sum_{i=1}^{n} \sigma_0 \sigma_i \tag{4.5}$$

where the lattice terms $\sigma_0, \sigma_1 \dots, \sigma_n$ are joins of variables and σ_0 and w have no variable in common. Then $\varepsilon \models_c^{\text{const}}$ dist.

In virtue of (4.1) this theorem implies Nation's result in [Na1], which asserts the same for $\models_c$ instead of $\models_c^{\text{const}}$. As a well-known example for ε occurring in Theorem 4.4 we mention Huhn's $(n-1)$-distributive law

$$x \sum_{i=1}^{n} y_i \le \sum_{j=1}^{n} (x \sum_{1 \le i \le n}^{i \ne j} y_i),$$

cf. Huhn [Hn1].

Proof. A large part of the proof we are going to present is almost the same as that of Nation's Theorem 3.7 in Jónnson [Jo1]. (We make [Jo1] our main reference because to follow the original proof in [Na1] the reader would have to consider several preliminary statements as well.) In fact, the lion's share of our proof is taken from [Jo1]. The main difference is that the original proof uses free algebras, what we do not have in our case. Therefore we need an appropriate substitute. This is what we will call a *locally free algebra*. First we define this tool and show its existence, then we borrow an argument from [Jo1] in a slightly modified form. When the proof in [Jo1] has an easy end, thanks to free algebras, we will have to work much more with our locally free algebras.

If F and B are algebras, $X \subseteq F$ and $Y \subseteq B$ are finite subsets, X generates F and each mapping $X \to Y$ extends to a homomorphism $F \to B$ then $F = F(X; Y \subseteq B)$ will be called a *locally free algebra* (with respect to X, Y and B). We claim that for each $B \in V$, each finite subset $Y \subseteq B$ with more than one element and each finite set X there is a locally free algebra $F = F(X; Y \subseteq B\rangle$ in V. Really, consider the direct power $\prod_{f \in M} B$ where M is the set of all $X \to Y$ mappings. Denoting the choice function $\langle f(x)\colon\ f \in M\rangle$ by $\hat{x}$, the subset $\hat{X} = \{\hat{x}\colon\ x \in X\}$ can be identified with X. Let F be the subalgebra of $\prod_{f \in M} B$ generated by $\hat{X}$. Clearly, any map $g\colon\ \hat{X} \to Y$ extends to the projection $F \to B$, $s \mapsto s(g)$, which is a homomorphism. This proves the existence of locally free algebras in V.

Whatever it looks trivial, we formulate and verify the following statement. After having seen Proposition 4.12, the last result in this chapter, the reader will hopefully agree that this precaution is not superfluous. Let p, p' and q be lattice terms on the variables $x_1, \dots, x_n$ such that $p' \le p$ holds in all lattices. Then

$$p \le q \quad \models_c^{\text{const}} \quad p' \le q. \tag{4.6}$$

Suppose $p \le q$ holds for congruences of an algebra A at e, $\alpha_1, \dots, \alpha_n \in \mathrm{Con}(A)$ and $a \in [e]p'(\alpha_1, \dots, \alpha_n)$. Then $p'(\alpha_1, \dots, \alpha_n) \subseteq p(\alpha_1, \dots, \alpha_n)$ and we have $a \in [e]p'(\alpha_1, \dots, \alpha_n) \subseteq [e]p(\alpha_1, \dots, \alpha_n) \subseteq [e]q(\alpha_1, \dots, \alpha_n)$, and (4.6) follows.

Now suppose that V is an SQ_{fin}-closed class such that ε holds for congruences of V at e. Following the original proof in [Jo1] we will define another identity ε'. Let X, Y and Z_i be the sets consisting of all those variables that occur in ε, in w and in σ_i, respectively. Then, with a more expressive but less precise notation, ε is just

$$(\sum Z_0) \cdot w(Y) \le \sum_{i=1}^{n} (\sum Z_0)(\sum Z_i). \tag{4.7}$$

We can fix a variable $x_0 \in Z_0 \setminus (Z_1 \cup \ldots \cup Z_n)$, for otherwise ε would hold in all lattices. By the same reason, none of the sets $Y_i = Y \setminus Z_i \quad (i = 1, \ldots, n)$ is empty. Put $w_i = \prod Y_i$, $i = 1, \ldots, n$. Denote $FL(Y)$ the lattice freely generated by Y. An easy induction on u shows that $w_i \le u$ or $u \le \sum (Y \cap Z_i)$ holds for every $u \in FL(Y)$. If $u = w$ then $w \le \sum (Y \cap Z_i)$ would imply $w \le \sum Z_i$, whence ε would hold in all lattices, a contradiction. Hence $w_i \le w$ in $FL(Y)$, and for $w' := w_1 + \ldots + w_n$, $w' \le w$ holds in all lattices. From the assumption and (4.6) we infer that the identity

$$\varepsilon': \qquad \sigma_0 w' \le \sum_{i=1}^{n} \sigma_0 \sigma_i,$$

in other words $(\sum Z_0)(\sum_{i=1}^{n} \prod Y_i) \le \sum_{i=1}^{n} (\sum Z_0)(\sum Z_i)$, holds for congruences of V at e.

Let $S = \{b, c\}$ with $b \ne c$, and let $F(S)$ denote an arbitrary (not necessarily locally free) algebra in V such that $S \subseteq F(S)$. Put $T = \{e, u_1, \ldots, u_n\}$ and let $F(T)$ denote the locally free algebra $F\bigl(\{u_1, \ldots, u_n\}; S \cup \{e\} \subseteq F(S)\bigr)$ in V. For technical reasons, e will often be denoted also by u_0. Now we are in the position to follow [Jo1] again. For each $x \in X$ we define an equivalence relation $\varphi(x)$ on the $(n+1)$-element set $T = \{e, u_1, \ldots, u_n\}$. (The assumption $b \ne c$ and the definition of a locally free algebra guarantees $|T| = n + 1$.) Let $\varphi(x_0)$ be the equivalence collapsing $e = u_0$ and u_n only. If $x \in X \setminus \{x_0\}$ then $\varphi(x)$ is defined by the property that for $0 \le i < j \le n$

$$\langle u_i, u_j \rangle \in \varphi(x) \iff \bigl((\forall k)(i < k \le j \Longrightarrow x \in Y_k\bigr).$$

For $x \in X$, let $\psi(x)$ denote the congruence of $F(T)$ generated by $\varphi(x)$. We extend ψ: $X \to \mathrm{Con}(F(T))$ to a lattice homomorphism of the free lattice $FL(X)$ to $\mathrm{Con}(F(T))$; this extension will also be denoted by ψ. Clearly $\psi(x_0)$ identifies e and u_n, whence so does $\psi(\sigma_0)$ as well. For $x \in Y_i$ (when necessarily $x \ne x_0$) $\psi(x)$ identifies u_{i-1} and u_i. I.e., $\langle u_{i-1}, u_i \rangle \in \prod_{x \in Y_i} \psi(x) = \psi(w_i)$, and we conclude $\langle u_0, u_n \rangle \in \psi(\sigma_0) \cap \psi(w') = \psi(\sigma_0 w')$. That is, $u_n \in [e]\psi(\sigma_0 w')$. Since ε' holds for congruences at e, $u_n \in [e]\psi(\sum_{i=1}^{n} \sigma_0 \sigma_i)$. Therefore there exist elements $v_0 = e$, $v_1, v_2, \ldots, v_m = u_n$ in $F(T)$, all in the same $\psi(\sigma_0)$-class, such that for each i with $0 < i \le m$ there is an integer $j(i)$ with $1 \le j(i) \le n$ for which

$$\langle v_{i-1}, v_i \rangle \in \psi(\sigma_{j(i)}).$$

Now let α, β and γ be the principal congruences of $F(S)$ generated by $\langle e, b \rangle$, $\langle e, c \rangle$ and $\langle b, c \rangle$, respectively. Let H_0 be the set of all $\{u_1, \ldots, u_n\} \to \{e, b, c\}$ maps such that $u_n \mapsto b$. By definitions, each member of H_0 extends to a (unique) homomorphism $F(T) \to F(S)$; let H be the set of these extensions of members of H_0.

From $Z_0 \cap Y_i \subseteq Z_0 \cap Y = \emptyset$ it follows that $\psi(x)$ is trivial for $x \in Z_0 \setminus \{x_0\}$.. Therefore $\psi(\sigma_0) = \psi(x_0)$ is the principal congruence generated by $\langle e, u_n \rangle$. Hence each member of H maps $\psi(\sigma_0)$ into α. In particular, all elements $\xi(v_i)$ with $0 \le i \le m$ and $\xi \in H$ belong to $[e]\alpha$.

We want to find homomorphisms $\xi_i \in H$ such that $\xi_i(v_{i-1}) = \xi_i(v_i)$. Since $\langle v_{i-1}, v_i \rangle \in \psi(\sigma_{j(i)})$, it suffices to choose ξ_i so that it maps $\psi(\sigma_{j(i)})$ onto the identity relation on $F(S)$, and since $\psi(\sigma_{j(i)})$ is generated by its restriction to T, it suffices to choose ξ_i so that any two $\psi(\sigma_{j(i)})$-equivalent members of T are mapped onto the same member of S. Therefore we define

$$\begin{aligned} \xi_i(u_k) &= e && \text{if } \langle e, u_k\rangle \in \psi(\sigma_{j(i)}),\\ \xi_i(u_k) &= b && \text{if } \langle u_k, u_n\rangle \in \psi(\sigma_{j(i)}),\\ \xi_i(u_k) &= c && \text{otherwise.} \end{aligned}$$

To see that this definition makes sense it is essential to note that $\psi(\sigma_{j(i)})$ does not identify e and u_n. This is so because if $x \in Z_{j(i)}$ then $x \notin Y_{j(i)}$, and hence $\varphi(x)$ is contained in the equivalence relation that partitions T into the two subsets $\{u_k\colon\ k < j(i)\}$ and $\{u_k\colon\ k \geq j(i)\}$.

For $\xi, \eta \in H$ we write $\xi\bar\beta\eta$ if $\langle \xi(y), \eta(y)\rangle \in \beta$ for all $y \in F(T)$, and define $\bar\gamma$ similarly. Next we show that

$$\text{(4.8)} \qquad \text{for any } \xi, \eta \in H \text{ there exist } \zeta, \zeta' \in H \text{ such that } \xi\bar\beta\zeta\bar\gamma\zeta'\bar\beta\eta.$$

To show this, for each $0 < k < n$ we have to define appropriate $\zeta(u_k), \zeta'(u_k) \in S \cup \{e\} = \{e, b, c\}$. These values will depend on $\xi(u_k)$ and $\eta(u_k)$. Since the possible values of $\xi(u_k)$ and $\eta(u_k)$ are e, b and c, this gives rise to nine cases. Thus, we can define $\zeta(u_k)$ and $\zeta'(u_k)$ by listing all the nine quadruples $\langle \xi(u_k), \zeta(u_k), \zeta'(u_k), \eta(u_k)\rangle$ which we permit:

$$\begin{array}{lll} \langle e, e, e, e\rangle, & \langle e, c, b, b\rangle, & \langle e, c, c, c\rangle,\\ \langle b, b, c, e\rangle, & \langle b, b, b, b\rangle, & \langle b, b, c, c\rangle,\\ \langle c, e, e, e\rangle, & \langle c, c, b, b\rangle, & \langle c, c, c, c\rangle. \end{array}$$

This proves (4.8).

Now (4.8) yields that, for $0 < i \leq m$, $\xi_{i-1}(v_{i-1})$ can be connected to $\xi_i(v_{i-1})$ by a sequence whose successive terms are always identified by either β or γ. But $\xi_i(v_{i-1}) = \xi_i(v_i)$, so this sequence connects $\xi_{i-1}(v_{i-1})$ to $\xi_i(v_i)$. Hence we obtain such a sequence which connects $e = \xi_0(e) = \xi_0(v_0)$ with $b = \xi_m(v_m) = \xi_m(u_n)$. Furthermore, all the members of this sequence are of the form $\zeta(v_k)\quad(\zeta \in H)$ and they are therefore in the α-class of e. This shows that

$$\text{(4.9)} \qquad \langle e, b\rangle \in (\alpha \cap \beta) + (\alpha \cap \gamma).$$

At this point the analogous argument in [Jo1] terminates, since (4.9) immediately implies the congruence distributivity of V, for $F(S)$ in [Jo1] is a free algebra with e being an additional free generator (not a constant) and (4.9) yields the existence of Jónsson terms characterizing congruence distributivity. Now we cannot deduce identities from (4.9) as Chajda [Ch1], following Jónsson [Jo2], did. (The slight difficulty is caused by the fact that V is not a variety in our case. It is not even a locally equational class in Hu's sense [Hu1], whence Pixley's local Mal'cev conditions from [Pi1] cannot be used.) Fortunately, some ideas from [Jo2] can still be used. Assume that $A \in V$, $\alpha', \beta', \gamma' \in \mathrm{Con}(A)$, $b' \in A \setminus \{e\}$ and

$$\langle e, b'\rangle \in \alpha' \cap (\beta' + \gamma').$$

We need to show that $\langle e, b'\rangle$ belongs to $\delta' = (\alpha' \cap \beta') + (\alpha' \cap \gamma')$. By the assumption there is an $m > 0$ (not necessarily the same as before) and there are elements $u_0 = e, u_1, \dots, u_m = b'$ in A such that $u_i \beta' u_{i+1}$ for i even and $u_i \gamma' u_{i+1}$ for i odd $(0 \le i < m)$. From now on let $F(S)$ be the locally free $F(S; \{u_0, u_1, \dots, u_m\} \subseteq A)$ algebra in V. From (4.9) we infer that there are elements $t_0 = e,\ t_1,\ \dots,\ t_r = b$ in $F(S)$ such that they all belong to $[e]\alpha$, $t_i \beta t_{i+1}$ for i even, and $t_i \gamma t_{i+1}$ for i odd $(0 \le i < r)$. Further, there exist binary terms q_i such that $t_i = q_i(c, b)$, $0 \le i \le r$. Let φ_i denote the unique $F(S) \to A$ homomorphism for which $\varphi_i(b) = b'$ and $\varphi_i(c) = u_i$, $0 \le i \le m$. (These φ_i exist by the local freeness of $F(S)$.) Observe that for any $x, y \in F(S)$ and $\mu \in \{\alpha, \beta, \gamma\}$

(4.10) if μ' collapses the φ_i-image of the generic pair of μ and $\langle x, y\rangle \in \mu$ then $\langle \varphi_i(x), \varphi_i(y)\rangle \in \mu'$.

Indeed, denoting the canonical $A \to A/\mu'$ homomorphism by κ we have $\mu \subseteq \ker(\kappa \circ \varphi_i)$, which implies (4.10).

Form (4.10) we infer that all the $\varphi_i(t_j)$, $0 \le i \le m$ and $0 \le j \le r$, belong to the same α'-class. Thus whenever two of them are found to be congruent modulo β' or γ' then they will be congruent module δ' as well. Observe that $\varphi_i(t_j) = \varphi_i(q_j(c, b)) = q_j(u_i, b')$. Therefore, for any j, $\varphi_0(t_j) = q_j(u_0, b')\ \beta'\ q_j(u_1, \beta')\ \gamma'\ q_j(u_2, b')\ \beta'\ q_j(u_3, \beta')\ \gamma' \dots\ q_j(u_m, \beta') = \varphi_m(t_j)$. This yields $\varphi_0(t_j)\delta'\varphi_m(t_j)$.

On the other hand, by (4.10), $\varphi_0(t_j)\beta'\varphi_0(t_{j+1})$ for j even and $\varphi_m(t_j)\gamma'\varphi_m(t_{j+1})$ for j odd. Thus $\varphi_0(t_j)\delta'\varphi_0(t_{j+1})$ for j even and $\varphi_m(t_j)\delta'\varphi_m(t_{j+1})$ for j odd. Therefore

$$e = \varphi_0(t_0)\delta'\varphi_0(t_1)\delta'\varphi_m(t_1)\delta'\varphi_m(t_2)\delta'\varphi_0(t_2)\delta'\varphi_0(t_3)\delta' \dots \\ \delta'\varphi_m(t_r) = \varphi_m(b) = b',$$

completing the proof. □

One might think that other classical results on $\models_c$ can similarly be strengthened to $\models_c^{\text{const}}$ results. Sometimes, e.g. in case of Day [Da1], this is true, but far from always. The following problem, which may look surprising at the first sight, would be completely trivial with $\models_c$ instead of $\models_c^{\text{const}}$. Let dist* denote the identity

$$(x + y)(x + z) \le x + yz,$$

the dual of the distributive law.

Problem 4.11. *([Cz1]) Is it true that* dist* $\models_c^{\text{const}}$ dist *?*

Although dist $\models$ dist* is a basic fact in lattice theory, related to the above problem we have

Proposition 4.12. *([Cz1])* dist $\models_c^{\text{const}}$ dist* *is false.*

Proof. Let V be the variety of meet-semilattices with 0. Then dist holds for the congruences of V at 0, cf. Chajda [Ch1, Example 3]. On the other hand, consider the seven element semilattice $A = \{a, b, c, ab, ac, bc, abc = 0\}$. Then the congruences α, β and γ corresponding to the respective partitions

$$\{\{a, b, ab, ac, bc, 0\}, \{c\}\},$$
$$\{\{a, c, ac\}, \{b\}, \{ab, bc, 0\}\}$$

and

$$\{\{a\},\{ac,ab,0\},\{b,c,bc\}\}$$

witness that dist* fails for congruences of A at 0. $\square$

CHAPTER V

QUASIVARIETIES OF SUBMODULE LATTICES

This chapter is based on [Cz6], [Cz7] and [CH2, Chapter 4]. By a ring we always mean a ring with unit element 1. For a ring R, the lattices $\mathrm{Sub}({}_RM)$ of R-modules ${}_RM$ are among the most important examples of modular lattices. A lattice L is called *representable by R-modules* if it is isomorphic to a sublattice of some $\mathrm{Sub}({}_RM)$. The class $\mathcal{L}(R)$ of all lattices representable by R-modules,

$$\mathcal{L}(R) = IS\{\mathrm{Sub}({}_RM)\colon\ {}_RM \text{ is an } R\text{-module}\},$$

where I and S are the operators of forming isomorphic copies and subalgebras, is known to be a quasivariety, cf. Makkai and McNulty [MM1]. In other words, the $\mathcal{L}(R)$ are axiomatizable by sets of lattice Horn sentences. Hence their study is, in a sense, equivalent to the study of lattice Horn sentences that hold in $\{\mathrm{Sub}({}_RM)\colon\ {}_RM \text{ is an } R\text{-module}\}$ classes.

As usual, R–**MOD** denotes the abelian category of (left) R-modules and R-linear homomorphisms. The fundamental theorem of quasivarieties $\mathcal{L}(R)$ asserts that

Theorem 5.A. *(Hutchinson [Ht2, Thm. 1, p. 108], for the whole proof see also [Ht1] and [Ht4].) For any two rings R and S with unit, $\mathcal{L}(R) \subseteq \mathcal{L}(S)$ if and only if there exists an exact embedding functor R–**MOD** $\to S$–**MOD**.*

So, this chapter may equally be considered as a study of exact embedding functors between module categories. Since there are many powerful methods for constructing exact embedding functors, the abelian category connection was quite useful for Hutchinson to achieve Theorem 5.B, to be presented soon, which is one of the main tools in this chapter. This permits us not to resort to category theory directly, so the reader need not know anything about abelian categories. While the first results on the $\mathcal{L}(R)$ have been achieved by a category theoretic approach (cf. Hutchinson [Ht1]) and only some of them have been reproved by other means (including Horn sentences, cf. [Cz5] and [CH1]), the proof of our forthcoming Theorem 5.2 indicates that now the role of Horn sentences is much more important.

We will consider rings with prime power characteristic p^k. (The consideration before Corollary 5.12 will indicate that among positive characteristics the prime powers are the interesting ones. Much less is known about the case of zero characteristic.) For any two such rings R and S, the lattice varieties, in fact congruence varieties, $H\mathcal{L}(R)$ and $H\mathcal{L}(S)$ are equal iff R and S have the same (prime power) characteristic, cf. [HC1, Cor. 2, p. 286]. The case $k = 1$ also simplifies: if R and S have the same prime characteristic p then $\mathcal{L}(R) = \mathcal{L}(S)$, cf. Hutchinson [Ht1, Thm. 5(6), p. 88] (or, for a different proof, [Cz5, Prop. 6.2]). For $k \geq 2$, much less is known about these quasivarieties. Thus the set

$$\mathbf{W}(p^k) = \{\mathcal{L}(R)\colon\ R \text{ is a ring with characteristic } p^k\}$$

will be interesting for us for $k \geq 2$ only. There are examples of rings R and S with characteristic 4 such that $\mathcal{L}(R) \neq \mathcal{L}(S)$, cf. Hutchinson [Ht1] and [CH1]. This means that $|\mathbf{W}(p^k)| \geq 2$ in general (at least for $p = 2$). Later, Hutchinson [Ht3] proved $|\mathbf{W}(p^k)| \geq 4$, and the author found countably many rings to show $|\mathbf{W}(p^k)| \geq \aleph_0$; in both cases the proof relied directly on Theorem 5.A. At this point it is worth mentioning that $|\mathbf{W}(p^k)| \leq 2^{\aleph_0}$, for there are $2^{\aleph_0}$ sets consisting of Horn sentences.

In the sequel, let $k \geq 2$ and a prime p be fixed. Suppose R is a ring with characteristic p^k. First we identify certain special two-sided ideals of R, which can be described by expressions not depending upon R. Recall that the set $\mathcal{I}(R)$ of two-sided ideals of R has a bounded modular lattice structure with join $X \vee Y = X + Y$ and meet $X \wedge Y = X \cap Y$. There are also products:

$$X \cdot Y = \{\sum_{i=1}^{n} x_i y_i\colon\ x_i \in X,\ y_i \in Y \text{ for } i = 1, \dots, n\},$$

usually written as just XY. In addition, we consider two unary operations for $\mathcal{I}(R)$, denoted by $\downarrow$ and $\uparrow$ and defined as the image and inverse image under multiplication by p :

$$\downarrow X = pX = \{px\colon\ x \in X\}$$
$$\uparrow X = \{y \in R\colon\ py \in X\}.$$

We also consider two nullary operation symbols $\mathbf{0}$ and $\mathbf{1}$ to denote the smallest and largest element of a bounded lattice. For $\mathcal{I}(R)$, $\mathbf{0}$ and $\mathbf{1}$ will denote $\{0\}$ and R. Let τ be the type $\langle \vee, \wedge, \cdot, \downarrow, \uparrow, \mathbf{0}, \mathbf{1} \rangle$ with respective arities $\langle 2, 2, 2, 1, 1, 0, 0 \rangle$. According to the above-mentioned notations, $\mathcal{I}(R)$ becomes an algebra of type τ. This way each nullary τ-term σ assigns an ideal σ_R of R. For example, $\sigma' = \uparrow \mathbf{0} \wedge \uparrow\downarrow \mathbf{1}$ and $\sigma'' = \uparrow \mathbf{0}$ are nullary τ-terms and, as it is easy to see, $\sigma'_R = \sigma''_R = \{x \in R\colon\ px = 0\} \in \mathcal{I}(R)$. Let $K(R)$ denote the set of all nullary τ-terms σ such that $\sigma_R = \mathbf{1}_R \ \ (= R)$ holds in $\mathcal{I}(R)$. As one of our main tools in this chapter, we have

Theorem 5.B. *(Hutchinson [CH2]) If $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$ then $K(R_1) \supseteq K(R_2)$.*

The set $\mathbf{W}(p^k)$ becomes a partially ordered set $\mathbf{W}(p^k) = \langle \mathbf{W}(p^k), \subseteq \rangle$ under inclusion. Hutchinson [CH2] has shown that $\mathbf{W}(p^k)$ has smallest and largest elements. Namely, the largest element of $\mathbf{W}(p^k)$ is $\mathcal{L}(\mathbf{Z}_{p^k})$ where $\mathbf{Z}_{p^k}$ is the factor ring of integers modulo p^k; the smallest element of $\mathbf{W}(p^k)$ is $\mathcal{L}(S_0)$ where

S_0 is the endomorphism ring of the additive group of the $\aleph_0$-th direct power of $\mathbf{Z}_p \times \mathbf{Z}_{p^2} \times \ldots \times \mathbf{Z}_{p^k}$. It is known from [CH2] that $\mathbf{W}(p^k)$ is closed under finite joins, taken in the lattice of all quasivarieties of lattices. (This fact also follows easily from the existence of Mal'cev conditions characterizing whether a given lattice Horn sentence holds for congruences in a congruence permutable variety, cf. Jónsson [Jo1, Thm. 9.16] or, for concretely given Mal'cev conditions, [Cz5] and [Cz15].) Yet, not much is known about $\mathbf{W}(p^k)$; the following assertion, the main result of this chapter, indicates that $\mathbf{W}(p^k)$ must have a complicated structure. For a set A let $P(A) = \langle P(A), \subseteq \rangle$ denote the complete Boolean lattice of all subsets of A.

Theorem 5.1. *(Czédli [CH2]) Let p be a prime and $k \geq 2$. Then $\mathbf{W}(p^k)$ has power of continuum. Further, $\mathbf{W}(p^k)$ has a sub-poset order-isomorphic to $P(A)$ for a countably infinite set A. In particular, $\mathbf{W}(p^k)$ has ascending and descending chains and antichains with $2^{\aleph_0}$ many elements.*

The theory of $\mathbf{W}(p^k)$ raises three questions naturally. The first of them is about the converse of Theorem 5.B.

Problem 5.A. *Does $K(R_1) \supseteq K(R_2)$ imply $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$?*

Problem 5.B. *Is $\mathbf{W}(p^k)$ closed with respect to arbitrary joins (taken in the lattice of all lattice quasivarieties)?*

Problem 5.C. *Is $\mathbf{W}(p^k)$ a complete lattice or at least a lattice?*

The analogous problems for the partially ordered set of lattice varieties $H\mathcal{L}(S)$, where the S are rings of any characteristic, have positive solutions, cf. [HC1].

The three problems above are not independent. Clearly, an affirmative answer to Problem 5.B (together with the mentioned fact that $\mathbf{W}(p^k)$ has a smallest element) would yield an affirmative answer to Problem 5.C. Hutchinson [Ht3] has shown that if the answer to Problem 5.A is "yes" then so is the answer to Problem 5.C. His argument, a bit complicated to be outlined here, initiated the following result, cf. [Cz7], even if it is not used in the final proof. Interestingly enough, we cannot solve problems 5.A and 5.B but we can say something about their conjunction.

Theorem 5.2. *At least one of Problems 5.A and 5.B has a negative answer.*

This statement strengthens the feeling that $\mathbf{W}(p^k)$ is complicated.

Proof of Theorem 5.1. We start with some definitions.

Definitions 5.3. Let $H_0 = \{n\colon n \geq 2k\}$. Abbreviate $\uparrow^{k-1}$ by $\Uparrow$, so

$$\Uparrow X = \{r \in R\colon p^{k-1} r \in X\}$$

for $X \in \mathcal{I}(R)$. Define nullary τ-terms $e_{j,n}$ for $n \in H_0$ by induction on j as follows:

$$\begin{aligned} e_{0,n} &= \mathbf{0}, \\ e_{j,n} &= (\Uparrow e_{j-1,n})^n \text{ for } 1 \leq j \leq n-1. \end{aligned}$$

For $n \in H_0$ let $d_n =\Uparrow e_{n-1,n}$, and say that τ_n is satisfied in a ring R of characteristic p^k iff $d_n = \mathbf{1}$ holds in $\mathcal{I}(R)$, i.e. $(d_n)_R = R$.

Definitions and Properties 5.4. Suppose H is a subset of H_0 such that $2k \in H$. Using the set

$$Y(H) = \{y_{n,i}\colon\ n \in H,\ 1 \le i \le n-1\},$$

we can form the commutative polynomial ring $F(H) = \mathbf{Z}_{p^k}[Y(H)]$ on $Y(H)$ with coefficients in $\mathbf{Z}_{p^k}$. Note that $Y(H)$ freely generates $F(H)$ in the variety of commutative unital rings with characteristic dividing p^k. Let $J(H)$ denote the ideal of $F(H)$ generated by

$$\{y_{n,i}^n - p^{k-1}y_{n,i-1}\colon\ n \in H,\ 1 \le i \le n-1\} \cup \{p^{k-1}y_{n,n-1}\colon\ n \in H\}.$$

Here and in the sequel, $y_{n,0} = 1$ in $F(H)$. Finally, define $R(H) = F(H)/J(H)$. So, $R(H)$ is a commutative ring with 1. Soon we will show that it is of characteristic p^k.

Next, we describe the structure of $R(H)$. Hereafter, H will denote a fixed subset of $H_0 = \{n\colon\ n \ge 2k\}$ containing $2k$.

Definitions and Properties 5.5. Let $B = \{\langle n,i\rangle\colon\ n \in H,\ 1 \le i \le n-1\}$, and let W denote the set of all functions (mappings) $w\colon\ B \to \mathbf{N}_0 = \{0,1,2,\dots\}$ such that $w(n,i) = 0$ for all but finitely many elements of B. Note that associative and commutative sums $w+z$ can be defined pointwise for $w, z \in W$. These $w \in W$ will be used to form expressions like $\prod_{\langle n,i\rangle\in B} y_{n,i}^{w(n,i)}$ a bit later.

Say that $w \in W$ has a *high power* if $w(n,i) \ge n$ for some $\langle n,i\rangle$ in B. Similarly, say that $w \in W$ has a *last variable* if $w(n,n-1) > 0$ for some $n \in H$. Let U denote the set of all functions u in W with no high power and no last variable. I.e., for $u \in W$, $u \in U$ iff $u(n,i) < n$ for all $\langle n,i\rangle \in B$ and $u(n,n-1) = 0$ for all $n \in H$. Let V denote the set of all functions $v \in W$ with no high power and at least one last variable (that is, $v(n,i) < n$ for all $\langle n,i\rangle \in B$, and $v(n,n-1) > 0$ for at least one $n \in H$). Let $\underline{\mathbf{U}}$ denote the set of all finite subsets of U, and $\underline{\mathbf{V}}$ the set of all finite subsets of V.

As usual, a *monomial* in $F(H)$ is a product

$$\alpha \prod_{\langle n,i\rangle\in B} y_{n,i}^{w(n,i)}$$

for $\alpha \in \mathbf{Z}_{p^k} \setminus \{0\}$ and $w \in W$. Every element of $F(H)$ is expressible as a sum of monomials obtained from distinct elements of W; the empty sum is 0 in $F(H)$ by convention. Let $x_{n,i} = y_{n,i} + J(H)$ in $R(H)$ for $\langle n,i\rangle \in B$, and $x_{n,0} = 1 + J(H) = 1_{R(H)}$ for $n \in H$. Let y^w denote product $\prod_{\langle n,i\rangle\in B} y_{n,i}^{w(n,i)}$ in $F(H)$ for $w \in W$, and $x^w = \prod_{\langle n,i\rangle\in B} x_{n,i}^{w(n,i)} = y^w + J(H)$ in $R(H)$.

A monomial αy^w in $F(H)$, $\alpha \in \mathbf{Z}_{p^k} \setminus \{0\}$ and $w \in W$, is called *reduced* if $w \in U$ or if $w \in V$ and $\alpha \in \{1, 2, \dots, p^{k-1} - 1\} \subset \mathbf{Z}_{p^k}$.

(5.5a) $F(H)$ is free as a $\mathbf{Z}_{p^k}$-module, with basis set $\{y^w \colon w \in W\}$.

(5.5b) Let c in W be given by $c(n, i) = 0$ for all $\langle n, i\rangle \in B$, so $c \in U$. Then $1y^c$ is the ring unit for $F(H)$, so $1x^c$ is the ring unit for $R(H)$.

(5.5c) If v is in V, then $p^{k-1}y^v$ is in $J(H)$, and for $\alpha \in \mathbf{Z}_{p^k}$, αy^v is in $\beta y^v + J(H)$ for some β in $\{0, 1, \dots, p^{k-1} - 1\}$. (See 5.4, and let β be the remainder of α after division by p^{k-1}.)

(5.5d) Suppose w in W has a variable $y_{n,i}$ with high power, and z in W satisfies $z(n, i) = w(n, i) - n$, $z(n, i - 1) = w(n, i - 1) + 1$ if $i > 1$, and $z(m, j) = w(m, j)$ otherwise. Then αy^w is in $p^{k-1}\alpha y^z + J(H)$, and αy^w is in $J(H)$ if z is not in U. (Use 5.4 and 5.5c, and note that $p^{2k-2} = 0$ in $\mathbf{Z}_{p^k}$.)

Now we show that every element of $R(H)$ is uniquely representable in $F(H)$ by a sum of reduced monomials obtained from distinct elements of $U \cup V$.

Lemma 5.6. *Each $f + J(H)$ in $R(H)$ $\quad$ ($f \in F(H)$, i.e. each element of $R(H)$) has a canonical form:*

$$f + J(H) = \sum_{u \in C} \alpha_u y^u + \sum_{v \in D} \beta_v y^v + J(H),$$

where $C \in \underline{\mathbf{U}}$, $D \in \underline{\mathbf{V}}$, $\alpha_u \in \mathbf{Z}_{p^k} \setminus \{0\}$ for all $u \in C$, and $\beta_v \in \{1, 2, \dots, p^{k-1} - 1\}$ for all $v \in D$. (By convention, $C = D = \emptyset$ represents 0 in $R(H)$.) The sets C and D and the coefficients α_u ($u \in C$) and β_v ($v \in D$) are uniquely determined.

Proof of Lemma 5.6. To show that every element of $R(H)$ is representable by a sum of reduced monomials it suffices to show that every non-reduced monomial αy^w in $F(H)$ is either in $J(H)$ or in $\delta y^z + J(H)$ for some reduced monomial δy^z. If w has no high power then it is in V since αy^w is not reduced and so αy^w is in $J(H)$ or in $\beta y^w + J(H)$ for βy^w reduced by 5.5c. If w has a high power then αy^w is in $J(H)$ or $\delta y^z + J(H)$ for a suitable reduced monomial δy^z by 5.5d and the above.

To show that C, D and all α_u and β_v are uniquely determined for a given $f + J(H)$, $f = \sum_{u \in C} \alpha_u y^u + \sum_{v \in D} \beta_v y^v$, it suffices to show that $C = D = \emptyset$ if f is in $J(H)$. But then $f = f_0$ in $F(H)$ where

$$f_0 = \sum_{\langle n,i\rangle \in P} g_{n,i} \cdot (y_{n,i}^n - p^{k-1} y_{n,i-1}) + \sum_{n \in Q} h_n \cdot p^{k-1} y_{n,n-1}$$

for finite subsets $P \subseteq B$ and $Q \subseteq H$, and elements $g_{n,i} \in F(H)$ for $\langle n, i\rangle \in P$ and $h_n \in F(H)$ for $n \in Q$.

If we express f_0 as a $\mathbf{Z}_{p^k}$-linear combination of basis elements in $F(H)$ by 5.5a then each monomial summand either has a high power or is a multiple of p^{k-1}. But then $D = \emptyset$ since a reduced βy^v for $\beta \in \{1, 2, \dots, p^{k-1} - 1\}$ and $v \in V$ is not a sum of such terms plus reduced monomials δy^w with $w \neq v$.

Suppose $C \neq \emptyset$, with $u \in C$. Since u contains no high powers and no last variables, there is a $\delta \in \mathbf{Z}_{p^k} \setminus \{0\}$ such that δy^u is a summand of a $\mathbf{Z}_{p^k}$-linear combination of basis elements equal to $g_{n,i} \cdot (-p^{k-1} y_{n,i-1})$ for some $\langle n, i \rangle \in P$, by 5.5a. Expressing $g_{n,i}$ as a $\mathbf{Z}_{p^k}$-linear combination of basis elements, it has a summand monomial $\kappa y^{u'}$ such that $-p^{k-1}\kappa = \delta$ in $\mathbf{Z}_{p^k}$ and u' in U satisfies $u'(m, j) = u(m, j)$, except that $u'(n, i-1) = u(n, i-1) - 1$ if $i > 1$. Define $u''\colon B \to \mathbf{N}_0$ by $u''(m, j) = u'(m, j)$ for $\langle m, j \rangle \neq \langle n, i \rangle$ and $u''(n, i) = u'(n, i) + n = u(n, i) + n$. Then $\kappa y^{u''}$ is a summand of $g_{n,i} \cdot y_{n,i}^n$ and $y_{n,i}$ is the only variable with a high power in u''. From $-p^{k-1}\kappa = \delta \neq 0$ we conclude that p does not divide κ in $\mathbf{Z}_{p^k}$. Since all the other summands of f_0 not obtained from $g_{n,i} \cdot y_{n,i}^n$ have either *another* variable with a high power or a coefficient divisible by p, $f = f_0$ cannot hold, and $C = \emptyset$. $\square$

Corollary 5.7. *$R(H)$ is of characteristic p^k, whence $\mathcal{L}(R(H)) \in \mathbf{W}(p^k)$.*

Proof. Use 5.5b and 5.6.

Lemma 5.8. *If $n \in H \subseteq H_0$ then*

$$y_{n,n-j} + J(H) \in (\Uparrow e_{j-1,n})_{R(H)} \tag{5.8a}$$

holds for $j = 1, 2, \ldots, n$, and τ_n is satisfied in $R(H)$.

Proof. First, verify 5.8a by induction. Note that $p^{k-1} y_{n,n-1} \in J(H)$ for $n \in H$, so $y_{n,n-1} + J(H)$ is in $(\Uparrow e_{0,n})_{R(H)}$. This proves 5.8a for $j = 1$. Now assume 5.8a as an induction hypothesis for $1 \le j < n$. Using 5.4 we have

$$p^{k-1} y_{n,n-(j+1)} + J(H) = y_{n,n-j}^n + J(H) \in (\Uparrow e_{j-1,n})_{R(H)}^n = (e_{j,n})_{R(H)},$$

and 5.8*a* for $j+1$ follows, completing the induction. By 5.8a with $j = n$ we obtain

$$1_{R(H)} = y_{n,0} + J(H) \in (\Uparrow e_{n-1,n})_{R(H)} = (d_n)_{R(H)},$$

from which it follows that τ_n holds in $R(H)$. $\square$

By close analysis, we will see that τ_n is not satisfied in $R(H)$ when n is in $H_0 \setminus H$. Theorem 5.1 then follows quickly.

Definitions and Properties 5.9. Fix H and n with $2k \in H \subseteq H_0$ and $n \in H_0 \setminus H$. For $0 \le j \le n-1$ and $w \in W$, say that the predicate last(j, w) holds iff there exists $m > n$, $m \in H$, such that

$$w(m, m-2) + w(m, m-3) + \ldots + w(m, m-j-1) \ge 1,$$

and last$^*(j, w)$ holds iff

$$\sum_{m \in H,\, m > n} \big(w(m, m-2) + w(m, m-3) + \ldots + w(m, m-j-1)\big) \ge 2.$$

By convention, any empty sum equals zero. In particular, the sums above are taken as empty if $j = 0$.

(5.9a) For all w, $\mathrm{last}(0, w)$ and $\mathrm{last}^*(0, w)$ do not hold.
(5.9b) If $\mathrm{last}(j, w)$ holds then so does $\mathrm{last}(s, w)$ for $j < s \le n - 1$.
(5.9c) If $\mathrm{last}(j, w)$ holds then so does $\mathrm{last}(j, w + z)$ for all $z \in W$. If $\mathrm{last}^*(j, w)$ holds then so does $\mathrm{last}^*(j, w + z)$ for all $z \in W$.
(5.9d) If $\mathrm{last}(j, w)$ and $\mathrm{last}(j, z)$ both hold then $\mathrm{last}^*(j, w+z)$ holds. If $\mathrm{last}^*(j, w)$ holds then $\mathrm{last}(j, w)$ holds.

Lemma 5.10. *Suppose n is in $H_0 \setminus H$ and L_j is the $\mathbf{Z}_{p^k}$-submodule of $R(H)$ generated by $W_j = U_{j1} \cup U_{j2} \cup V_0$ for $j = 0, 1, \dots, n-1$ where*

$$U_{j1} = \{1x^u\colon\ u \in U \textit{ and } \mathrm{last}(j, u)\},$$
$$U_{j2} = \{px^z\colon\ z \in U \textit{ and not } \mathrm{last}(j, z)\},$$
$$V_0 = \{1x^v\colon\ v \in V\}.$$

Then L_j is an ideal of $R(H)$ which satisfies

$$(\Uparrow e_{j,n})_{R(H)} \subseteq L_j \tag{5.10a}$$

for $j = 0, 1, \dots, n-1$, and τ_n is not satisfied in $R(H)$.

Proof of Lemma 5.10. Assume the hypothesis and suppose that $0 \le j \le n-1$. By 5.5a, L_j is an ideal if αx^{w+z} is in L_j for all $w \in W$ and $\alpha x^z \in W_j$. By 5.5d, x^{w+z} is a multiple of p^{k-1} if $w + z$ has a high power, and then αx^{w+z} is in L_j by 5.6. If $w + z$ has a last variable and no high power then $1x^{w+z}$ is in $V_0 \subseteq L_j$. Otherwise, $w + z$ is in U, whence $z \in U$. Then αx^{w+z} is in L_j if $\mathrm{last}(j, w + z)$, and also if not $\mathrm{last}(j, w + z)$ because then not $\mathrm{last}(j, z)$ holds by 5.9c and $\alpha = p$ by 5.6. Therefore each L_j is an ideal of $R(H)$.

Next, verify 5.10a by induction on j. If $j = 0$ then 5.10a follows from 5.6, 5.9a and $\Uparrow \mathbf{0} = \downarrow \mathbf{1}$ for $\mathbf{Z}_{p^k}$. So, assume 5.10a as an induction hypothesis, $0 \le j < n-1$. Let K denote the $\mathbf{Z}_{p^k}$-submodule of $R(H)$ generated by $K^* \cup V_0$ where

$$K^* = \{1x^u\colon\ u \in U \text{ and } \mathrm{last}^*(j, u)\} \cup \{px^u\colon\ u \in U \text{ and } \mathrm{last}(j + 1, u)\}.$$

To prove that $L_j^n \subseteq K$, it suffices by ring distributivity to show that αx^w is in K for $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ and $w = w_1 + w_2 + \dots + w_n$ where $\alpha_s x^{w_s}$ is in W_j for $s = 1, 2, \dots, n$. Since 0 is in K, we assume $\alpha x^w \ne 0$.

Suppose that w has no high power. If w has a last variable then $1x^w \in V_0$, hence $\alpha x^w \in K$. Suppose w has no last variable, so $w \in U$. Then no w_s has a last variable, hence all $\alpha_s x^{w_s}$ are in $U_{j1} \cup U_{j2}$ by 5.6. But $\alpha = 0$ if more than k elements α_s are equal to p, hence by 5.6 at least $n - k \ge k \ge 2$ elements $\alpha_s x^{w_s}$ are in U_{j1}. Since $\mathrm{last}(j, w_s)$ for two or more s values implies $\mathrm{last}^*(j, w)$ by 5.9c and 5.9d, αx^w is in K.

Now suppose that w has a high power, say $w(m, i) \ge m$ for some $\langle m, i\rangle \in B$. Define $z \in W$ by $z(m, i) = w(m, i) - m$, $z(m, i - 1) = w(m, i - 1) + 1$ if $i > 1$, and $z(s,t) = w(s,t)$ otherwise. By 5.5d, $\alpha x^w = p^{k-1}\alpha x^z$, and $\alpha x^w \neq 0$ implies that $z \in U$ and p does not divide α, hence no $\alpha_s x^{w_s}$ is in U_{j2}. If some $\alpha_s x^{w_s}$ is in U_{j1}, then last(j,w) holds by 5.9c, so either last(j,z) holds or $m > n$ and $m-j-1 \leq i \leq m-2$, so last$(j+1,z)$ holds by 5.9b or because $z(m,i-1) > 0$, hence $\alpha x^w = p^{k-1}\alpha x^z$ is in K. Otherwise, $\alpha_s x^{w_s}$ is in V_0 for all $s \leq n$. Then $z \in U$ implies that $i = m-1$, $w_s(m,m-1) > 0$ for all $s \leq n$, $w(m,m-1) = m$ and $z(m,m-2) > 0$. But then $m \geq n$, and so $m > n$ since $m \in H$ and $n \notin H$. Therefore last$(1,z)$ holds, and $\alpha x^w = p^{k-1}\alpha x^z$ is in K by 5.9b. This completes the proof that $L_j^n \subseteq K$.

Suppose $f \in\Uparrow (L_j^n)$ and $f = \sum_{u\in C}\alpha_u x^u + \sum_{v\in D}\beta_v x^v$ as in 5.6. Using 5.5c, $p^{k-1}f = \sum_{u\in C} p^{k-1}\alpha_u x^u \in L_j^n \subseteq K$. By the uniqueness in 5.6, either $p \mid \alpha_u$ or last$^*(j,u)$ holds or last$(j+1,u)$ holds for each u in C. If $p \mid \alpha_u$, obviously $\alpha_u x^u \in L_{j+1}$. Now last$^*(j,u)$ implies last$(j+1,u)$ by 5.9b and 5.9d, and so $\alpha_u x^u$ is in L_{j+1} in the last two cases. Then $V_0 \subseteq L_{j+1}$ implies $f \in L_{j+1}$, so

$$(\Uparrow e_{j+1,n})_{R(H)} = \Uparrow ((\Uparrow e_{j,n})^n)_{R(H)} \subseteq \Uparrow (L_j^n) \subseteq L_{j+1},$$

using 5.10a and the above. This completes the induction, and so 5.10a holds for $0 \leq j \leq n-1$.

By 5.10a with $j = n-1$ we obtain $(d_n)_{R(H)} = (\Uparrow e_{n-1,n})_{R(H)} \subseteq L_{n-1}$. Since $1_{R(H)}$ is not in L_{n-1} by 5.5b and 5.6, τ_n is not satisfied in $R(H)$. $\square$

Now, armed with several preliminary statements, not much is left from proving Theorem 5.1. The argument runs as follows. By 5.7, we can assign $\mathcal{L}(R(H)) \in \mathbf{W}(p^k)$ to any H as above. Suppose $\{2k\} \subseteq H_j \subseteq H_0$ for $j = 1,2$.

We claim that

$$H_1 \supseteq H_2 \iff \mathcal{L}(R(H_1)) \subseteq \mathcal{L}(R(H_2)). \tag{5.11}$$

To show the forward implication, suppose $H_1 \supseteq H_2$. Then the obvious ring homomorphism

$$F(H_2) \to F(H_1) \to F(H_1)/J(H_1)$$

annihilates $J(H_2)$ by 5.4. Hence $R(H_1)$ is a homomorphic image[1] of $R(H_2)$, and we can apply Hutchinson [Ht1, Proposition 2] (cf. also [Cz5, Corollary 6.1]) to conclude $\mathcal{L}(R(H_1)) \subseteq \mathcal{L}(R(H_2))$. To verify the reverse implication, suppose $\mathcal{L}(R(H_1)) \subseteq \mathcal{L}(R(H_2))$. In virtue of Theorem 5.B we obtain $K(R(H_1)) \supseteq K(R(H_2))$. Now 5.8 and 5.10 yield $n \in H_i \iff \tau_n \in K(R(H_i))$, whence $H_1 \supseteq H_2$ follows.

Now 5.11 clearly shows that the partially ordered set $\langle\{H\colon\ 2k \in H \subseteq H_0\},\supseteq\rangle$ is embedded in $\mathbf{W}(p^k)$. But $\langle\{H\colon\ 2k \in H \subseteq H_0\},\supseteq\rangle \cong \langle\{H\colon\ H \subseteq H_0 \setminus \{0\}\},\supseteq\rangle = \langle P(H_0 \setminus \{0\}),\supseteq\rangle \cong \langle P(H_0 \setminus \{0\}),\subseteq\rangle$. Thus, taking $A = H_0 \setminus \{0\}$ we have seen that $\mathbf{W}(p^k)$ has a subset order-isomorphic to $P(A) = \langle P(A),\subseteq\rangle$. Clearly, $P(A)$ has huge chains; put $A = \mathbf{Q}$, the set of rational numbers, consider the chain of

[1] by a homomorphism we mean a 1-preserving ring homomorphism

Dedekind cuts, i.e. $\{(r] \cap \mathbf{Q}\colon\ r \text{ is a real number}\}$. If $A = \mathbf{Z}$, the set of integers, and $\mathbf{N}$ is the set of positive integers then $\{(\mathbf{N} \setminus X) \cup (-X)\colon\ X \subseteq N\}$ is an antichain in $P(A)$ with continuously many elements. (Here $-X$ denotes $\{-x\colon\ x \in X\}$.) This completes the proof of Theorem 5.1 $\square$

In Hutchinson [Ht1, Thm. 4] the case of composite characteristic is completely reduced to the case of prime power characteristic. Indeed, [Ht1, Thm. 4] asserts that for $m = p_1^{t_1} \dots p_s^{t_s}$, where the p_i are distinct primes, and R and S of characteristic m we have $\mathcal{L}(R) \subseteq \mathcal{L}(S)$ iff $\mathcal{L}(R/p_i^{t_i}R) \subseteq \mathcal{L}(S/p_i^{t_i}S)$ for every i. Hence it is not hard to conclude that for $m = p_1^{t_1} \dots p_s^{t_s}$, where the p_i are distinct primes, the partially ordered set (perhaps a lattice?) $\mathbf{W}(m) = \{\mathcal{L}(R)\colon\ R \text{ is a ring with characteristic } m\}$ is isomorphic to $\mathbf{W}(p_1^{t_1}) \times \dots \times \mathbf{W}(p_s^{t_s})$. For each i, if $t_i = 1$ then $\mathbf{W}(p_i^{t_i})$ is a singleton, cf. Hutchinson [Ht1, Thm. 5(6), p. 88] (or, for a different proof, [Cz5, Prop. 6.2]). Thus, Theorem 5.1 leads to

Corollary 5.12. *([CH2]) Let m be a positive integer. If m is square-free then $\mathbf{W}(m)$ is a singleton. Otherwise, if $p^2 \mid m$ for some prime p, $\mathbf{W}(m)$ has a sub-poset order-isomorphic to $P(A)$ for a countably infinite set A.*

Proof of Theorem 5.2. The proof is based on certain lattice Horn sentences $\chi(m,p)$, which might be of separate interest. Note that $\chi(2,2)$, the simplest particular case of $\chi(m,p)$, appeared in [CH1]. However, the work with $\chi(m,p)$ and especially the way to find it proved much more difficult than one could expect from the particular case $\chi(2,2)$. Our proof is divided into several lemmas.

First we define appropriate rings. The ring of integers modulo p^k will be denoted by $\mathbf{Z}_{p^k}$. For a given n let F_n denote the polynomial ring

$$\mathbf{Z}_{p^k}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n].$$

Let I_n be the ideal generated by

$$\begin{aligned}&\{\,\xi_i\eta_i - p^{k-1}\xi_{i-1} : 1 \le i \le n\,\} \cup \{\,p\eta_i : 1 \le i \le n\,\} \cup \\ &\quad \{\,p^{k-1}\xi_n\,\} \cup \{\,\xi_i\xi_j : 1 \le i \le n,\ 1 \le j \le n\,\} \cup \{\,\eta_i\eta_j : 1 \le i \le n,\ 1 \le j \le n\,\} \\ &\qquad\qquad\qquad\qquad \cup \{\,\xi_i\eta_j : 1 \le i \le n,\ 1 \le j \le n,\ i \ne j\,\},\end{aligned}$$

where $\xi_0 = 1$. Put $R_n = F_n/I_n$, $\quad x_i = \xi_i + I_n$, $\quad y_i = \eta_i + I_n$. Note that $x_0 = 1$. By the definition of R_n we have

$$\text{(5.13)} \quad \begin{aligned}&x_iy_i = p^{k-1}x_{i-1}, \quad y_iy_j = 0, \quad x_ix_j = 0, \quad x_iy_l = 0, \quad p^kx_i = 0, \\ &\qquad p^{k-1}x_n = 0, \quad p\,y_i = 0 \qquad \text{for } i,j,l \in \{\,1,2,\dots n\,\},\ i \ne l.\end{aligned}$$

Lemma 5.14. *The elements x_i ($i = 0, \dots, n-1$), x_n and y_i ($i = 1, \dots, n$) are of respective additive order p^k, p^{k-1} and p. Further, the additive group of R_n is the direct sum of the additive cyclic subgroups generated by these elements. In other words, each element of R_n has a unique canonical form*

$$\sum_{i=0}^{n-1} \alpha_i x_i + \beta x_n + \sum_{i=1}^{n} \gamma_i y_i \tag{5.15}$$

where $\alpha_i \in \{0, 1, \dots, p^k - 1\}$, $\beta \in \{0, 1, \dots, p^{k-1} - 1\}$ and $\gamma_i \in \{0, 1, \dots, p - 1\}$. The rules of computation in R_n are (1) together with the axioms of unital commutative rings of characteristic p^k.

Proof. It suffices to show the uniqueness of (5.15); the rest is clear. Assume that $0 \in R_n$ is of the form (5.15). Then, by the definition of I_n, we have

$$\begin{aligned} \sum_{i=0}^{n-1} \alpha_i \xi_i + \beta \xi_n + \sum_{i=1}^{n} \gamma_i \eta_i &= \sum_{i=1}^{n} f_i \cdot (\xi_i \eta_i - p^{k-1} \xi_{i-1}) + \sum_{i=1}^{n} g_i \cdot p \eta_i \\ &+ g_0 p^{k-1} \xi_n + \sum_{i=1}^{n} \sum_{j=1}^{n} h_{ij} \cdot \xi_i \xi_j + \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij} \cdot \eta_i \eta_j + \sum_{i=1}^{n} \sum_{\substack{l=1 \\ l \neq i}}^{n} s_{ij} \cdot \xi_i \eta_l \end{aligned} \tag{5.16}$$

where f_i, g_i, h_{ij}, r_{ij}, $s_{ij} \in F_n$. We treat the elements of F_n as polynomials in the usual canonical form. Hence these polynomials are sums of uniquely determined summands and each summand consists of uniquely determined factors (i.e., powers of indeterminants) and a unique coefficient (from $\mathbf{Z}_{p^k}$). Suppose we have performed the operations on the right-hand-side of (5.16). Then each summand on the right-hand-side in which η_i is the only indeterminant has a coefficient divisible by p. Therefore $\gamma_i = 0$ for all i. We obtain $\beta = 0$ similarly.

Suppose $\alpha_i \neq 0$ for some i. The only source of ξ_i on the right is $f_{i+1} \cdot (\xi_{i+1} \eta_{i+1} - p^{k-1} \xi_i)$. Since p^k does not divide α_i, the constant δ in f_{i+1} is not divisible by p. But then $\delta \xi_{i+1} \eta_{i+1}$ cannot be cancelled by other summands. This contradiction completes the proof. $\square$

Before describing $K(\mathbf{Z}_{p^k})$ we make the set $\{0, 1, 2, \dots k\}$ into an algebra of type τ via putting $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $\uparrow x = \min\{x+1, k\}$, $\downarrow x = \max\{x-1, 0\}$, $\mathbf{0} = 0$, $\mathbf{1} = k$ and $x \cdot y = \max\{x + y - k, 0\}$. (To avoid confusion, the ordinary product of x and y will be denoted by the concatenation xy.) Denoting the set of nullary τ-terms by $\mathcal{P}_0$, let h be the map associating with any element of $\mathcal{P}_0$ its value in the above-defined algebra $\{0, 1, 2, \dots k\}$.

Lemma 5.17. $K(\mathbf{Z}_{p^k}) = \{\sigma \in \mathcal{P}_0 : h(\sigma) = k\}$.

Proof. An easy induction on the length of σ yields that the value of σ in $\mathcal{I}(\mathbf{Z}_{p^k})$ is $p^{k-h(\sigma)} \mathbf{Z}_{p^k} = \downarrow^{k-h(\sigma)} \mathbf{Z}_{p^k}$, whence the lemma follows. $\square$

Lemma 5.18. $\bigcap_{n=1}^{\infty} K(R_n) = K(\mathbf{Z}_{p^k})$.

Proof. For $0 \le t \le n-1$ and $0 \le j \le k$ we consider the following subsets of R_n:

$$\begin{aligned}
A^{(n)}_{j,t} &= \{\, p^i x_l : 1 \le l < n-t,\ i \ge k-j,\ i \ge 0 \,\},\\
B^{(n)}_{j,t} &= \{\, p^i x_l : n-t \le l \le n-1,\ i+l \ge n-t+k-j-1,\ i \ge 0 \,\},\\
C^{(n)}_{j,t} &= \{\, p^i x_n : i \ge k-j-1,\ i \ge 0 \,\},\\
D^{(n)}_{j,t} &= \{ y_l : 1 \le l \le n,\ j > 0 \,\} \quad \text{and}\\
E^{(n)}_{j,t} &= \{\, p^i : i \ge k-j \,\} \cup A^{(n)}_{j,t} \cup B^{(n)}_{j,t} \cup C^{(n)}_{j,t} \cup D^{(n)}_{j,t}\,.
\end{aligned}$$

Note that $D^{(n)}_{j,t} = \{\, y_1, \dots, y_n \}$ for $j > 0$ and $D^{(n)}_{0,t} = \emptyset$. Let $I^{(n)}_{j,t}$ be the additive subgroup of R_n generated by $E^{(n)}_{j,t}$. With the help of Lemma 5.14 it is not hard to see that the $I^{(n)}_{j,t}$ are ideals of R_n, $I^{(n)}_{k,t} = R_n$, $\quad 0 \le t_1 \le t_2 \le n-1$ implies $I^{(n)}_{j,t_1} \subseteq I^{(n)}_{j,t_2}$, and $0 \le j_1 \le j_2 \le k$ implies $I^{(n)}_{j_1,t} \subseteq I^{(n)}_{j_2,t}$. Further, $\downarrow I^{(n)}_{j,t} \subseteq I^{(n)}_{\downarrow j,t}$, and $\uparrow I^{(n)}_{j,t} \subseteq I^{(n)}_{\uparrow j,t}$. Now we claim that $I^{(n)}_{j,t} \cdot I^{(n)}_{s,t} \subseteq I^{(n)}_{j\cdot s,t+1}$. Suppose $a \in E^{(n)}_{j,t}$ and $b \in E^{(n)}_{s,t}$. It suffices to check $ab \in E^{(n)}_{j\cdot s,t+1}$. We omit the straightforward but long details and consider only the case $a \in B^{(n)}_{j,t}$ and $b \in D^{(n)}_{s,t}$. Then $a = p^i x_l$, $\quad n-t \le l \le n-1$, $\quad i+l \ge n-t+k-j-1$ and $s > 0$. We may assume that $b = y_l$ as otherwise $ab = 0$. We conclude $ab = p^{i+k-1} x_{l-1}$, $\quad n-(t+1) \le l-1 \le n-1$ and $(i+k-1)+(l-1) = i+l+k-2 \ge n-t+k-j-1+k-2 = n-(t+1)+k-(j+1-k)-1 \ge n-(t+1)+k-(j+s-k)-1 \ge n-(t+1)+k-j\cdot s-1$, yielding $ab \in B^{(n)}_{j\cdot s,t+1} \subseteq E^{(n)}_{j\cdot s,t+1}$.

For a τ-term $\sigma \in \mathcal{P}_0$ let σ_{R_n} denote the value of σ in $\mathcal{I}(R_n)$. The *length* $|\sigma|$ of σ is defined via induction: $|\,\mathbf{0}\,| = |\,\mathbf{1}\,| = 1$, $|\uparrow \sigma| = |\downarrow \sigma| = |\sigma| + 1$, $|\sigma_1 \vee \sigma_2| = |\sigma_1 \wedge \sigma_2| = |\sigma_1 \cdot \sigma_2| = |\sigma_1| + |\sigma_2| + 1$. The inclusions among the $I^{(n)}_{j,t}$ we have already established yield

$$\sigma_{R_n} \subseteq I^{(n)}_{h(\sigma),|\sigma|}\,, \qquad \text{provided } |\sigma| < n\,, \tag{5.18a}$$

via an easy induction on $|\sigma|$.

Now the proof of Lemma 5.18 will be completed easily. Suppose that $\sigma \notin K(\mathbf{Z}_{p^k})$. Then $h(\sigma) \le k-1$ by Lemma 5.17. Choose an n with $n > |\sigma| + 2$. Then, by (5.18a) and Lemma 5.14,

$$\sigma_{R_n} \subseteq I^{(n)}_{h(\sigma),|\sigma|} \subseteq I^{(n)}_{k-1,|\sigma|} \subseteq I^{(n)}_{k-1,n-2} \not\ni 1,$$

whence $\sigma \notin K(R_n)$. Therefore $\bigcap_{l=1}^{\infty} K(R_l) \not\supset K(\mathbf{Z}_{p^k})$.

Conversely, an easy induction on $|\sigma|$ yields $\sigma_{R_n} \supseteq \downarrow^{k-h(\sigma)} R_n$. In particular, if $h(\sigma) = k$ then $\sigma_{R_n} = R_n$. Hence Lemma 5.17 yields $\bigcap_{l=1}^{\infty} K(R_l) \supseteq K(\mathbf{Z}_{p^k})$, proving Lemma 5.18. $\square$

Now let $m = p^{k-1}$. On the set of variables $\{x, y, z, t\}$ we define the following lattice terms:

$$\begin{aligned} r &= (x \vee y) \wedge (z \vee t), & h_0 &= g_0 = t, & h_i' &= (h_i \vee y) \wedge (x \vee z) \\ h_{i+1} &= (h_i' \vee r) \wedge (x \vee t), & g_i' &= (g_i \vee x) \wedge (y \vee z), & g_{i+1} &= (g_i' \vee r) \wedge (y \vee t), \\ r_0 &= (h_{m-1} \vee z) \wedge y, & q_0 &= x \vee z \vee g_{p-1}, & q &= r_0 \vee x. \end{aligned}$$

Let $\chi(m,p)$ denote the lattice Horn sentence

$$r_0 \le q_0 \implies r \le q\,.$$

Although the proofs of the following two lemmas are "tamed" by now, a long development (going through [Cz17], [Cz18], [Cz15], [CD1], [Cz5] and [CH1]) was necessary to achieve the present technique.

Lemma 5.19. *$\chi(m,p)$ does not hold in $\mathcal{L}(\mathbf{Z}_{p^k})$.*

Proof. Let M be the $\mathbf{Z}_{p^k}$-module freely generated by $\{f_1, f_2, f_3\}$. Consider the submodules $x = [f_2]$, $y = [f_1 - f_2]$, $z = [f_3]$, $t = [f_1 - f_3]$. An easy calculation gives $r = [f_1]$. (We do not make a notational distinction between lattice terms and the submodules obtained from them by substituting the *submodules* x, y, z, t for their variables.) It is not hard to check, via induction on i, that $h_i' = [(i+1)f_2 - f_3]$, $h_i = [f_1 + if_2 - f_3]$, $g_i' = [(i+1)f_1 - (i+1)f_2 - f_3]$, $g_i = [(i+1)f_1 - if_2 - f_3]$. These equations yield $r_0 = \{\alpha(f_1 - f_2) : m\alpha = 0\} = [p(f_1 - f_2)]$, $q_0 = [pf_1,\, f_2,\, f_3]$, $q = [pf_1,\, f_2]$. Therefore $\chi(m,p)$ does not hold in $\mathrm{Su}(M)$. $\square$

Lemma 5.20. *$\chi(m,p)$ holds in $\mathcal{L}(R_n)$ for every $n \ge 1$.*

Proof. Assume that x, y, z, t are submodules of an R_n-module M such that $r_0 \subseteq q_0$, and let $f_1 \in M$ be an arbitrary element of r. Our aim is to show $f_1 \in q$. Since $f_1 \in r = (x + y) \cap (z + t)$, we can choose $f_2, f_3 \in M$ such that $f_2 \in x$, $f_1 - f_2 \in y$, $f_3 \in z$, $f_1 - f_3 \in t$. An easy calculation, essentially the same as in the previous lemma, gives $(i+1)f_2 - f_3 \in h_i'$, $f_1 + if_2 - f_3 \in h_i$, and $\{\alpha(f_1 - f_2) : m\alpha = 0\} \subseteq r_0$. In particular, $x_n(f_1 - f_2) \in r_0$.

Now let us suppose that $x_j(f_1 - f_2) \in r_0$ for some $j > 0$. We intend to show $x_{j-1}(f_1 - f_2) \in r_0$; then $f_1 - f_2 = x_0(f_1 - f_2) \in r_0$ follows by (downward) induction on j. From $r_0 \subseteq q_0$ we infer $x_j(f_1 - f_2) \in q_0 = x + z + g_{p-1}$. Hence there exist elements e_0 and e_1 in M such that $e_0 \in x$, $e_1 - e_0 \in z$ and $x_j(f_1 - f_2) - e_1 \in g_{p-1} = (g_{p-2}' + r) \cap (y + t)$. This implies the existence of two elements, say e_2^{p-1} and $e_4^{p-1} \in M$ such that $e_1 - e_4^{p-1} \in y$, $x_j(f_1 - f_2) - e_4^{p-1} \in t$, $e_1 - e_2^{p-1} \in g_{p-2}'$, and $x_j(f_1 - f_2) - e_2^{p-1} \in r$. Continuing this parsing and denoting $x_j(f_1 - f_2)$ by e_1^p we obtain that there exist elements $e_l^i \in M$ for $i = 1, 2, \ldots, p-1$ and $l = 1, 2, \ldots, 6$ such that for $i \in \{1, 2, \ldots, p-1\}$

$$\begin{gathered} e_1 - e_3^i \in y, \quad e_1 - e_4^i \in y, \quad e_2^i - e_3^i \in z, \quad e_4^i - e_1^{i+1} \in t, \quad e_1^i - e_2^i \in x, \\ e_2^i - e_5^i \in x, \quad e_1^{i+1} - e_5^i \in y, \quad e_2^i - e_6^i \in z, \quad e_1^{i+1} - e_6^i \in t, \quad e_1 - e_1^1 \in t. \end{gathered}$$

Clearly, $e_1^p = x_j(f_1 - f_2) \in y$. Let us observe that x contains $u_0 = x_j f_2 + e_0 + \sum_{i=1}^{p-1}(e_2^i - e_1^i)$. But $u_0 = \sum_{i=1}^{p-2}(e_2^i - e_6^i) + \sum_{i=1}^{p-2}(e_6^i - e_1^{i+1}) - (x_j(f_1 - f_2) - e_4^{p-1}) + x_j(f_1 - f_3) + x_j f_3 + (e_0 - e_1) + (e_1 - e_1^1) + (e_2^{p-1} - e_6^{p-1}) + (e_6^{p-1} - e_1^p) + (e_1^p - e_4^{p-1})$, whence $u_0 \in r$. Now $u_0 \in x$ and $u_0 \in r$ imply $u_0 \in h_i$ for all $i > 0$. In particular, $u_0 \in h_{m-1}$. Let $u_i = e_0 - e_1 - e_2^i + e_3^i$ for $1 \le i \le p-1$. We have, for $i > 0$, $u_i = e_0 - (e_1 - e_3^i) - e_1^p + \sum_{l=i}^{p-1}(e_1^{l+1} - e_5^l) + \sum_{l=i}^{p-1}(e_5^l - e_2^l) + \sum_{l=i+1}^{p-1}(e_2^l - e_1^l) \in x + y$ and $u_i = (e_0 - e_1) - (e_2^i - e_3^i) \in z$, whence $u_i \in r$. Let $v_i = e_1 + e_1^i - e_3^i$. Since $e_1^1 - e_3^1 = (e_1^1 - e_1) + (e_1 - e_3^1) \in y + t$ and, for $i > 1$, $e_1^i - e_3^i = (e_1^i - e_4^{i-1}) - (e_1 - e_4^{i-1}) + (e_1 - e_3^i) \in y + t$, we have $v_i = (e_1 - e_4^{p-1}) + (e_4^{p-1} - e_1^p) + e_1^p + (e_1^i - e_3^i) \in y + t$. But $v_i = e_0 - (e_0 - e_1) + (e_1^i - e_2^i) + (e_2^i - e_3^i) \in x + z$, whence $v_i \in h_0'$ $(i = 1, 2, \ldots, p-1)$. For $1 \le i \le p-1$ let $w_i = e_0 + e_1^i - e_2^i$. From $w_i = v_i + u_i \in h_0' + r$ and $w_i = e_0 + (e_1^i - e_2^i) \in x$ we infer that $w_i \in h_1$. This together with $w_i \in x$ yield $w_i \in h_{m-1}$.

Now $x_{j-1}(f_1 - f_2) \in y$ and, by $y_j x_j = m x_{j-1}$ and $p y_j = 0$, $x_{j-1}(f_1 - f_2) = x_{j-1}(f_1 + (m-1) f_2 - f_3) - y_j u_0 - \sum_{i=1}^{p-1} y_j w_i + x_{j-1} f_3 \in h_{m-1} + z$. Thus $x_{j-1}(f_1 - f_2) \in r_0$, as intended.

Finally, $f_1 = (f_1 - f_2) + f_2 \in r_0 + x = q$ completes the proof of Lemma 5.20. $\square$

Armed with the previous lemmata we can complete the proof Theorem 5.2 as follows. Let us assume that Problem 5.A has an affirmative answer. We claim that

$$\bigvee_{n=1}^{\infty} \mathcal{L}(R_n) = \mathcal{L}(\mathbf{Z}_{p^k}) \tag{5.21}$$

where the join is formed in $(\mathbf{W}(p^k); \subseteq)$. Since $K(R_n) \supseteq K(\mathbf{Z}_{p^k})$ by Lemma 5.18, we obtain $\mathcal{L}(R_n) \subseteq \mathcal{L}(\mathbf{Z}_{p^k})$, for every n, by the assumption. (Note that $\mathcal{L}(R_n) \subseteq \mathcal{L}(\mathbf{Z}_{p^k})$ also follows from Theorem 5.A.) On the other hand, suppose $\mathcal{L}(S) \in \mathbf{W}(p^k)$ and, for all n, $\mathcal{L}(R_n) \subseteq \mathcal{L}(S)$. Theorem 5.B yields $K(R_n) \supseteq K(S)$. From Lemma 5.18 we conclude $K(\mathbf{Z}_{p^k}) = \bigcap_{n=1}^{\infty} K(R_n) \supseteq K(S)$, and the assumption on Problem 5.A gives $\mathcal{L}(\mathbf{Z}_{p^k}) \subseteq \mathcal{L}(S)$. This proves (5.21).

Now if Problem 5.B had an affirmative answer then (5.21) would be true even in the lattice of all quasivarieties of lattices. But this would contradict Lemmas 5.19 and 5.20. The proof of Theorem 5.2 is complete. $\square$

The Horn sentence $\chi(m, p)$ in the above proof has four variables. The identities $\Delta(m, n)$ from [HC1], which were used in Chapter II, have four variables, too. On the other hand, there are important lattices Horn sentences, e.g. the modular law or the join semidistributivity $x \vee y = x \vee z \Longrightarrow x \vee y = x \vee (y \wedge z)$ and its dual, that have three variables only. Less variables, one or two, necessarily lead to a trivial Horn sentence, which either holds in all lattices or holds only in the one element lattice. In the rest of the chapter we deal with the problem whether having four variables in our case is really necessary. Since submodule lattices are modular, the following assertion implies that yes, four cannot be reduced.

Proposition 5.22. *Let χ be a lattice Horn sentence on three variables. Then either χ is a consequence of the modular law or χ together with modularity imply distributivity.*

Proof. Let χ be of the form

$$p_1 \leq q_1 \ \& \ \ldots \ \& \ p_k \leq q_k \Longrightarrow p \leq q$$

where $k \geq 0$ and $p_1, \ldots\ p_k$, p, $q_1, \ldots\ q_k$ and q are lattice term on the set $\{x, y, z\}$ of variables. ($k = 0$ is allowed, then χ is a lattice identity.) Let $M = F_{\mathbf{M}}(x, y, z)$ denote the modular lattice freely generated by $\{x, y, z\}$. Then, by a classical result of Dededind [De1], M has twenty-eight elements, and the reader is assumed to be familiar with its diagram (cf. also Grätzer [Gr1, page 39] or any other textbook).

Suppose χ is not a consequence of the modular law. We intend to show that χ *together with* modularity implies distributivity. Hence, without loss of generality, we may assume that $p_1, \ldots\ p_k$, p, $q_1, \ldots\ q_k$ and q are element of M. If our statement is true for χ_1 and χ_2 and χ is equivalent (modulo lattice theory) to the conjunction of χ_1 and χ_2 the statement is also true for χ. On the other hand, in any lattice and for arbitrary lattice terms r, r_1 and r_2, $r_1 + r_2 \leq r$ holds iff $r_1 \leq r$ and $r_2 \leq r$, and dually. This allows us to make the following assumption: $p_1, \ldots p_k, p \in M$ are join-irreducible elements and $q_1, \ldots q_k, q \in M$ are meet-irreducible elements. We may also suppose that $p_1 \not\leq q_1, \ldots, p_k \not\leq q_k$ and $p \not\leq q$ in M and, in M, $p \leq p_i$ simultaneously with $q_i \leq q$ holds for no $i \in \{1, \ldots, k\}$ as otherwise χ would automatically hold in all modular lattices or some of the $p_i \leq q_i$ could be omitted from the premise of χ. Finally, we assume that χ holds in $\mathbf{2}$, the two-element lattice, for otherwise χ would trivially imply distributivity. Therefore, by this assumption, χ holds in every distributive lattice, for each distributive lattice is a subdirect power of $\mathbf{2}$.

Now let $u = xy + xz + yz$, $v = (x+y)(x+z)(y+z)$, $a = u + xv$, $b = u + yv$, and $c = u + zv$. Then $M_3 = [u, v] = \{u, a, b, c, v\}$ is the only diamond, i.e. five element nondistributive sublattice, in M. Let ϑ denote the congruence of M corresponding to the partition $\{[x(y+z), x+yz], [y(x+z), y+xz], [z(x+y), z+xy], [0, u], [v, 1]\}$. We claim that, for any congruence $\psi \in \mathrm{Con}(M)$, M/ψ is distributive iff $\psi \not\subseteq \vartheta$. Indeed, if $\psi \subseteq \vartheta$ then $M_3 \cong M/\vartheta$ is a homomorphic image of M/ψ, whence M/ψ is not distributive. Conversely, if $\psi \not\subseteq \vartheta$ then, as the blocks of ψ are intervals, there are $d, e \in M$ such that $\langle d, e \rangle \in \psi \setminus \vartheta$ and e covers d. Further, $\{d, e\} \subseteq M_3$ can be assumed, for the only other case is symmetric and/or dual to $d = a + x$, $e = v + x$, whence $\langle a, v \rangle = \langle vd, ve \rangle \in \psi \setminus \vartheta$. Since $\psi \cap M_3^2 \neq 0$ (the smallest congruence) and M_3 is a simple lattice, ψ includes γ, the congruence of M generated by M_3^2. Hence M/ψ is a homomorphic image of M/γ, which is a distributive (moreover, a free distributive) lattice. Thus M/ψ is distributive.

Now let ϑ_i denote the congruence of M generated by $\langle p_i, p_i q_i \rangle$. Suppose $\vartheta_i \not\subseteq \vartheta$ holds for some $i \in \{1, 2, \ldots, k\}$. If L is an arbitrary modular lattice, $x_1, y_1, z_1 \in L$ and $p_j(x_1, y_1, z_1) \leq q_j(x_1, y_1, z_1)$ for $j = 1, \ldots, k$ then ϑ_i is included in the kernel of the *surjective* homomorphism $M \to [x_1, y_1, z_1]$, $x \mapsto x_1$, $y \mapsto y_1$, $z \mapsto$

z_1. Hence $[x_1, y_1, z_1]$, as a homomorphic image of M/ϑ_i, is distributive, yielding $p(x_1, y_1, z_1) \le q(x_1, y_1, z_1)$. This shows that χ holds in L.

From now on let us assume that $\vartheta_i \subseteq \vartheta$ holds for all $i \in \{1, \dots, k\}$. Then the premise of χ holds for $a, b, c \in M_3$. Really, considering the homomorphism τ: $M \to M_3$, $x \mapsto a$, $y \mapsto b$, $z \mapsto c$, from $\langle p_i, p_i q_i \rangle \in \vartheta = \ker\tau$ we obtain

$$\begin{aligned} p_i(a,b,c) &= p_i(x\tau, y\tau, z\tau) = p_i(x,y,z)\tau = p_i\tau = (p_i q_i)\tau \\ &= p_i(x\tau, y\tau, z\tau) q_i(x\tau, y\tau, z\tau) = p_i(a,b,c) q_i(a,b,c) \end{aligned}$$

for $i = 1, \dots, k$. If $p(a,b,c) \not\le q(a,b,c)$ then χ fails in M_3. Hence χ and modularity imply distributivity. Therefore we assume that $p(a,b,c) \le q(a,b,c)$. In virtue of our former assumptions, a quick glance at M shows that, apart form symmetry and duality, $p = x$ and $q = y + z$. Since χ holds in $\{0,1\}$, a distributive sublattice of M, but $1 = p(1,0,0) \not\le q(1,0,0) = 0$, there is an $i \in \{1, \dots, k\}$ such that $p_i(1,0,0) \not\le q_i(1,0,0)$, i.e. $p_i(1,0,0) = 1$ and $q_i(1,0,0) = 0$. Considering the homomorphism φ: $M \to \{0,1\}$, $x \mapsto 1$, $y \mapsto 0$, $z \mapsto 0$, $\ker\varphi$ has only two blocks: $[x,1]$ and $[0, y+z]$. Since $p_i\varphi = p_i(x,y,z)\varphi = p_i(x\varphi, y\varphi, z\varphi) = p_i(1,0,0) = 1 = 1\varphi$ and $q_i\varphi = 0 = 0\varphi$, we have $p_i \in [x,1]$ and $q_i \in [0, y+z]$. Thus the contradiction $p \le p_i$ and $q_i \le q$ completes the proof. $\square$

CHAPTER VI

INVOLUTION LATTICES

Based on [CC1], [Cz10], [Cz13] and [CS1], the present chapter is devoted to involution lattices. The main result of the chapter, Theorem 6.3, is about a Horn sentence in certain related involution lattices, of course. However, to give a proper motivation to the main result and to give an application (cf. Corollary 6.5), involution lattices as related structures will be studied from some other aspects, too. A quadruplet $L = \langle L; \vee, \wedge, {}^* \rangle$ is called an *involution lattice* if $L = \langle L; \vee, \wedge \rangle$ is a lattice and ${}^*\colon L \to L$ is a lattice automorphism such that $(x^*)^* = x$ holds for all $x \in L$. To present a natural example, let us consider an algebra A. A binary relation $\rho \subseteq A^2$ is called a *quasiorder* of A if ρ is reflexive, transitive and compatible. (Sometimes we consider a set A rather than an algebra, then all relations are compatible.) Defining $\rho^* = \{\langle x, y\rangle\colon\ \langle y, x\rangle \in \rho\}$, the set $\mathrm{Quord}(A)$ of quasiorders of A becomes an involution lattice $\mathrm{Quord}(A) = \langle \mathrm{Quord}(A); \vee, \wedge, {}^*\rangle$, where $\wedge$ is the intersection and $\vee$ is the transitive closure of the union. These involution lattices were studied by Chajda and Pinus [CP3]. For an involution lattice I, the subalgebra $\{x \in I\colon\ x^* = x\}$ is a lattice if we forget about the (trivial) involution operation. In particular, $\{\rho \in \mathrm{Quord}(A)\colon\ \rho^* = \rho\}$ is just the congruence lattice of A. For a lattice L, the direct square L^2 of L becomes an involution lattice if we define $\langle x, y\rangle^* = \langle y, x\rangle$ for $\langle x, y\rangle \in L^2$. The involution lattice arising from the congruence lattice $\mathrm{Con}(A)$ of A this way will be denoted by $\mathrm{Con}^2(A)$. There are many more examples for involution lattices as related structures, e.g., the ideal lattice of a ring with involution, the lattice of all semigroup varieties, the lattice of clones over a two-element set (the so-called Post lattice), etc., but only $\mathrm{Con}^2(A)$ and $\mathrm{Quord}(A)$ of them will be studied in this work.

Motivated by the classical Grätzer — Schmidt Theorem [GS1], Chajda and Pinus [CP3] asked which involution lattices I are isomorphic to $\mathrm{Quord}(A)$. Some partial answer to this question is given in the following four theorems. Note that an obvious necessary condition on I is that it has to be algebraic as a lattice. The simplest case, when the involution is trivial (i.e. $x^* = x$ for all x), is settled in

Theorem 6.1. *([CC1] and Pinus [Pn1], independently.) Let I be an algebraic involution lattice such that $x^* = x$ for all x. Then there exists an algebra A such that $I \cong \mathrm{Quord}(A)$.*

Proof. We will use the yeast graph construction given by Pudlák and Tůma [PT1] which gives an algebra with $\mathrm{Con}(A) \cong I$ and we will show $\mathrm{Con}(A) = \mathrm{Quord}(A)$ only. The graph construction in Chapter 1 of [PT1] is much more general than needed here, so we describe only as much of it as necessary. Let $J = \langle J; \vee, {}^* \rangle$ be a semilattice with involution. The elements of J will be denoted by lowercase Greek letters. Let V be a nonempty set, let $P_2(V)$ denote the set of two-element subsets of V and let $E \subseteq J \times P_2(V)$. An element $\langle \alpha, \{a, b\} \rangle$ of E will mostly be denoted by $\langle a, \alpha, b \rangle$; of course $\langle a, \alpha, b \rangle = \langle b, \alpha, a \rangle$ and $a \neq b$. A pair $G = \langle V, E \rangle$ is called a J-graph or simply graph if, for any $a, b \in V$ and $\alpha, \beta \in J$, $\langle a, \alpha, b \rangle, \langle a, \beta, b \rangle \in E$ implies $\alpha = \beta$. The elements of V are called vertices while the elements of E are called edges. Here α resp. a, b are called the colour resp. endpoints of the edge $\langle a, \alpha, b \rangle$. The endpoints of an edge uniquely determine its colour. Our graphs will often have two distinguished vertices referred to as left and right endpoints. Given two graphs, $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$, a map $f \colon V_1 \to V_2$ is called a homomorphism if for every $\langle a, \alpha, b \rangle \in E_1$ either $f(a) = f(b)$ or $\langle f(a), \alpha, f(b) \rangle \in E_2$. Isomorphisms, endomorphisms and automorphisms are the usual particular cases of this notion.

With any positive integer k and $\langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle \in J^k$ we associate a graph $R(\alpha_1, \dots, \alpha_k)$, called arc, such that the vertex set of $R(\alpha_1, \dots, \alpha_k)$ is $\{a_0, a_1, \dots, a_k\}$ and the edge set is $\{\langle a_0, \alpha_1, a_1 \rangle, \langle a_1, \alpha_2, a_2 \rangle, \dots, \langle a_{k-1}, \alpha_k, a_k \rangle, \langle a_k, \alpha_1, a_{k+1} \rangle, \langle a_{k+1}, \alpha_2, a_{k+2} \rangle, \dots, \langle a_{2k-1}, \alpha_k, a_{2k} \rangle\}$. The vertices a_0 resp. a_{2k} are the left resp. right endpoints of $R(\alpha_1, \dots, \alpha_k)$. Given an $\alpha \in J$, we define a graph $C(\alpha)$, called α-cell, as follows. We start with $C_0(\alpha) = \langle \{b_0, b_1\}, \{\langle b_0, \alpha, b_1 \rangle\} \rangle$. I.e., $C_0(\alpha)$ consists of two vertices, which are its endpoints, and a single α-coloured edge connecting them. For each $k \geq 1$ and for each $\langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle \in J^k$ such that $\alpha \leq \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_k$ let us take (an isomorphic copy of) the arc $R(\alpha_1, \alpha_2, \dots, \alpha_k)$. The arcs we consider must be disjoint from each other and from $C_0(\alpha)$ as well. Now identifying the left endpoints of these arcs with b_0 and their right endpoints with b_1 we obtain $C(\alpha)$. The vertices b_0 and b_1 are the left and right endpoints of $C(\alpha)$, respectively, and the edge $\langle b_0, \alpha, b_1 \rangle$ is called the base edge of $C(\alpha)$. Let us cite from [PT1] that $C(\alpha)$ admits an automorphism interchanging its endpoints. Indeed, we obtain a desired automorphism by mapping the vertices of $R(\alpha_1, \alpha_2, \dots, \alpha_k)$ to the vertices of $R(\alpha_k, \alpha_{k-1}, \dots, \alpha_1)$ in the reverse order.

Now, for all $k \geq 0$ and $\alpha \in J$ we define a graph $G_n(\alpha) = \langle V_n(\alpha), E_n(\alpha) \rangle$ via induction on n as follows. Let $G_0(\alpha)$ be the α-cell $C(\alpha)$ and let $E_{-1}(\alpha) = \emptyset$. We obtain $G_{n+1}(\alpha)$ from $G_n(\alpha)$ as follows. For each edge $\langle a, \beta, b \rangle \in E_n(\alpha) \setminus E_{n-1}(\alpha)$ we take (an isomorphic copy of) the β- cell $C(\beta)$. These cells, even those associated with distinct edges of the same colour, must be disjoint form each other and from $G_n(\alpha)$. Now, for each $\langle a, \beta, b \rangle \in E_n(\alpha) \setminus E_{n-1}(\alpha)$ at the same time, let us identify a resp. b with the left resp. right endpoint of (the copy of) $C(\beta)$ associated with this edge. (In other words, to each edge in $E_n(\alpha) \setminus E_{n-1}(\alpha)$ we glue the base edge of a cell with the same colour, and we use disjoint cells for distinct edges.) The graph we have obtained is $G_{n+1}(\alpha)$.

Now $V_0(\alpha) \subseteq V_1(\alpha) \subseteq V_2(\alpha) \subseteq \dots$ and $E_0(\alpha) \subseteq E_1(\alpha) \subseteq E_2(\alpha) \subseteq \dots$, so we can define $V(\alpha) = \bigcup_{n=0}^{\infty} V_n(\alpha)$, $E(\alpha) = \bigcup_{n=0}^{\infty} E_n(\alpha)$, and let $G(\alpha) = G_\infty(\alpha)$ denote the graph $\langle V(\alpha), E(\alpha) \rangle$. The base edge and the endpoints of $G(\alpha)$ are that of $G_0(\alpha) = C(\alpha)$, respectively. Since $G_0(\alpha) = C(\alpha)$ has an automorphism interchanging its endpoints, a trivial induction shows that so does $G(\alpha) = G_\infty(\alpha)$ as well.

Now we are ready to define the last of our graphs, denoted by $G(J)$. For each $\alpha \in J$ let us take (a copy of) $G(\alpha)$ such that $G(\alpha)$ and $G(\beta)$ be disjoint when $\alpha \neq \beta$. Identifying the left endpoints of these $G(\alpha)$ to a single vertex we obtain $G(J) = \langle V(J), E(J)\rangle$.

Let us consider the algebra $A = \langle V(J), F\rangle$ where F is the set of endomorphisms of the graph $G(J)$. Further, let J be the set of nonzero compact elements of I. It is well-known, cf. Grätzer and Schmidt [GS1] or Grätzer [GR2, p. 22], that the ideal lattice $\mathcal{I}(J)$ of J is isomorphic to I. (Here the empty set is also considered an ideal.) Consequently, the first chapter of [PT1] yields that I is isomorphic to $\mathrm{Con}(A)$. (Indeed, the "quadricle" $\langle J, \leq, D, \mathcal{I}\rangle$ in [PT1] corresponds to $\langle J, =, D, \mathcal{I}(J)\rangle$ in our case where $D = \{\langle \alpha, \{\alpha_1, \ldots, \alpha_k\}\colon \alpha \in J,\ \{\alpha_1, \ldots, \alpha_k\} \subseteq J,\ \alpha \leq \alpha_1 \vee \ldots \vee \alpha_k\}$.) So we have to show that every quasiorder of A is symmetric, i.e. a congruence.

Suppose ρ is a quasiorder of A, $a \neq b \in A$ and $\langle a, b\rangle \in \rho$. It is shown in [PT1], cf. RC 5 and the proof of Lemma 1.9, that there is a "path" from a to b, i.e. a sequence

$$\langle c_0, \alpha_1, c_1\rangle, \langle c_1, \alpha_2, c_2\rangle, \ldots, \langle c_{k-1}, \alpha_k, c_k\rangle \in E(J)$$

of edges such that $c_0 = a$, $c_k = b$, and for $i = 1, 2, \ldots, k$ there is an $f_i \in F$ with $\{f_i(a), f_i(b)\} = \{c_{i-1}, c_i\}$. We want to show the existence of a $g_i \in F$ such that $g_i(a) = c_i$ and $g_i(b) = c_{i-1}$. For a fixed i let u resp. v denote the left resp. right endpoints of $G(\alpha_i)$, and let h be an endomorphism of $G(\alpha_i)$ interchanging them. Clearly, the map

$$f^{(1)}\colon V(J) \to V(J), \qquad x \mapsto \begin{cases} h(x), & \text{if } x \in V(\alpha_i) \\ v, & \text{if } x \notin V(\alpha_i) \end{cases}$$

belongs to F and interchanges u and v. By [PT1], cf. RC 4 of Theorem 1.6, there are $f^{(2)}, f^{(3)} \in F$ such that $\{f^{(2)}(u), f^{(2)}(v)\} = \{c_{i-1}, c_i\}$ and $\{f^{(3)}(c_{i-1}), f^{(3)}(c_i)\} = \{u, v\}$. Since F is closed with respect to composition, both $f^{(2)}f^{(1)}f^{(3)}f_i$ and $f^{(2)}f^{(3)}f_i$ belong to F, and one of them is an appropriate g_i.

Since the g_i preserve ρ, we obtain $\langle c_i, c_{i-1}\rangle = \langle g_i(a), g_i(b)\rangle \in \rho$, and $\langle b, a\rangle = \langle c_k, c_0\rangle \in \rho$ follows by transitivity. $\square$

When the involution is not assumed to be trivial, much less is known. The quasiorders of an algebra A are called 3-permutable if $\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta$ holds for any $\alpha, \beta \in \mathrm{Quord}(A)$.

Theorem 6.2. *([CC1]) For any finite distributive involution lattice I there exists a finite algebra A such that $I \cong \mathrm{Quord}(A)$ and, in addition, the quasiorders of A are 3-permutable.*

Proof. Let J be the set of join-irreducible elements of I, 0 is included. For each $a \in J \setminus \{0\}$ we define a unary operation

$$f_a\colon J \to J, \qquad x \mapsto \begin{cases} 0, & \text{if } x = a, \\ a^*, & \text{if } x \neq a. \end{cases}$$

Let us call a map g: $J \to J$ a contraction of J if $g(x) \leq x$ holds for all $x \in J$. Let F consist of all contractions of J and all f_a, $a \in J \setminus \{0\}$. Consider the algebra $A = \langle J; F\rangle$; we intend to show that I and $\mathrm{Quord}(A)$ are isomorphic.

A subset Y of J is called hereditary if for any $x \in J$ and $y \in Y$ if $x \leq y$ then $x \in Y$. Let $\mathcal{H}(J)$ denote the set of nonempty hereditary subsets of J. It is well-known, cf. Grätzer [Gr1, Theorem II.1.9 on page 61], that the map $a \mapsto \{x \in J\colon\ x \leq a\}$ is a lattice isomorphism from I to the lattice $\mathcal{H}(J) = \langle \mathcal{H}(J); \cup, \cap\rangle$. Clearly, $\mathcal{H}(J)$ becomes an involution lattice by defining $Y^* = \{y^*\colon\ y \in Y\}$ and the above-mentioned map preserves this involution. So it suffices to prove that the map ψ: $\mathcal{H}(J) \to \mathrm{Quord}(A)$, $Y \mapsto (Y \times Y^*) \cup \{\langle x, x\rangle\colon\ x \in J\}$ is an isomorphism. Clearly, $\psi(Y)$ is reflexive, transitive and preserved by all contractions of J. To show that f_a preserves $\psi(Y)$ suppose that $\langle u, v\rangle \in \psi(Y)$ and, without loss of generality, $f_a(u) \neq f_a(v)$. Then either $f_a(u) = 0$, $u = a$ and $\langle f_a(u), f_a(v)\rangle = \langle 0, a^*\rangle \in \psi(Y)$ since $a = u \in Y$, or $f_a(v) = 0$, $v = a$ and $\langle f_a(u), f_a(v)\rangle = \langle a^*, 0\rangle \in \psi(Y)$ since $a^* = v^* \in (Y^*)^* = Y$. Thus $\psi(Y)$ is a quasiorder of A. Clearly, ψ is meet-preserving, whence it is monotone. Assume that $\langle u, v\rangle \in \psi(X \cup Y)$ and $u \neq v$. Then $u \in X \cup Y$, $v \in (X \cup Y)^* = X^* \cup Y^*$. There are four cases depending on the location of u and v but each of these cases can be treated similarly, so we detail the case $u \in Y$, $v \in X^*$ only. Then $\langle u, 0\rangle \in \psi(Y)$ and $\langle 0, v\rangle \in \psi(X)$, so by reflexivity we obtain $\langle u, v\rangle \in \psi(X) \circ \psi(Y) \circ \psi(X) \subseteq \psi(X) \vee \psi(Y)$ and $\langle u, v\rangle \in \psi(Y) \circ \psi(X) \circ \psi(Y) \subseteq \psi(X) \vee \psi(Y)$. Besides proving that ψ is join-preserving, this also shows that $\psi(X)$ and $\psi(Y)$ 3-permute. Clearly, $\psi(X^*) = (\psi(X))^*$, therefore ψ is a homomorphism. If $x \in Y \setminus X$ then $\langle x, 0\rangle \in \psi(Y) \setminus \psi(X)$, whence ψ is injective.

To prove surjectivity, assume that $\rho \in \mathrm{Quord}(A)$ and let $X = \{x \in J\colon\ \langle x, 0\rangle \in \rho\}$ and $Y = \{y \in J\colon\ \langle 0, y\rangle \in \rho\}$. Thanks to the fact that ρ is preserved by the contractions we conclude that $X, Y \in \mathcal{H}(J)$. If $x \in X \setminus \{0\}$ then $\langle 0, x^*\rangle = \langle f_x(x), f_x(0)\rangle \in \rho$, whence $x = (x^*)^* \in Y^*$. Similarly, if $y \in Y \setminus \{0\}$ then $\langle y^*, 0\rangle = \langle f_y(0), f_y(y)\rangle \in \rho$, whence $y^* \in X$ gives $y \in X^*$. From $X \subseteq Y^*$ and $Y \subseteq X^*$ we obtain $Y = X^*$.

Now, to show that $\rho = \psi(X)$, suppose $a \neq b$ and $\langle a, b\rangle \in \rho$. Then $\langle b^*, 0\rangle = \langle f_b(a), f_b(b)\rangle \in \rho$ gives $b^* \in X$, i.e. $b \in X^*$, while $\langle 0, a^*\rangle = \langle f_a(a), f_a(b)\rangle \in \rho$ gives $a^* \in Y$, i.e. $a \in Y^* = X$, yielding $\langle a, b\rangle \in X \times X^* \subseteq \psi(X)$. Conversely, suppose that $a \neq b$ and $\langle a, b\rangle \in \psi(X)$. Then, by definitions and $Y = X^*$, $\langle a, 0\rangle \in \rho$ and $\langle 0, b\rangle \in \rho$, yielding $\langle a, b\rangle \in \rho$ by transitivity. $\square$

We remark that if the quasiorders of all algebras in a given variety V are 3-permutable then $\mathrm{Con}(A) = \mathrm{Quord}(A)$ for all $A \in V$, cf. Chajda and Rachůnek [CR1]. Sharpening Whitman's result in [Wh1], Jónsson [Jo3] has shown that each modular lattice L has a type 2 representation. We say that an involution lattice I has a type 2 representation if for some set A the involution lattice $\mathrm{Quord}(A)$ has a subalgebra S isomorphic to I such that $\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta$ holds for any $\alpha, \beta \in S$.

Theorem 6.A. *([CC1]) Each distributive involution lattice L has a type 2 representation.*

For a partial algebra $A = \langle A, F\rangle$, a reflexive and symmetric relation $\rho \subseteq A^2$ is called a quasiorder of A provided for any $f \in F$, say n-ary, and $\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle \in \rho$ if both $f(a_1, \dots, a_n)$ and $f(b_1, \dots, b_n)$ are defined then $\langle f(a_1, \dots, a_n), f(b_1, \dots, b_n)\rangle \in \rho$. The quasiorders of a partial algebra A still constitute an algebraic involution lattice $\mathrm{Quord}(A)$ under the set-theoretic inclusion and $\rho^* = \{\langle x, y\rangle\colon \langle y, x\rangle \in \rho\}$, but the join is not the transitive closure of the union in general.

Theorem 6.B. *([CC1]) For any algebraic involution lattice I there is a partial algebra A such that I is isomorphic to* $\mathrm{Quord}(A)$.

The previous four theorems naturally lead to the question whether every algebraic involution lattice is isomorphic to $\mathrm{Quord}(A)$ for some algebra A. The affirmative answer would imply that any involution lattice I could be embedded in $\mathrm{Quord}(A)$ for some set A, for I is embedded in the (algebraic) involution lattice of its lattice ideals. Unfortunately, as the next few lines witness, this is not the case.

On the set $\{x, y, z, t, u, v, w\}$ of variables let us define the following involution lattice terms

$$\begin{aligned} s_1 &= (z \vee u) \wedge (u^* \vee x \vee z^* \vee t^*),\\ s_2 &= (y \vee w) \wedge (y^* \vee x \vee v^* \vee w^*),\\ s_3 &= (y \vee s_1) \wedge (u^* \vee x \vee z^* \vee t^*),\\ s_4 &= (u \vee s_2) \wedge (y^* \vee x \vee v^* \vee w^*). \end{aligned}$$

Theorem 6.3. *([Cz10]) The Horn sentence*

$$x \le y \vee u \;\&\; y \le z \vee t \;\&\; u \le v \vee w \Longrightarrow x \le s_3 \vee s_4 \vee z^* \vee w^*$$

holds in $\mathrm{Quord}(A)$ *for any set A but does not hold in all involution lattices.*

Proof. Let χ be an arbitrary Horn sentence for involution lattices. Then, without loss of generality, χ is of the form

$$p_1(x, x^*) \le q_1(x, x^*) \;\&\; \dots \;\&\; p_t(x, x^*) \le q_t(x, x^*) \Longrightarrow p(x, x^*) \le q(x, x^*)$$

where $x = \langle x_1, x_2, \dots, x_n\rangle$, $x^* = \langle x_1{}^*, x_2{}^*, \dots, x_n{}^*\rangle$ and p_i, q_i, p, q are lattice terms. Let $y = \langle y_1, y_2, \dots, y_n\rangle$, and consider the lattice Horn sentence $\hat{\chi}$:

$$\begin{aligned} p_1(x, y) \le q_1(x, y) \;\&\; p_1(y, x) \le q_1(y, x) \;\&\; \dots \;\&\; p_t(x, y) \le q_t(x, y)\\ \&\; p_t(y, x) \le q_t(y, x) \Longrightarrow p(x, y) \le q(x, y). \end{aligned}$$

Claim 6.4. *([Cz10]) χ holds in all involution lattices iff $\hat{\chi}$ holds in all lattices.*

Suppose $\hat{\chi}$ holds in all lattices, L is an involution lattice, $a \in L^n$ and $p_i(a, a^*) \leq q_i(a, a^*)$ for $i = 1, 2, \dots, t$. Denoting a^* by b we obtain

$$p_i(b, a) = p_i(a, b)^* \leq q_i(a, b)^* = q_i(b, a),$$

whence the premise of $\hat{\chi}$ holds for $\langle a, b\rangle$ and $p(a, a^*) = p(a, b) \leq q(a, b) = q(a, a^*)$ follows.

Conversely, assume that χ holds in all involution lattices, L is a lattice, $a, b \in L^n$, and $p_i(a, b) \leq q_i(a, b)$, $p_i(b, a) \leq q_i(b, a)$ for $i = 1, \dots, t$. The congruence ϑ generated by

$$\{\langle p_i(x, y), p_i(x, y) \wedge q_i(x, y)\rangle\colon\ 1 \leq i \leq t\} \quad \cup \\ \{\langle p_i(y, x), p_i(y, x) \wedge q_i(y, x)\rangle\colon\ 1 \leq i \leq t\}$$

in the free lattice $F = F(x_1, \dots, x_n, y_1, \dots, x_n)$ is clearly included in the kernel of the lattice homomorphism $\varphi\colon\ F \to L$, $x_i \mapsto a_i$, $y_i \mapsto b_i$. Consider the automorphism $\psi\colon\ F \to F$, $x_i \mapsto y_i$, $y_i \mapsto x_i$. Then ψ preserves ϑ, for it preserves the set generating ϑ. Therefore the map $\kappa\colon\ F/\vartheta \to F/\vartheta$, $[u]\vartheta \mapsto [\psi(u)]\vartheta$ is a lattice automorphism. Thus we can consider F/ϑ as an involution lattice where $v^* = \kappa(v)$. Then $([x_i]\vartheta)^* = [y_i]\vartheta$, and the premise of χ holds for $[x]\vartheta$. From the assumption on χ we infer $p([x]\vartheta, [y]\vartheta) = p([x]\vartheta, ([x]\vartheta)^*) \leq q([x]\vartheta, ([x]\vartheta)^*) = q([x]\vartheta, [y]\vartheta)$, and the canonical lattice homomorphism $F/\vartheta \to L$, $[u]\vartheta \mapsto \varphi(u)$ yields $p(a, b) \leq q(a, b)$. This proves Claim 6.4.

Therefore, to decide if χ holds in all involution lattices, it suffices to deal with $\gamma = \hat{\chi}$. There are several known algorithms for the word problem of lattices, cf. Dean [De2], Evans [Ev1], McKinsey [Mc1], and [Cz11]; we have chosen [Cz11], which seems to give the simplest and fastest algorithm. To point out that the closure operator T is needed only for a few subsets and can be determined fast, we cite the algorithm given in [Cz11]. First we have to reduce γ into an equivalent "canonical" form

$$\bigwedge M_1 \leq \bigvee J_1 \ \&\ \dots\ \&\ \bigwedge M_r \leq \bigvee J_r \Longrightarrow \bigwedge M \leq z$$

such that $M_1, \dots, M_r$, $J_1, \dots, J_r$, M are subsets of the set $X = \{z_1, z_2, \dots, z_s\}$ of variables occurring in γ and $z \in X$. Sometimes we write z_0 instead of M. For $j = 0, 1, 2, \dots$ we define a map T_j from $X \cup \{M\}$ to the power set of X by the following induction. Let $T_0(M) = M$, $T_0(u) = \{u\}$ for $u \in X$, and let

$$T_{j+1}(u) := T_j(u) \cup \bigcup_{\substack{0<i\leq r \\ M_i \subseteq T_j(u)}} \bigcap_{v \in J_i} T_j(v)$$

for $u \in X \cup \{M\}$. By finiteness, there is a (smallest) j'such that $T_{j'+1}(u) = T_{j'}(u)$ holds for all $u \in X \cup \{M\}$. In other words, $T_{j'}$ is the smallest T such that

$$v \in T(v), \quad M \subseteq T(M) \quad \text{and} \quad T(u) = T(u) \cup \bigcup_{\substack{0<i\leq r \\ M_i \subseteq T(u)}} \bigcap_{v \in J_i} T(v)$$

holds for all $u \in X \cup \{M\}$ and $v \in X$, and this formula leads to a more effective algorithm than the previous one. Now γ holds in all lattices iff $z \in T_{j'}(M)$ $(= T_{j'}(z_0))$.

Based on the algorithm described above, the author has developed a Turbo Pascal program for personal computers (Borland's Turbo Pascal version 4.0 — 7.0), which reduces γ to a canonical form and tests if γ holds in all lattices or not. This program can be found in directory WPROBLEM on the floppy disk attached to this dissertation. The directory contains the source file and the executable file of the program, and a file with our Horn sentence. (The program takes its input from a file; this makes the program much easier to use. Upon request, the program outputs the details of calculations into another file.) But first of all, this directory contains a README.1ST file, this is where the interested reader is expected to start. Now, in contrast with Chapter III, the reader can also check the result manually, it would take half a day or so. For this reason, the printed output of the program is enclosed. (For technical reasons it is at the end of the present chapter.) Beside the calculations it contains some comments explaining what is going on.

Now, if χ is the Horn sentence in the Theorem and $\gamma = \hat{\chi}$ then $s = 34$, $r = 74$, and it takes about a second for this program to manifest that γ does not hold in all lattices. Therefore χ does not hold in all involution lattices.

Now, in order to show the other statement of the Theorem, let A be a set, x, y, z, t, u, v, $w \in \text{Quord}(A)$, suppose that the premise of the Horn sentence χ in the Theorem holds for these quasiorders, and let $\langle a_0, a_1 \rangle \in x$. Since $x \leq y \vee u$, there are elements $a_0 = b_0, b_1, \ldots, b_n = a_1$ in A such that $\langle b_i, b_{i+1} \rangle \in y$ for i even and $\langle b_i, b_{i+1} \rangle \in u$ for i odd. By reflexivity, we may assume that $n \geq 4$ and n is even. Since $y \leq z \vee t$ and $u \leq v \vee w$, there are elements $b_i = c_{i0}, c_{i1}, \ldots, c_{ik} = b_{i+1}$ for i even and $b_i = d_{i0}, d_{i1}, \ldots, d_{ik} = b_{i+1}$ for i odd such that $k \geq 4$, k is even, k does not depend on i, $\langle c_{ij}, c_{i,j+1} \rangle \in z$ for j even, $\langle c_{ij}, c_{i,j+1} \rangle \in t$ for j odd, $\langle d_{ij}, d_{i,j+1} \rangle \in v$ for j even, and $\langle d_{ij}, d_{i,j+1} \rangle \in w$ for j odd. Now, for $i = 1, 3, 5, \ldots, n-3$, $b_i \; u \; b_{i+1} = c_{i+1,0} \; z \; c_{i+1,1}$ and $b_i = c_{i-1,k} \; t^* \; c_{i-1,k-1} \; z^* \; c_{i-1,k-2} \; t^* \; c_{i-1,k-3} \; z^* \ldots z^* \; c_{i-1,0} = b_{i-1} \; u^* \; b_{i-2} = c_{i-3,k} \; t^* \; c_{i-3,k-1} \; z^* \; c_{i-3,k-2} \; t^* \; c_{i-3,k-3} \; z^* \ldots z^* \; c_{i-3,0} = b_{i-3} \ldots b_0 = a_0 \; x \; a_1 = b_n \; u^* \; b_{n-1} = c_{n-2,k} \; t^* \; c_{n-2,k-1} \; z^* \; c_{n-2,k-2} \; t^* \; c_{n-2,k-3} \; z^* \ldots z^* \; c_{n-2,0} = b_{n-2} \; u^* \; b_{n-3} \ldots b_{i+2} = c_{i+1,k} \; t^* \; c_{i+1,k-1} \; z^* \; c_{i+1,k-2} \; t^* \; c_{i+1,k-3} \; z^* \ldots t^* \; c_{i+1,1}$, whence we conclude $\langle b_i, c_{i+1,1} \rangle \in s_1$. The rest of the following four formulas follow similarly:

$$\begin{aligned} \langle b_i, c_{i+1,1}\rangle &\in s_1 && \text{for } i = 1, 3, 5, \dots, n-3;\\ \langle d_{i-1,k-1}, b_{i+1}\rangle &\in s_2 && \text{for } i = 2, 4, 6, \dots, n-2;\\ \langle b_{i-1}, c_{i+1,1}\rangle &\in s_3 && \text{for } i = 1, 3, 5, \dots, n-3; \quad \text{and}\\ \langle d_{i-1,k-1}, b_{i+2}\rangle &\in s_4 && \text{for } i = 2, 4, 6, \dots, n-2. \end{aligned}$$

In virtue of these formulas we obtain $a_0 = b_0\ s_3\ c_{21}\ z^*\ c_{20} = b_2\ s_3\ c_{41}\ z^*\ c_{40} = b_4 \dots b_{n-2} = d_{n-3,k}\ w^*\ d_{n-3,k-1}\ s_4\ b_n = a_1$, yielding $\langle a_0, a_1\rangle \in s_3 \vee s_4 \vee z^* \vee w^*$. Hence χ holds in $\mathrm{Quord}(A)$. $\square$

The description of quasiorders of a lattice L is due to Szabó [Sz1]. The computations proving this result were quite long; perhaps this is the reason that [Sz1] has never been published. To demonstrate the usefulness of involution lattices, we are going to give a much simpler proof.

Let I denote an involution lattice and let $L = \{x \in I\colon x^* = x\}$ be regarded as a lattice. As previously, L^2 is an involution lattice.

Theorem 6.4. *(Czédli [CS1]) Assume that I is a distributive involution lattice and $\rho \in I$ such that $\rho \wedge \rho^* = 0$ and $\rho \vee \rho^* = 1$. Then*

$$u\colon I \to L^2, \quad \gamma \mapsto \langle (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*), (\gamma \wedge \rho^*) \vee (\gamma^* \wedge \rho)\rangle$$

is an isomorphism. The inverse of u is the isomorphism

$$v\colon L^2 \to I, \quad \langle \alpha, \beta\rangle \mapsto (\alpha \wedge \rho) \vee (\beta \wedge \rho^*).$$

Proof. The map $\kappa\colon I \to [0,\rho]\times[0,\rho^*]$, $x \mapsto \langle x\wedge\rho, x\wedge\rho^*\rangle$ is a lattice isomorphism with inverse $\kappa'\colon [0,\rho]\times[0,\rho^*] \to I$, $\langle x, y\rangle \mapsto x \vee y$ by Grätzer [Gr1, Thm. 14 on p. 169 plus the remark after it]. Since $(x \vee x^*)^* = x \vee x^*$, $\lambda\colon x \mapsto x \vee x^*$ is a $[0,\rho] \to L$ map. Consider the map $\lambda'\colon L \to [0,\rho]$, $x \mapsto x \wedge \rho$. By distributivity, for $x \in [0,\rho]$ we have $\lambda'(\lambda(x)) = (x \vee x^*) \wedge \rho = (x \wedge \rho) \vee (x^* \wedge \rho) = x \vee (x \wedge \rho^*)^* = x \vee (x \wedge \rho \wedge \rho^*)^* = x \vee 0 = x$, and for $y \in L \subseteq I$ we obtain $\lambda(\lambda'(y)) = (y\wedge\rho)\vee(y\wedge\rho)^* = (y\wedge\rho)\vee(y^*\wedge\rho^*) = (y\wedge\rho)\vee(y\wedge\rho^*) = y\wedge(\rho\vee\rho^*) = y\wedge 1 = y$. Since both λ and λ' are monotone, they are reciprocal lattice isomorphisms. Similarly, $\mu\colon [0,\rho^*] \to L$, $x \mapsto x \vee x^*$ and $\mu'\colon L \to [0,\rho^*]$, $x \mapsto x \wedge \rho^*$ are reciprocal lattice isomorphisms as well. Thus, $u = (\lambda \times \mu) \circ \kappa$ is a lattice isomorphism with inverse $\kappa' \circ (\lambda' \times \mu') = v$. Clearly, both u and v preserve the involution operation *. This proves that u and v are isomorphisms and inverses of each other. $\square$

Now let A be a lattice or, more generally, assume that A has a lattice reduct such that the basic operations of A are monotone with respect to the lattice order. Denoting the lattice order by ρ, we have $\rho \wedge \rho^* = 0$ and $\rho \vee \rho^* = 1$ in $\mathrm{Quord}(A)$. Put $I = \mathrm{Quord}(A)$, then $L = \mathrm{Con}^2(A)$. It is known that $\mathrm{Quord}(A)$ is distributive. This was first proved in [CL2, Corollary 5.2 and pages 53–54] and rediscovered in Chajda and Pinus [CP3]. Therefore Theorem 6.4 applies and gives a satisfactory description of (members of) $\mathrm{Quord}(A)$:

Corollary 6.5. *([CS1], Szabó [Sz1]) The quasiorders of a lattice A are exactly the relations of the form $(\alpha \wedge \rho) \vee (\beta \wedge \rho^*)$ where ρ denotes the (natural) lattice order of A and $\alpha, \beta \in \mathrm{Con}(A)$. Moreover, the maps*

$$\tau_A\colon \mathrm{Quord}(A) \to \mathrm{Con}^2(A), \quad \gamma \mapsto \langle (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*), (\gamma \wedge \rho^*) \vee (\gamma^* \wedge \rho) \rangle$$

and

$$\nu_A\colon \mathrm{Con}^2(A) \to \mathrm{Quord}(A), \quad \langle \alpha, \beta \rangle \mapsto (\alpha \wedge \rho) \vee (\beta \wedge \rho^*),$$

where $\wedge$, $\vee$ and * are taken in the domain of τ_A resp. ν_A, are reciprocal isomorphisms between the involution lattices $\mathrm{Quord}(A)$ and $\mathrm{Con}(A)$.

From this result it is quite straightforward to derive

Corollary 6.C. *(Szabó [CS1], Szabó [Sz1], for finite lattices [CHS1]) Every compatible (partial) order γ of a lattice A is induced by a subdirect representation of A as a subdirect product of A_1 and A_2 such that $\langle x, y \rangle \in \gamma$ iff $x_1 \le y_1$ in A_1 and $x_2 \ge y_2$ in A_2. Conversely, any relation derived from a subdirect decomposition this way is a compatible order of A.*

Proof. For $\langle \alpha, \beta \rangle \in \mathrm{Con}^2(A)$, suppose $\nu_A(\langle \alpha, \beta \rangle)$ is an ordering. Then $\nu_A(\langle \alpha, \beta \rangle) \wedge \nu_A(\langle \alpha, \beta \rangle)^* = 0$. Computing by distributivity and using $\alpha^* = \alpha$, $\beta^* = \beta$ we obtain $0 = \nu_A(\langle \alpha, \beta \rangle) \wedge \nu_A(\langle \alpha, \beta \rangle)^* = ((\alpha \wedge \rho) \vee (\beta \wedge \rho^*)) \wedge ((\alpha \wedge \rho) \vee (\beta \wedge \rho^*))^* = ((\alpha \wedge \rho) \vee (\beta \wedge \rho^*)) \wedge ((\alpha \wedge \rho^*) \vee (\beta \wedge \rho)) = (\alpha \wedge \rho \wedge \rho^*) \vee (\alpha \wedge \beta \wedge \rho^*) \vee (\alpha \wedge \beta \wedge \rho) \vee (\beta \wedge \rho \wedge \rho^*) = (\alpha \wedge 0) \vee (\alpha \wedge \beta \wedge (\rho \vee \rho^*)) \vee (\beta \wedge 0) = \alpha \wedge \beta \wedge 1 = \alpha \wedge \beta$. From $\alpha \wedge \beta = 0$ we infer that A is a subdirect product of $A_1 = A/\beta$ and $A_2 = A/\alpha$. Since $\nu_A(\langle \alpha, \beta \rangle) = 0 \vee 0 \vee (\alpha \wedge \rho) \vee (\beta \wedge \rho^*) = (\alpha \wedge \beta) \vee (\rho \wedge \rho^*) \vee (\alpha \wedge \rho) \vee (\beta \wedge \rho^*) = (\alpha \vee \rho^*) \wedge (\beta \vee \rho)$, it is not hard to see that $\nu_A(\langle \alpha, \beta \rangle)$ is induced by this subdirect decomposition. The rest of the corollary is evident. □

Remarks. The fact that $\mathrm{Quord}(A)$ and $\mathrm{Con}^2(A)$ are isomorphic via τ_A and ν_A was first proved in [Sz1]. However, the present approach is much shorter.

It is worth mentioning that here we have proved a bit more than stated (and so did Szabó in [Sz1], too). Indeed, let $\mathcal{L}$ be a variety with two distinguished binary terms $\vee$ and $\wedge$ in its language such that the reduct $\langle A; \vee, \wedge \rangle$ is a lattice for each $A \in \mathcal{L}$ and all basic operations of A are monotone with respect to the natural ordering of this lattice. E.g., $\mathcal{L}$ can be the variety of lattice-ordered semigroups or that of involution lattices. Then, for $A \in \mathcal{L}$, Corollaries 6.5 and 6.C are still valid.

Note that describing the compatible orders is an interesting task also for semilattices; this was done by Kolibiar [Ko1].

Oddly enough, the whole investigation leading to Theorem 6.4 and Corollaries 6.5 and 6.C was initiated by a preprint of Tischendorf and Tůma [TT1] where they show that congruence lattices of lattices are, up to isomorphism, exactly the distributive algebraic lattices. Using this result and Corollary 6.5 it would not be difficult to give an abstract characterization for the involution lattices $\mathrm{Quord}(A)$, A being a lattice. Unfortunately, a gap has been found in their proof.

At this point the author wishes to express his thanks to E. Tamás Schmidt, who draw his attention to the mistake in [TT1]. His warning came in the right time: not too soon to break down our enthusiasm which led to Theorem 6.4 and its corollaries, and not too late to prevent the author from drawing a possibly false conclusion from [TT1] in this dissertation.

For lattices A the fact $\mathrm{Quord}(A) \cong \mathrm{Con}^2(A)$ can be stated in a stronger form. Let us fix a prevariety $\mathcal{L}$ of lattices. I.e., $\mathcal{L}$ is a class closed under forming sublattices, homomorphic images and finite direct products. (As for the next theorem, the best choice for $\mathcal{L}$ is the variety of all lattices. However, for the sake of what comes later, we will formulate the next theorem for arbitrary prevarieties.) $\mathcal{L}$ will be considered a category in which the morphisms are the *surjective* lattice homomorphisms. The category of all involution lattices with all homomorphisms will be denoted by $\mathcal{V}$. For $A, B \in \mathcal{L}$ and a morphism $f\colon A \to B$, let

$$\mathrm{Quord}(f)\colon \mathrm{Quord}(B) \to \mathrm{Quord}(A), \quad \gamma \mapsto \{\langle x,y\rangle \in A^2\colon \langle f(x), f(y)\rangle \in \gamma\}$$

and

$$\mathrm{Con}^2(f)\colon \mathrm{Con}^2(B) \to \mathrm{Con}^2(A) \quad \langle \alpha,\beta\rangle \mapsto \langle \hat f(\alpha), \hat f(\beta)\rangle,$$

where $\hat f(\delta) = \{\langle x,y\rangle \in A^2\colon \langle f(x), f(y)\rangle \in \delta\}$. Then Quord and Con^2 are contravariant $\mathcal{L} \to \mathcal{V}$ functors. As before, for $A \in \mathcal{L}$ let

$$\tau_A\colon \mathrm{Quord}(A) \to \mathrm{Con}^2(A), \quad \gamma \mapsto \langle (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*), (\gamma \wedge \rho^*) \vee (\gamma^* \wedge \rho)\rangle$$

and

$$\nu_A\colon \mathrm{Con}^2(A) \to \mathrm{Quord}(A), \quad \langle \alpha,\beta\rangle \mapsto (\alpha \wedge \rho) \vee (\beta \wedge \rho^*),$$

where ρ is the lattice order of A.

Theorem 6.6. *(Czédli [CS1]) τ is a natural equivalence from the functor* Quord *to the functor* Con^2. *The inverse of τ is ν:* $\mathrm{Con}^2 \to \mathrm{Quord}$.

Proof. It suffices to show that that τ is a natural transformation. Then it will be a natural equivalence by Corollary 6.5, and so will be its inverse, ν.

Assume that $\varphi\colon A \to B$ is a surjective lattice homomorphism, and let φ_q and φ_c denote $\mathrm{Quord}(\varphi)$ and $\mathrm{Con}^2(\varphi)$, respectively. We have to show that the following diagram

$$\begin{array}{ccc}
\mathrm{Quord}(B) & \xrightarrow{\varphi_q} & \mathrm{Quord}(A) \\
{\scriptstyle \tau_B}\big\downarrow & & \big\downarrow{\scriptstyle \tau_A} \\
\mathrm{Con}^2(B) & \xrightarrow{\varphi_c} & \mathrm{Con}^2(A)
\end{array}$$

commutes. Let $\gamma \in \mathrm{Quord}(B)$ and $\delta := \varphi_q(\gamma)$; we have to show that φ_c sends $\tau_B(\gamma) = \langle (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*), (\gamma \wedge \rho^*) \vee (\gamma^* \wedge \rho)\rangle$ to $\tau_A(\delta) = \langle (\delta \wedge \rho) \vee (\delta^* \wedge \rho^*), (\delta \wedge \rho^*) \vee (\delta^* \wedge \rho)\rangle$. This means that, for any $x, y \in A$, $\langle x,y\rangle \in (\delta \wedge \rho) \vee (\delta^* \wedge \rho^*)$ iff $\langle \varphi(x), \varphi(y)\rangle \in (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*)$, and similarly for the second components which we will not be detailed. Suppose $\langle \varphi(x), \varphi(y)\rangle \in (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*)$. Then there is an $n \geq 1$ and there are elements $b_0 = \varphi(x), b_1, b_2, \ldots, b_{2n} = \varphi(y)$ in B

such that $\langle b_i, b_{i+1}\rangle \in \gamma \wedge \rho$ for i even and $\langle b_i, b_{i+1}\rangle \in \gamma^* \wedge \rho^*$ for i odd, $i < 2n$. Let $a'_0 = x$, $a'_{2n} = y$, and for $i = 1, \dots, 2n-1$ let $a'_i \in A$ such that $\varphi(a'_i) = b_i$. Put $a_i = a'_i$ for i even and $a_i = a'_i \vee a'_{i-1} \vee a'_{i+1}$ for i odd. For i odd we obtain $\varphi(a_i)=\varphi(a'_i) \vee \varphi(a'_{i-1}) \vee \varphi(a'_{i+1}) = b_i \vee b_{i-1} \vee b_{i+1} = b_i$, whence $\varphi(a_i) = b_i$ holds for all i. Since $\langle a_i, a_{i+1}\rangle \in \delta \wedge \rho$ for i even and $\langle a_i, a_{i+1}\rangle \in \delta^* \wedge \rho^*$ for i odd, we conclude $\langle x, y\rangle = \langle a_0, a_{2n}\rangle \in (\delta \wedge \rho) \vee (\delta^* \wedge \rho^*)$. The converse implication being straightforward we have shown that τ is a natural transformation. $\square$

In the sequel, for a fixed prevariety $\mathcal{L}$ of lattices, we will investigate the natural equivalences Quord $\to \mathrm{Con}^2$. One natural equivalence, τ, is given in Theoerem 6.6, and clearly we can obtain another one by interchanging the components of the pair in the formula defining τ_A. Evidently, the map $\psi \to \psi \circ \tau$ from the class of $\mathrm{Con}^2 \to \mathrm{Con}^2$ natural equivalences to the class of Quord $\to \mathrm{Con}^2$ natural equivalences is a bijection. Therefore it suffices to describe the class $T(\mathcal{L})$ of natural equivalences from the contravariant functor Con^2: $\mathcal{L} \to \mathcal{V}$ to the same functor. We are able to describe $T(\mathcal{L})$ for some very small prevarieties $\mathcal{L}$ only. The fact that $|T(\mathcal{L})|$ heavily depends on $\mathcal{L}$ for these small $\mathcal{L}$ indicates that we are far from describing $T(\mathcal{L})$ for all $\mathcal{L}$.

From now on let $\mathcal{L}$ be a prevariety *consisting of finite lattices only*. Let $\mathcal{S} = \mathcal{S}(\mathcal{L})$ be the class of subdirectly irreducible lattices belonging to $\mathcal{L}$. Note that the one-element lattice is not considered subdirectly irreducible. A pair $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2\rangle$ of subclasses of $\mathcal{S}$ is said to be an *H-partition of $\mathcal{S}$* if $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{S}$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, and for any $i = 1, 2$, $A \in \mathcal{D}_i$ and $B \in \mathcal{S}$ if B is a homomorphic image of A then $B \in \mathcal{D}_i$. An H-partition $\mathcal{D}$ is called trivial if $\mathcal{D}_1 = \emptyset$ or $\mathcal{D}_2 = \emptyset$. Since the $\mathcal{D}_i$ are closed under isomorphism and we consider finite lattices only, the H-partitions of $\mathcal{S}$ form a set.

We always have at least two natural equivalences from Con^2 to Con^2. The identical $\mathrm{Con}^2 \to \mathrm{Con}^2$ natural equivalence will be denoted by id; id_A is the identical $\mathrm{Con}^2(A) \to \mathrm{Con}^2(A)$ map for each $A \in \mathcal{L}$. Defining inv_A: $\mathrm{Con}^2(A) \to \mathrm{Con}^2(A)$, $x \to x^*$, it is easy to see that inv: $\mathrm{Con}^2 \to \mathrm{Con}^2$ is also a natural equivalence.

With an H-partition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2\rangle$ we associate a transformation (in fact a natural equivalence) $\psi = \psi(\mathcal{D})$: $\mathrm{Con}^2 \to \mathrm{Con}^2$ as follows. Let $A \in \mathcal{L}$ and choose $\alpha_1, \alpha_2 \in \mathrm{Con}(A)$ such that $\alpha_1 \wedge \alpha_2 = 0$, A/α_1 is isomorphic to a (finite) subdirect product of some lattices from $\mathcal{D}_1$ and A/α_2 is isomorphic to a (finite) subdirect product of some lattices from $\mathcal{D}_2$. (The case $\alpha_i = 1$ is allowed since the empty subdirect product is defined to be the one-element lattice. We will show soon that α_1 and α_2 exist and they are uniquely determined.) Let

$$\psi_A\colon \mathrm{Con}^2(A) \to \mathrm{Con}^2(A) \quad \langle \gamma, \delta\rangle \mapsto \langle (\gamma \vee \alpha_1) \wedge (\delta \vee \alpha_2), (\delta \vee \alpha_1) \wedge (\gamma \vee \alpha_2)\rangle.$$

Conversely, given a natural equivalence ψ: $\mathrm{Con}^2 \to \mathrm{Con}^2$, we define $\mathcal{D} = \mathcal{D}(\psi) = \langle \mathcal{D}_1, \mathcal{D}_2\rangle$ by $\mathcal{D}_1 = \{A \in \mathcal{S}\colon \psi_A = \mathrm{id}_A\}$ and $\mathcal{D}_2 = \{A \in \mathcal{S}\colon \psi_A = \mathrm{inv}_A\}$.

Theorem 6.7. *([Cz13]) Given a prevariety $\mathcal{L}$ of finite lattices, the map $\mathcal{D} \mapsto \psi(\mathcal{D})$ from the set of H-partitions of S to the set of $\mathrm{Con}^2 \to \mathrm{Con}^2$ natural equivalences is a bijection. The map $\psi \mapsto \mathcal{D}(\psi)$ is the inverse of this bijection.*

Proof. First we make some observations for an arbitrary natural equivalence $\psi\colon \mathrm{Con}^2 \to \mathrm{Con}^2$. For $A, B \in \mathcal{L}$ and a surjective homomorphism $f\colon A \to B$ with kernel $\mu \in \mathrm{Con}(A)$ let $\hat{f}$ denote the canonical lattice embedding $\mathrm{Con}(B) \to \mathrm{Con}(A)$, $\alpha \mapsto \{\langle x, y\rangle\colon \langle f(x), f(y)\rangle \in \alpha\}$. Then $\mathrm{Con}^2(f)\colon \langle \alpha, \beta\rangle \mapsto \langle \hat{f}(\alpha), \hat{f}(\beta)\rangle$. Let us consider the following diagram

$$\begin{array}{ccc} \mathrm{Con}^2(B) & \xrightarrow{\psi_B} & \mathrm{Con}^2(B) \\ {\scriptstyle \mathrm{Con}^2(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Con}^2(f)} \\ \mathrm{Con}^2(A) & \xrightarrow{\psi_A} & \mathrm{Con}^2(A) \end{array} \tag{6.8}$$

This diagram is commutative by the definition of a natural equivalence. Therefore, for any $\langle \gamma, \delta\rangle \in \mathrm{Con}^2(B)$ we have

$$\mathrm{Con}^2(f)(\psi_B(\langle \gamma, \delta\rangle)) = \psi_A(\langle \hat{f}(\gamma), \hat{f}(\delta)\rangle). \tag{6.9}$$

Since $\psi_B(\langle 0, 0\rangle) = \langle 0, 0\rangle$, we obtain from (6.9) that $\psi_A(\langle \mu, \mu\rangle) = \langle \mu, \mu\rangle$. But any member of $\mathrm{Con}(A)$ is the kernel of an appropriate surjective homomorphism, so we obtain that

$$\psi_A(\langle \beta, \beta\rangle) = \langle \beta, \beta\rangle \tag{6.10}$$

holds for every $\beta \in \mathrm{Con}(A)$. Now let $\psi_A^{(1)}(\langle \gamma, \delta\rangle)$ resp. $\psi_A^{(2)}(\langle \gamma, \delta\rangle)$ denote the first resp. second component of $\psi_A(\langle \gamma, \delta\rangle)$. Since ψ_A is monotone, $\psi_A(\langle \hat{f}(\gamma), \hat{f}(\delta)\rangle) \geq \psi(\langle \mu, \mu\rangle) = \langle \mu, \mu\rangle$. Therefore, factoring both sides of (6.9) by μ componentwise, we obtain

$$\psi_B(\langle \gamma, \delta\rangle) = \big\langle \psi_A^{(1)}(\langle \hat{f}(\gamma, \hat{f}(\delta)\rangle)/\mu, \psi_A^{(2)}(\langle \hat{f}(\gamma, \hat{f}(\delta)\rangle)/\mu\big\rangle. \tag{6.11}$$

I.e., ψ_A determines ψ_B for any homomorphic image B of A. For $\langle \gamma, \beta\rangle \in \mathrm{Con}^2(A)$ such that $\langle \gamma, \beta\rangle \geq \langle \mu, \mu\rangle$, we can rewrite (6.11) with the help of (6.9) into the following form:

$$\psi_A(\langle \gamma, \delta\rangle) = \big\langle \hat{f}(\psi_B^{(1)}(\langle \gamma/\mu, \delta/\mu\rangle)), \hat{f}(\psi_B^{(2)}(\langle \gamma/\mu, \delta/\mu\rangle))\big\rangle. \tag{6.12}$$

Now we assert that

$$(\forall A \in \mathcal{S})(\psi_A = \mathrm{id}_A \text{ or } \psi_A = \mathrm{inv}_A). \tag{6.13}$$

Let $\mu \in \mathrm{Con}(A)$ be the monolith of A. To prove (6.13), first we observe that since ψ_A is monotone, bijective, and leaves $\langle \mu, \mu\rangle$ fixed, ψ_A permutes the subset

$$Y = \{\langle u, v\rangle\colon \langle u, v\rangle \not\geq \langle \mu, \mu\rangle\}$$

of $\mathrm{Con}^2(A)$. Since $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are the only maximal elements of Y, ψ_A either interchanges these two elements or leaves both elements fixed. Suppose

$\psi_A(\langle 0,1\rangle) = \langle 1,0\rangle$. (This assumption will soon imply $\psi_A = \mathrm{inv}_A$ while the case $\psi_A(\langle 0,1\rangle) = \langle 0,1\rangle$, not to be detailed, gives $\psi_A = \mathrm{id}_A$ analogously.) Let us compute, using (6.10) frequently: $\psi_A(\langle \mu,1\rangle) = \psi_A(\langle 0,1\rangle \vee \langle \mu,\mu\rangle) = \psi_A(\langle 0,1\rangle) \vee \psi_A(\langle \mu,\mu\rangle) = \langle 1,0\rangle \vee \langle \mu,\mu\rangle = \langle 1,\mu\rangle$; applying the involution operation to both sides we conclude $\psi_A(\langle 1,\mu\rangle) = \langle \mu,1\rangle$; for $\langle \alpha,\beta\rangle \geq \langle \mu,\mu\rangle$ we have $\psi_A(\langle \alpha,\beta\rangle) = \psi_A\big((\langle \mu,1\rangle \vee \langle \alpha,\alpha\rangle) \wedge (\langle 1,\mu\rangle \vee \langle \beta,\beta\rangle)\big) = (\psi_A(\langle \mu,1\rangle) \vee \psi_A(\langle \alpha,\alpha\rangle)) \wedge (\psi_A(\langle 1,\mu\rangle) \vee \psi_A\langle \beta,\beta\rangle)) = (\langle 1,\mu\rangle \vee \langle \alpha,\alpha\rangle) \wedge (\langle \mu,1\rangle \vee \langle \beta,\beta\rangle) = \langle \beta,\alpha\rangle$; for any $\gamma \in \mathrm{Con}(A)$ we obtain $\psi_A(\langle \gamma,0\rangle) = \psi_A(\langle 1,0\rangle \wedge \langle \gamma,\mu\rangle) = \psi_A(\langle 1,0\rangle) \wedge \psi_A(\langle \gamma,\mu\rangle) = \langle 0,1\rangle \wedge \langle \mu,\gamma\rangle = \langle 0,\gamma\rangle$; and $\psi_A(\langle 0,\gamma\rangle) = \langle \gamma,0\rangle$ follows similarly. Having taken all elements of $\mathrm{Con}^2(A)$ into consideration we have shown that $\psi_A = \mathrm{inv}_A$. This proves (6.13).

Armed with (6.11) and (6.13) we conclude that $\mathcal{D} = \mathcal{D}(\psi)$ is an H-partition, provided ψ is a natural equivalence.

Now let us assume that $\mathcal{D}$ is an H-partition, and let $\psi = \psi(\mathcal{D})$. We have to show that ψ is a natural equivalence. We claim that

(6.14) If $C \in \mathcal{S}$ is a homomorphic image of $A \in \mathcal{L}$ such that A is isomorphic to a subdirect product of finitely many $B_i \in \mathcal{D}_j$ then $C \in \mathcal{D}_j$.

Indeed, by the assumptions there are $\gamma, \beta_1, \ldots, \beta_n \in \mathrm{Con}(A)$ such that $A/\beta_i \in \mathcal{D}_j$, $A/\gamma \cong C$ and $\bigwedge_{i=1}^n \beta_i = 0$. By distributivity we have $\gamma = \gamma \vee 0 = \gamma \vee \bigwedge_{i=1}^n \beta_i = \bigwedge_{i=1}^n (\gamma \vee \beta_i)$. Since C is subdirectly irreducible, γ is meet-irreducible in $\mathrm{Con}(A)$ and we obtain $\gamma = \gamma \vee \beta_i$, i.e. $\gamma \geq \beta_i$ for some i. Therefore $C \cong A/\gamma$ is a homomorphic image of $A/\beta_i \in \mathcal{D}_j$. This yields $C \in \mathcal{D}_j$, proving (6.14).

Now let $A \in \mathcal{L}$ and let $\alpha_1, \alpha_2 \in \mathrm{Con}(A)$ be the congruences from Theorem 6.7. (I.e., A/α_j is a subdirect product of some members of $\mathcal{D}_j$, $j = 1,2$, and $\alpha_1 \wedge \alpha_2 = 0$.) We assert that

$$\alpha_1 \vee \alpha_2 = 1. \tag{6.15}$$

Suppose this is not the case. Then $A/(\alpha_1 \vee \alpha_2)$ is not the one-element lattice, whence it has a homomorphic image C in $\mathcal{S}$. (Indeed, $A/(\alpha_1 \vee \alpha_2)$ is a subdirect product of some lattices in $\mathcal{S}$ and C can be any of the factors of this subdirect decomposition.) But then, by (6.14), C belongs to $\mathcal{D}_j$ for $j = 1$ and $j = 2$ since it is a homomorphic image of A/α_j. This contradicts $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, proving (6.15).

Now we claim that

(6.16) α_1 and α_2 exist and they are uniquely determined.

If $0 \in \mathrm{Con}(A)$ is meet-irreducible, i.e. $A \in \mathcal{S}$, then let $\langle \alpha_1, \alpha_2\rangle$ be $\langle 0,1\rangle$ or $\langle 1,0\rangle$ depending on $A \in \mathcal{D}_1$ or $A \in \mathcal{D}_2$, respectively. Otherwise 0 is the meet $\beta_1 \wedge \ldots \wedge \beta_k$ of some meet-irreducible congruences β_i, and we may put

$$\alpha_j = \bigwedge_{\substack{i=1\\ A/\beta_i \in \mathcal{D}_j}}^{k} \beta_i, \quad j = 1,2.$$

Now, having seen the existence, suppose that besides α_1, α_2 the pair α_1', α_2' also satisfies the corresponding definition. Hence there are congruences $\gamma_i, \gamma_j, \delta_k, \delta_\ell \in \mathrm{Con}(A)$ such that

$$\bigwedge_{i\in J} \gamma_i = \alpha_1, \ \bigwedge_{j\in J'} \gamma_j = \alpha_1', \ \bigwedge_{k\in K} \delta_k = \alpha_2, \ \bigwedge_{\ell\in K'} \delta_\ell = \alpha_2'.$$

and $A/\gamma_i, A/\gamma_j \in \mathcal{D}_1$, $A/\delta_k, A/\delta_\ell \in \mathcal{D}_2$. Put $\alpha_1'' = \alpha_1 \wedge \alpha_1'$ and $\alpha_2'' = \alpha_2 \wedge \alpha_2'$. From $\alpha_1 \wedge \alpha_2 = 0$ we have $\alpha_1'' \wedge \alpha_2'' = 0$. Since

$$\bigwedge_{i\in J\cup J'} \gamma_i = \alpha_1'', \ \bigwedge_{k\in K\cup K'} \delta_k = \alpha_2'',$$

the pair α_1'', α_2'' also meets the requirements of the definition. We obtain from (6.15) that $\alpha_1 \vee \alpha_2 = 1$ and $\alpha_1'' \vee \alpha_2'' = 1$. By distributivity, $\alpha_1 = \alpha_1 \wedge 1 = \alpha_1 \wedge (\alpha_1'' \vee \alpha_2'') = (\alpha_1 \wedge \alpha_1'') \vee (\alpha_1 \wedge \alpha_2'')$. But $\alpha_1 \wedge \alpha_2'' \le \alpha_1 \wedge \alpha_2 = 0$, whence $\alpha_1 = \alpha_1 \wedge \alpha_1''$. Hence $\alpha_1 = \alpha_1''$, and $\alpha_2 = \alpha_2''$ follows similarly. Therefore $\alpha_1 \le \alpha_1'$ and $\alpha_2 \le \alpha_2'$, and the reverse inequalities follow similarly. This yields (6.16).

Now we are ready to prove that $\psi = \psi(\mathcal{D})$ is a natural equivalence. Suppose $f\colon A \to B$ is a surjective lattice homomorphism with kernel $\mu \in \mathrm{Con}(A)$; we have to show that the diagram (6.8) commutes. Consider the congruences $\alpha_1, \alpha_2 \in \mathrm{Con}(A)$ resp. $\alpha_1', \alpha_2' \in \mathrm{Con}(B)$ occurring in the definition of ψ_A resp. ψ_B. For $i = 1, 2$, $(A/\mu)/((\alpha_i \vee \mu)/\mu) \cong A/(\alpha_i \vee \mu)$ can be decomposed into a subdirect product of finitely many members of $\mathcal{S}$. These subdirectly irreducible factors are homomorphic images of $A/(\alpha_i \vee \mu)$, so they are homomorphic images of A/α_i as well. By (6.14), they all belong to $\mathcal{D}_i$. Further, $(\alpha_1 \vee \mu) \wedge (\alpha_2 \vee \mu) = (\alpha_1 \wedge \alpha_2) \vee \mu = 0 \vee \mu = \mu$ yields $(\alpha_1 \vee \mu)/\mu \wedge (\alpha_2 \vee \mu)/\mu = 0$. Therefore we infer from (6.16) that $\alpha_1' = (\alpha_1 \vee \mu)/\mu$ and $\alpha_2' = (\alpha_2 \vee \mu)/\mu$.

Now let $\langle \gamma', \delta' \rangle \in \mathrm{Con}^2(B)$, and denote $\hat{f}(\gamma')$ and $\hat{f}(\delta')$ by γ and δ, respectively. Then $\mathrm{Con}^2(f)(\langle \gamma', \delta' \rangle) = \langle \gamma, \delta \rangle$. To check the commutativity of (6.8) we have to show that $\mathrm{Con}^2(f)$ sends $\psi_B(\langle \gamma', \delta' \rangle) = \big\langle (\gamma' \vee \alpha_1') \wedge (\delta' \vee \alpha_2'), (\delta' \vee \alpha_1') \wedge (\gamma' \vee \alpha_2') \big\rangle$ to $\psi_A(\langle \gamma, \delta \rangle) = \big\langle (\gamma \vee \alpha_1) \wedge (\delta \vee \alpha_2), (\delta \vee \alpha_1) \wedge (\gamma \vee \alpha_2) \big\rangle$. Since $\hat{f}\colon \mathrm{Con}(B) \to \mathrm{Con}(A)$ is a lattice homomorphism and sends $\alpha_i', \gamma', \delta'$ to $\alpha_i \vee \mu, \gamma, \delta$ respectively, $\mathrm{Con}^2(f)\big(\psi_B(\langle \gamma', \delta' \rangle)\big) = \big\langle (\gamma \vee \alpha_1 \vee \mu) \wedge (\delta \vee \alpha_2 \vee \mu), (\delta \vee \alpha_1 \vee \mu) \wedge (\gamma \vee \alpha_2 \vee \mu) \big\rangle$. But this equals $\psi_A(\langle \gamma, \delta \rangle)$ by $\gamma \ge \mu$ and $\delta \ge \mu$, indeed. We have seen that ψ is a natural transformation.

Clearly, ψ_A is monotone and preserves the operation *. So, in order to show that it is a lattice isomorphism, it suffices to show that $\psi_A \circ \psi_A = \mathrm{id}_A$. Let us compute for $\langle\gamma,\delta\rangle \in \mathrm{Con}^2(A)$, using first modularity, then (6.15) and distributivity:

$$\begin{aligned}
\psi_A \circ \psi_A(\langle\gamma,\delta\rangle) &= \psi_A\big(\langle(\gamma\vee\alpha_1)\wedge(\delta\vee\alpha_2),(\delta\vee\alpha_1)\wedge(\gamma\vee\alpha_2)\rangle\big) = \\
&\big\langle(((\gamma\vee\alpha_1)\wedge(\delta\vee\alpha_2))\vee\alpha_1)\wedge(((\delta\vee\alpha_1)\wedge(\gamma\vee\alpha_2))\vee\alpha_2), \\
&\quad(((\delta\vee\alpha_1)\wedge(\gamma\vee\alpha_2))\vee\alpha_1)\wedge(((\gamma\vee\alpha_1)\wedge(\delta\vee\alpha_2))\vee\alpha_2)\big\rangle = \\
&\big\langle((\gamma\vee\alpha_1)\wedge(\delta\vee\alpha_2\vee\alpha_1))\wedge((\delta\vee\alpha_1\vee\alpha_2)\wedge(\gamma\vee\alpha_2)), \\
&\quad((\delta\vee\alpha_1)\wedge(\gamma\vee\alpha_2\vee\alpha_1))\wedge((\gamma\vee\alpha_1\vee\alpha_2)\wedge(\delta\vee\alpha_2))\big\rangle = \\
&\big\langle((\gamma\vee\alpha_1)\wedge(\delta\vee 1))\wedge((\delta\vee 1)\wedge(\gamma\vee\alpha_2)), \\
&\quad((\delta\vee\alpha_1)\wedge(\gamma\vee 1))\wedge((\gamma\vee 1)\wedge(\delta\vee\alpha_2))\big\rangle = \\
&\big\langle(\gamma\vee\alpha_1)\wedge(\gamma\vee\alpha_2),(\delta\vee\alpha_1)\wedge(\delta\vee\alpha_2)\big\rangle = \\
&\big\langle\gamma\vee(\alpha_1\wedge\alpha_2),\delta\vee(\alpha_1\wedge\alpha_2)\big\rangle = \big\langle\gamma\vee 0,\delta\vee 0\big\rangle = \langle\gamma,\delta\rangle,
\end{aligned}$$

indeed. Thus, for every $A \in \mathcal{L}$, ψ_A is an isomorphism, whence $\psi = \psi(\mathcal{D})$ is a natural equivalence.

It is straightforward from the definitions that for any H-partition $\mathcal{D}$ we have $\mathcal{D}(\psi(\mathcal{D})) = \mathcal{D}$.

Now let us assume that ψ is a natural equivalence and let $\psi' = \psi(\mathcal{D}(\psi))$. We have to show that, for any $A \in \mathcal{L}$, $\psi_A = \psi'_A$. This is clear if $A \in \mathcal{S}$; assume this is not the case. Suppose A is a finite subdirect product of members of $\mathcal{D}_j$ for some $j = 1, 2$. We claim that

$$\psi_A = \mathrm{id}_A \text{ for } j = 1 \text{ and } \psi_A = \mathrm{inv}_A \text{ for } j = 2. \tag{6.17}$$

To show (6.17), observe that $0 = \bigwedge_{i=1}^n \beta_i$ holds in $\mathrm{Con}(A)$ for some β_i such that $A/\beta_i \in \mathcal{D}_j$ for all i. We will detail the case $j = 2$ only, for the case $j = 1$ is quite similar. For any $\langle\gamma,\delta\rangle \in \mathrm{Con}^2(A)$ we obtain $\langle\gamma,\delta\rangle \vee 0 = \langle\gamma,\delta\rangle \vee \bigwedge_{i=1}^n \langle\beta_i,\beta_i\rangle = \bigwedge_{i=1}^n (\langle\gamma,\delta\rangle \vee \langle\beta_i,\beta_i\rangle) = \bigwedge_{i=1}^n \langle\gamma\vee\beta_i,\delta\vee\beta_i\rangle$, i.e.,

$$\langle\gamma,\delta\rangle = \bigwedge_{i=1}^n \langle\gamma\vee\beta_i,\delta\vee\beta_i\rangle. \tag{6.18}$$

Since $\psi_{A/\beta_i} = \psi'_{A/\beta_i} = \mathrm{inv}_{A/\beta_i}$, (6.12) yields $\psi_A(\langle\gamma\vee\beta_i,\delta\vee\beta_i\rangle) = \langle\delta\vee\beta_i,\gamma\vee\beta_i\rangle$, whence (6.17) follows easily from (6.18).

Now let $A \in \mathcal{L}$ be arbitrary and let $\langle\gamma,\delta\rangle \in \mathrm{Con}^2(A)$. Similarly to (6.18) we have

$$\langle\gamma,\delta\rangle = \langle\gamma\vee\alpha_1,\delta\vee\alpha_1\rangle \wedge \langle\gamma\vee\alpha_2,\delta\vee\alpha_2\rangle. \tag{6.19}$$

From (6.12) and (6.17) we obtain $\psi_A(\langle\gamma\vee\alpha_1,\delta\vee\alpha_1\rangle) = \langle\gamma\vee\alpha_1,\delta\vee\alpha_1\rangle$ and $\psi_A(\langle\gamma\vee\alpha_2,\delta\vee\alpha_2\rangle) = \langle\delta\vee\alpha_2,\gamma\vee\alpha_2\rangle$. Therefore $\psi_A(\langle\gamma,\delta\rangle) = \psi'_A(\langle\gamma,\delta\rangle)$ follows from (6.19), completing the proof. $\square$

Since any finite lattice has a simple homomorphic image, we immediately obtain

Corollary 6.20. *([Cz13]) Given a prevariety $\mathcal{L}$ of finite lattices, if two* $\mathrm{Con}^2 \to \mathrm{Con}^2$ *natural equivalences coincide on every simple lattice of $\mathcal{L}$ then they coincide on the whole $\mathcal{L}$.*

Now let $\mathcal{L}$ be a prevariety generated by a finite set K of finite lattices[1]. By a celebrated result of Jónsson [Jo2], each subdirectly irreducible lattice in $\mathcal{L}$ is a homomorphic image of a sublattice of some lattice in K. Therefore, apart from isomorphic copies, $\mathcal{S} = \mathcal{S}(\mathcal{L})$ is finite. This short argument proves

Corollary 6.21. *([Cz13]) There is an algorithm which produces for any finite set K of finite lattices, as input, a description of the (necessarily finitely many) natural equivalences from the functor* Quord: $\mathcal{L} \to \mathcal{V}$ *(or, equivalently, from the functor* Con^2: $\mathcal{L} \to \mathcal{V}$*) to the functor* Con^2: $\mathcal{L} \to \mathcal{V}$ *where $\mathcal{L}$ denotes the prevariety generated by K.*

In virtue of Corollary 6.21 it is quite easy to present some examples. Let $t(\mathcal{L}) = |T(\mathcal{L})|$, the number of natural equivalences from the functor Quord: $\mathcal{L} \to \mathcal{V}$ to the functor Con^2: $\mathcal{L} \to \mathcal{V}$. By M_n and N_5 we denote the modular lattice of height two with exactly n atoms and the five-element non-modular lattice, respectively. For $1 \le n < \infty$ let $\mathcal{L}_n$ resp. $\mathcal{L}'_n$ be the prevariety generated by M_{n+1} resp. $\{M_{n+1}, N_5\}$. Note that $\mathcal{L}_1$ is the class of finite distributive lattices. Clearly, $\mathcal{L}_{\aleph_0} = \bigcup_{n=1}^{\infty} \mathcal{L}_n$ and $\mathcal{L}'_{\aleph_0} = \bigcup_{n=1}^{\infty} \mathcal{L}'_n$ are prevarieties, too.

Example 6.22. *([Cz13]) For $n = 1, 2, \dots, \aleph_0$, $t(\mathcal{L}_n) = t(\mathcal{L}'_n) = 2^n$.*

The straightforward proof, based on Corollary 6.21 and the aforementioned result of Jónsson, will be omitted.

To conclude this chapter with an open problem we mention that $t(\{\text{all finite lattices}\})$, $t(\{\text{all lattices}\})$ and $t(\{\text{all distributive lattices}\})$ are still unknown.

[1] $\mathcal{L}$ is just the class of finite lattices of the variety generated by K; this follows from Jónsson [Jo2].

Now, as promised in the proof of Theorem 6.3, we enclose the result of our computer program. The roman text is the original output of the program, which was developed to escort [Cz11]. The italic text, like this, contains additional information.

The input file

This file must obey certain syntactical rules. These rules are displayed by the program. They allow comments after a left curly bracket in each line. As for the variables of the Horn sentence, s1 and s2 are subterms in the conclusion but now treated as variables.
This is just to shorten the calculations. This has nothing to do with Claim 6.4, this is applied to the LATTICE Horn sentence obtained from the involution lattice Horn sentence, so in this case we do not have to add involuted inequalities to the premise. Below, xi, yi, ... stand for x^*, y^*, ...

```
\name
The Horn sentence from Theorem 6.3
\variables
x  y  z  t  u  v  w  s1  s2
xi yi zi ti ui vi wi s1i s2i
\premises
 x < y + u
 xi< yi+ui        {To each inequality we have to add }
 y < z + t        { its involuted one, by Claim 6.4}
 yi< zi+ti
 u < v +w
 ui< vi+wi
 s1 < (u +z ) * (ui+x +zi+ti)   {We have to tell what is }
 (u +z ) * (ui+x +zi+ti) < s1   {denoted by s1 and s2 }
 s2 < (y +w ) * (yi+x +wi+vi)   {Here we do not have to}
 (y +w ) * (yi+x +wi+vi) < s2   {add involuted premises!}
\conclusion
x < ((y+s1)*(ui+x+zi+ti))+((u+s2)*(yi+x+wi+vi)) zi+wi
\end
```

The output file of the program

Some unnecessary details are deleted, but enough is left to check the result. Where the program printed too long lines, we had to break the line into smaller pieces. In this case the line(s) continuing the original one are generously indented. The original variables are denoted the same way as in the input file. There are some extra variables which are necessary to bring the Horn sentence into a canonical form via a trivial algorithm (or cf [Cz11] for details). These extra variables are prefixed by @.

Name = The Horn sentence from the paper *[Cz10]*

The premises are the following
(The original resp. reduced resp. intermediate ones are prefixed by # resp. R resp. &. The reduced premises are obtained via an easy recursion, whose current depth is indicated by the dots. Any reduced or intermediate

premise comes from the (nearest) one above which has less (generally one less) dots before it.)

```
# x<y+u
R .x<y+u
# xi<yi+ui
R .xi<yi+ui
# y<z+t
R .y<z+t
# yi<zi+ti
R .yi<zi+ti
# u<v+w
R .u<v+w
# ui<vi+wi
R .ui<vi+wi
# s1<(u+z)*(ui+x+zi+ti)
& .s1<(u+z)*(ui+x+zi+ti)
R ..s1<u+z
R ..s1<ui+x+zi+ti
# (u+z)*(ui+x+zi+ti)<s1
& .(u+z)*(ui+x+zi+ti)<s1
& ...u+z<@19
R ....u<@19
R ....z<@19
R ...@19<u+z
& ...ui+x+zi+ti<@20
R ....ui<@20
R ....x<@20
R ....zi<@20
R ....ti<@20
R ...@20<ui+x+zi+ti
R ..@19*@20<s1
# s2<(y+w)*(yi+x+wi+vi)
& .s2<(y+w)*(yi+x+wi+vi)
R ..s2<y+w
R ..s2<yi+x+wi+vi
# (y+w)*(yi+x+wi+vi)<s2
& .(y+w)*(yi+x+wi+vi)<s2
& ...y+w<@21
R ....y<@21
R ....w<@21
R ...@21<y+w
& ...yi+x+wi+vi<@22
R ....yi<@22
R ....x<@22
R ....wi<@22
R ....vi<@22
R ...@22<yi+x+wi+vi
R ..@21*@22<s2
```

The conclusion is:

```
# x<((y+s1)*(ui+x+zi+ti))+((u+s2)*(yi+x+wi+vi))+zi+wi
```

The rest of derived premises come from the conclusion:

```
& .@23<((y+s1)*(ui+x+zi+ti))+((u+s2)*(yi+x+wi+vi))+zi+wi
& ...(y+s1)*(ui+x+zi+ti)<@24
& .....y+s1<@25
R ......y<@25
R ......s1<@25
```

```
R .....@25<y+s1
& .....ui+x+zi+ti<@26
R ......ui<@26
R ......x<@26
R ......zi<@26
R ......ti<@26
R .....@26<ui+x+zi+ti
R ....@25*@26<@24
& ...@24<(y+s1)*(ui+x+zi+ti)
R ....@24<y+s1
R ....@24<ui+x+zi+ti
& ...(u+s2)*(yi+x+wi+vi)<@27
& .....u+s2<@28
R ......u<@28
R ......s2<@28
R .....@28<u+s2
& .....yi+x+wi+vi<@29
R ......yi<@29
R ......x<@29
R ......wi<@29
R ......vi<@29
R .....@29<yi+x+wi+vi
R ....@28*@29<@27
& ...@27<(u+s2)*(yi+x+wi+vi)
R ....@27<u+s2
R ....@27<yi+x+wi+vi
& ..@23<@24*@27*zi*wi
R ...@23<@24
R ...@23<@27
R ...@23<zi
R ...@23<wi
& .((y+s1)*(ui+x+zi+ti))+((u+s2)*(yi+x+wi+vi))+zi+wi<@23
& ..(y+s1)*(ui+x+zi+ti)<@23
& ....y+s1<@30
R .....y<@30
R .....s1<@30
R ....@30<y+s1
& ....ui+x+zi+ti<@31
R .....ui<@31
R .....x<@31
R .....zi<@31
R .....ti<@31
R ....@31<ui+x+zi+ti
R ...@30*@31<@23
& ..(u+s2)*(yi+x+wi+vi)<@23
& ....u+s2<@32
R .....u<@32
R .....s2<@32
R ....@32<u+s2
& ....yi+x+wi+vi<@33
R .....yi<@33
R .....x<@33
R .....wi<@33
R .....vi<@33
R ....@33<yi+x+wi+vi
R ...@32*@33<@23
R ..zi<@23
R ..wi<@23
```

Further details of the computation

Total number of variables=33
Number of reduced premises=74

The reduced premises are (those indicated with R previously):
(The inequalities below constitute the premise of the `canonical' Horn sentence.)

```
  1: x<y+u
  2: xi<yi+ui
  3: y<z+t
  4: yi<zi+ti
  5: u<v+w
  6: ui<vi+wi
  7: s1<z+u
  8: s1<x+zi+ti+ui
  9: u<@19
 10: z<@19
 11: @19<z+u
 12: ui<@20
 13: x<@20
 14: zi<@20
 15: ti<@20
 16: @20<x+zi+ti+ui
 17: @19*@20<s1
 18: s2<y+w
 19: s2<x+yi+vi+wi
 20: y<@21
 21: w<@21
 22: @21<y+w
 23: yi<@22
 24: x<@22
 25: wi<@22
 26: vi<@22
 27: @22<x+yi+vi+wi
 28: @21*@22<s2
 29: y<@25
 30: s1<@25
 31: @25<y+s1
 32: ui<@26
 33: x<@26
 34: zi<@26
 35: ti<@26
 36: @26<x+zi+ti+ui
 37: @25*@26<@24
 38: @24<y+s1
 39: @24<x+zi+ti+ui
 40: u<@28
 41: s2<@28
 42: @28<u+s2
 43: yi<@29
 44: x<@29
 45: wi<@29
 46: vi<@29
 47: @29<x+yi+vi+wi
 48: @28*@29<@27
```

```
 49: @27<u+s2
 50: @27<x+yi+vi+wi
 51: @23<@24
 52: @23<@27
 53: @23<zi
 54: @23<wi
 55: y<@30
 56: s1<@30
 57: @30<y+s1
 58: ui<@31
 59: x<@31
 60: zi<@31
 61: ti<@31
 62: @31<x+zi+ti+ui
 63: @30*@31<@23
 64: u<@32
 65: s2<@32
 66: @32<u+s2
 67: yi<@33
 68: x<@33
 69: wi<@33
 70: vi<@33
 71: @33<x+yi+vi+wi
 72: @32*@33<@23
 73: zi<@23
 74: wi<@23
```

The assignment of variables is:
  1:x   2:y   *(This is not so important here, so the rest is deleted!)*
Conclusion left side: x *(This in $M=z_0$ in the proof.)*
Conclusion right side: @23 *(This is z in the proof.)*

The closures after 0 steps are:
(Note that the recursive formula of [Czedli: On the word problem of lattices, Periodica Math. Hungar. 23 (1991) 49-58] is used in a bit faster form. Namely, when the (j+1)-th closure of {x} is known then the program uses that rather than the j-th closure.) *(That is, the algorithm described in the proof of Thm, 6.3 is used.)*
        x   [x]
        y   [y]   *(etc. Here the identical map T_0 is given. So the details are cancelled. So are the intermediate steps, for only the last step is important for us.)*
Below, $T = T_j$ is given. The reader can check even manually that the algorithm, which is monotone in its `variable' T, keeps T_j fixed, and the result follows.)

The closures after 4 steps are:
(This is the last step. If the program asserts that the Horn sentence fails then one can use the following data to prove it. Indeed, it suffices to check that the closure S given below is a fixed point of the formula given in [Cz11]. Then S >= T and q not in T({p}) follows from q not in S({p}).)

    x [x @20 @22 @26 @29 @31 @33]

```
    y   [y @21 @25 @30 ]
    z   [z @19 ]  (E.g., this means T(z)={z,@19}, etc.)
    t   [t ]
    u   [u @19 @28 @32 ]
    v   [v ]
    w   [w @21 ]
   s1   [s1 zi wi @19 @20 @22 @23 @24 @25 @26 @27 @28 @29
                        @30 @31 @32 @33 ]
   s2   [s2 zi wi @20 @21 @22 @23 @24 @25 @26 @27 @28 @29
                        @30 @31 @32 @33 ]
   xi   [xi @20 @22 @26 @29 @31 @33 ]
   yi   [yi @20 @22 @26 @29 @31 @33 ]
   zi   [zi wi @20 @22 @23 @24 @25 @26 @27 @28 @29 @30 @31
                        @32 @33 ]
   ti   [ti @20 @26 @31 ]
   ui   [ui @20 @22 @26 @29 @31 @33 ]
   vi   [vi @22 @29 @33 ]
   wi   [zi wi @20 @22 @23 @24 @25 @26 @27 @28 @29 @30 @31
                        @32 @33 ]
  s1i   [s1i ]
  s2i   [s2i ]
  @19   [@19 ]
  @20   [@20 @26 @31 ]
  @21   [@21 ]
  @22   [@22 @29 @33 ]
  @23   [zi wi @20 @22 @23 @24 @25 @26 @27 @28 @29 @30 @31
                        @32 @33 ]
  @24   [zi wi @20 @22 @23 @24 @25 @26 @27 @28 @29 @30 @31
                        @32 @33 ]
  @25   [@25 @30 ]
  @26   [@20 @26 @31 ]
  @27   [zi wi @20 @22 @23 @24 @25 @26 @27 @28 @29 @30 @31
                        @32 @33 ]
  @28   [@28 @32 ]
  @29   [@22 @29 @33 ]
  @30   [@25 @30 ]
  @31   [@20 @26 @31 ]
  @32   [@28 @32 ]
  @33   [@22 @29 @33 ]
```

The Horn sentence does not hold in all lattices.
Indeed, now M in the proof of Theorem 6.3 corresponds to x, z corresponds to @23, and @23 is not in the closure of x, cf. the first row starting with x.

CHAPTER VII

COALITION LATTICES

Now, based on [CP1], [Cz14] and [CP2], we are going to deal with coalition lattices. In game theory or in the mathematics of human decision making the following situation is frequently considered, cf. e.g. Peleg [Pe1]. Given a finite set P, for example we may think of P as a set of political parties, and each $x \in P$ has a certain strength measured on a numerical scale that we may think of as the number of votes x receives. Subsets of P are called *coalitions*. The strength of a coalition is the sum of strengths of its members. Let $\mathcal{L}(P)$ stand for the set of all coalitions. The relation "stronger or equally strong" is a quasiorder on P and also on $\mathcal{L}(P)$. The quasiorder on P has some influence on the quasiorder on $\mathcal{L}(P)$. Sometimes, like before the election in our example, all we have is a quasiorder or, more frequently, a partial order on P, supplied e.g. by a public opinion poll. Yet, as we will see, this often suffices to build some algebraic structure on $\mathcal{L}(P)$.

From now on, let $P = \langle P, \leq \rangle$ be a fixed finite quasiordered set, i.e., $\leq$ is a reflexive and transitive relation on the finite set P. For $x, y \in P$, $x > y$ means that $y \leq x$ and $x \not\leq y$. For undefined terminology the reader is referred to Grätzer [Gr1]. Even without explicit mentioning, all sets occurring in this paper are assumed to be finite. The set of all subsets, alias coalitions, of P is denoted by $\mathcal{L}(P)$. For $X, Y \in \mathcal{L}(P)$, a map φ: $X \to Y$ is called an *extensive map* if φ is injective and for every $x \in X$ we have $x \leq \varphi(x)$. Let $X \leq Y$ mean that there exists an extensive map $X \to Y$; this definition turns $\mathcal{L}(P)$ into a quasiordered set $\mathcal{L}(P) = \langle \mathcal{L}(P), \leq \rangle$. Using singleton coalitions one can easily see that P is a partially ordered set, in short a poset, iff $\mathcal{L}(P)$ is a poset. Our main result, Theorem 7.2, describes the posets P for which $\mathcal{L}(P)$ is a lattice. However, to achieve more generality without essentially lengthening the proof, Theorem 7.2 will be concluded from its generalization Theorem 7.1 for quasiorders.

Definition. A quasiordered set P is called *upper bound free*, in short UBF, if for any $a, b, c \in P$ we have

$$((a \leq c) \;\&\; (b \leq c)) \implies ((a \leq b) \text{ or } (b \leq a)).$$

The equivalence classes of the equivalence generated by $\leq_P$ will be called the *components* of P. If P is an UBF poset and has only one component then P is called a *tree*. A poset is called a *forest* if its components are trees. Clearly, a finite poset is a forest iff it is UBF. Let $\overline{P} = \langle \overline{P}, \leq \rangle$ denote the poset obtained from P in the canonical way, i.e., consider the intersection $\sim$ of $\leq_P$ with its inverse, let

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$\overline{P}$ consist of the classes of the equivalence relation $\sim$, and for $A, B \in \overline{P}$ let $A \le B$ mean that $a \le b$ for some $a \in A$ and $b \in B$. For $x \in P$ the $\sim$-class of x will be denoted by $\bar{x}$. Sometimes, for $x \in P$ and $Y \in \overline{P}$, we write $x \le Y$ or $x > Y$ instead of $\bar{x} \le Y$ or $\bar{x} > Y$, respectively. P is called a *quasilattice* if each two-element subset of P has an infimum and a supremum in P. (The infimum and supremum is defined only up to the equivalence $\sim$!) Equivalently, P is a quasilattice iff $\overline{P}$ is a lattice. Following Chajda [Ch2], cf. also Chajda and Kotrle [CK1], an algebra $\langle L; \vee, \wedge\rangle$ is called a *q-lattice* if both binary operations are associative and commutative, and the identities $x \vee (x \wedge y) = x \vee x$, $x \vee (y \vee y) = x \vee y$, their duals, and the identity $x \vee x = x \wedge x$ hold. In Chajda [Ch2], the well-known connection between lattices as posets and lattices as algebraic structures is generalized to a similar connection between quasilattices and q-lattices. Hence our first theorem indicates that q-lattices are relevant tools to study coalitions.

Theorem 7.1. *(Czédli [CP1]) For a finite quasiordered set P, $\mathcal{L}(P)$ is a quasilattice iff P is upper bound free.*

As already indicated, this theorem instantly yields

Theorem 7.2. *([CP1]) For a finite poset P, $\mathcal{L}(P)$ is a lattice iff P is a forest.*

In contrast with many other related lattices, it is not so evident that coalitions form a lattice. In [CP1], Theorem 7.2 is proved in two distinct ways, representing the authors' separate approaches. (There is no essential difference in complexity between these approaches.) Since $\mathcal{L}(P)$ has a least element, the empty coalition, one possibility is to show the existence of suprema. This will be presented in this chapter. (Fortunately, this approach works for q-lattices as well.) On the other hand, P is the largest element of $\mathcal{L}(P)$, whence it suffices to show the existence of infima. This was done by the second author, who obtained

Proposition 7.A. *(Pollák [CP1]) Let P be a forest, $k \ge 2$, and for $A_1, \dots, A_k \in \mathcal{L}(P)$ let $M = \{b_1 \wedge \dots \wedge b_k \colon b_1 \in A_1, \dots, b_k \in A_k$, and the infimum $b_1 \wedge \dots \wedge b_k$ exists in $P\}$. If M is empty (in particular when one of the A_i is empty) then $\bigwedge_{i=1}^k A_i = \emptyset$. If M is non-empty then choose a maximal element $c = a_1 \wedge \dots \wedge a_k$ in M where the a_i belong to A_i such that, for every i, $c \in A_i \implies c = a_i$. Let $A_i' = A_i \setminus \{a_i\}$ for $i = 1, \dots, k$, $P' = P \setminus \{c\}$, and put $C' = \bigwedge_{i=1}^k A_i'$ in $\mathcal{L}(P')$. Then $\bigwedge_{i=1}^k A_i = C' \cup \{c\}$ in $\mathcal{L}(P)$.*

The proof of Theorem 7.1 will give an effective construction of suprema in $\mathcal{L}(P)$. (When P is a forest and therefore $\mathcal{L}(P)$ is a lattice then constructing suprema is much easier than in case of quasilattices.)

Proposition 7.3. *([CP1]) For any finite quasiordered set P, $\mathcal{L}(P)$ is selfdual. In fact, the map $\mathcal{L}(P) \to \mathcal{L}(P)$, $X \mapsto P \setminus X$ is a dual automorphism.*

In virtue of Proposition 7.3 we have

$$A_1 \vee \dots \vee A_k = \overline{\overline{A}_1 \wedge \dots \wedge \overline{A}_k}, \tag{7.4}$$

and dually. This offers a way of deducing suprema from infima and vice versa. In practical computations this can be useful e.g. when the $\overline{A}_i = P \setminus A_i$ have only a few elements. However, Proposition 7.A gives a better view of infima for lattices $\mathcal{L}(P)$ than the dual of (7.4), and the author does not think that (7.4) would make the proof of Proposition 7.A easier.

To give a better picture of coalition lattices and also for later references, we cite two further assertions from [CP1] without proofs; the first of them is quite easy while the second could also be derived from the forthcoming Theorem 7.9.

Proposition 7.B. *Let T_1, T_2, ..., T_s be the components of the quasiordered set P. Then $\mathcal{L}(P) = \langle \mathcal{L}(P), \le \rangle$ is isomorphic to the direct product of the $\mathcal{L}(T_i)$, $1 \le i \le s$.*

Proposition 7.C. *Let P be a finite forest. Then the lattice $\mathcal{L}(P)$ is distributive iff $\mathcal{L}(P)$ is modular iff every tree of P is a chain.*

Before formulating further results, let us prove what has already been stated.

Proof of Theorem 7.1. Let us suppose first that $\mathcal{L}(P)$ is a quasilattice, and $a \le c$, $b \le c$ hold for $a, b, c \in P$. Let U be a supremum of $\{a\}$ and $\{b\}$ in $\mathcal{L}(P)$. Since $\{a\} \le \{c\}$ and $\{b\} \le \{c\}$, we have $U \le \{c\}$, whence $|U| \le 1$. On the other hand, $|U| \ge 1$ by $\{a\} \le U$. Thus U is a singleton, say $\{d\}$. From $\{a\} \le U = \{d\}$ and $\{b\} \le U = \{d\}$ we infer $a \le d$ and $b \le d$. Since $\{a, b\}$ is an upper bound of $\{a\}$ and $\{b\}$, we obtain $\{d\} = U \le \{a, b\}$, yielding $d \le b$ or $d \le a$. By transitivity, $a \le b$ or $b \le a$. I.e., P is upper bound free.

To prove the converse, let us assume that P is UBF. Then so is $\overline{P}$. Let $\overline{P}_1$ be the set of maximal elements of the forest $\overline{P}$. If $\overline{P} \setminus \overline{P}_1$ is not empty then let $\overline{P}_2$ denote the set of its maximal elements, etc.; if $\overline{P} \setminus (\overline{P}_1 \cup \ldots \cup \overline{P}_{i-1})$ is not empty then let $\overline{P}_i$ denote the set of its maximal elements. Then $\overline{P}$ is partitioned in finitely many subsets $\overline{P}_1$, ..., $\overline{P}_r$. For $1 \le i \le r$ let $P_i = \{x \in P\colon \bar{x} \in \overline{P}_i\}$; now P is the union of the pairwise disjoint P_i, $1 \le i \le r$. The set $\{x \in P_1 \cup \ldots \cup P_i\colon x \ge B$ holds for no $B \in \overline{P}_i\}$ will be denoted by Q_i.

Now, for given coalitions A_1, ..., A_k, we intend to define a sequence $\emptyset = C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_r = C$ of coalitions such that $C_i = C \cap (P_1 \cup \ldots \cup P_i)$ and C is a supremum of $\{A_1, \ldots, A_k\}$. Suppose $i > 0$ and C_{i-1} has already been defined. For given $B \in \overline{P}_i$ and $1 \le j \le k$ we define the following numbers.

$$\begin{aligned} \gamma_i(B) &= |\{x \in C_{i-1}\colon x > B\}|, \\ \nu_i(j, B) &= |\{x \in A_j\colon x \ge B\}|, \\ \delta_i(j, B) &= \nu_i(j, B) - \gamma_i(B), \\ \lambda_i(B) &= \max\{0, \delta_i(1, B), \delta_i(2, B), \ldots, \delta_i(k, B)\}. \end{aligned}$$

Let us choose a subset $S_i(B)$ of B such that $|S_i(B)| = \lambda_i(B)$. (We will soon prove that this choice is possible.) We define C_i by

$$C_i = C_{i-1} \cup \bigcup_{B \in \overline{P}_i} S_i(B).$$

Denote $A_j\cap(P_1\cup\ldots\cup P_i)$ by $A_j^{(i)}$ and consider the following induction hypothesis

$$(H(i)) \qquad A_j^{(i)} \le C_i \text{ for all } j \text{ and } \lambda_i(B) \le |B| \text{ for all } B \in \overline{P}_i.$$

Note that $\lambda_i(B) \le |B|$ is necessary to make the choice of $S_i(B)$ possible.

For $i=1$, $\gamma_1(B)=0$ and $\nu_1(j,B)=|A_j\cap B|\le|B|$ imply $\lambda_1(B)\le|B|$. Since $|A_j\cap B| = \nu_1(j,B) = \delta_1(j,B) \le \lambda_1(B) = |S_1(B)|$, we can chose an injection ψ_B: $A_j\cap B\to S_1(B)$. Clearly,

$$\bigcup_{B\in\overline{P}_1} \psi_B\colon\ A_j^{(1)} \to C_1$$

is an extensive map. This proves $H(1)$.

Now, for $1\le i\le r$, suppose $H(i-1)$. For $B\in\overline{P}_i$, the existence of extensive maps $\alpha_j^{(i-1)}$: $A_j^{(i-1)} \to C_{i-1}$, which necessarily map $\{x\in A_j\colon\ x>B\}$ into $\{x\in C_{i-1}\colon\ x>B\}$, yields $|\{x\in A_j\colon\ x>B\}|\le|\{x\in C_{i-1}\colon\ x>B\}|$ for any j. Using this inequality we can estimate: $\delta_i(j,B)=\nu_i(j,B)-\gamma_i(B)=|\{x\in A_j\colon\ x\ge B\}|-|\{x\in C_{i-1}\colon\ x>B\}|=|\{x\in A_j\colon\ x>B\}\cup(A_j\cap B)|-|\{x\in C_{i-1}\colon\ x>B\}|=|A_j\cap B|+|\{x\in A_j\colon\ x>B\}|-|\{x\in C_{i-1}\colon\ x>B\}|\le|A_j\cap B|\le|B|$. Therefore $\lambda_i(B)\le|B|$, indeed.

Now, for a fixed j and arbitrary $B\in\overline{P}_i$, we will define an extensive map $\varphi_B=\varphi_{j,B}$: $\{x\in A_j\colon\ x\ge B\}\to\{x\in C_i\colon\ x\ge B\}$. Since $|\{x\in A_j\colon\ x\ge B\}|=\nu_i(j,B)=\gamma_i(B)+\delta_i(j,B)\le\gamma_i(B)+\lambda_i(B)=|\{x\in C_{i-1}\colon\ x>B\}|+|C_i\cap B|=|\{x\in C_i\colon\ x>B\}\cup(C_i\cap B)|=|\{x\in C_i\colon\ x\ge B\}|$, i.e.,

$$|\{x\in A_j\colon\ x\ge B\}|\le|\{x\in C_i\colon\ x\ge B\}|, \tag{7.5}$$

the restriction of $\alpha_j^{(i-1)}$ to the set $\{x\in A_j\colon\ x\ge B\}\cap A_j^{(i-1)}$ can be extended to an injective map φ_B: $\{x\in A_j\colon\ x\ge B\}\to\{x\in C_i\colon\ x\ge B\}$. For any $y\in\{x\in A_j\colon\ x\ge B\}$ either $y\in A_j^{(i-1)}$ and $\varphi_B(y)=\alpha_j^{(i-1)}(y)\ge y$ or $y\in B$, whence φ_B is an extensive map. Let $\alpha_j^{(i)}$ be the union of $\alpha_j^{(i-1)}$ and all the φ_B, $B\in\overline{P}_i$. Then $\alpha_j^{(i)}$: $A_j^{(i)}\to C_i$. Since, by the UBF property, Q_i and the sets $\{x\in C_i\colon\ x\ge B\}$, $B\in\overline{P}_i$, are pairwise disjoint, $\alpha_j^{(i)}$ is injective and therefore it is an extensive map. Hence $A_j^{(i)}\le C_i$, proving $H(i)$.

We have seen that the definition of $C=C_r$ is correct and, by $H(r)$, C is an upper bound of the A_j, $1\le j\le k$.

Now let $D\in\mathcal{L}(P)$ be an arbitrary upper bound of the A_j, $1\le j\le k$. We have to show that $C\le D$. By the assumption, there are extensive maps μ_j: $A_j\to D$. Let $D_i=D\cap(P_1\cup\ldots\cup P_i)$. We will define extensive maps τ_i: $C_i\to D_i$ for $i=1,2,\ldots,r$ via induction, and $C=C_r\le D_r=D$ will follow evidently.

For each $B\in\overline{P}_1$ such that $B\cap C=B\cap C_1=S_1(B)$ is non-empty, choose a j with $|S_1(B)|=\lambda_1(B)=\delta_1(j,B)$. Then $|A_j\cap B|=\nu_1(j,B)-0=\delta_1(j,B)=|S_1(B)|=|C_1\cap B|$. Since μ_j clearly maps $A_j\cap B$ into $D_1\cap B$, $|C_1\cap B|=|A_j\cap B|\le|D_1\cap B|$.

Therefore we can choose an injective map β_B: $C_1 \cap B \to D_1 \cap B$. Let β_B denote the empty map when $B \cap C = \emptyset$. Define τ_1 as the union of the β_B, $B \in \overline{P}_1$. Clearly, τ_1: $C_1 \to D_1$ is an extensive map.

Now, for $1 < i \leq r$, suppose we already have an extensive map τ_{i-1}: $C_{i-1} \to D_{i-1}$; we define τ_i as follows. For $B \in \overline{P}_i$, if $|C_i \cap B| = \lambda_i(B) = 0$, then let κ_B be the restriction of τ_{i-1} to the set $\{x \in C_{i-1}\colon\ x > B\} = \{x \in C_i\colon\ x \geq B\}$. Otherwise choose a j such that $|C_i \cap B| = \lambda_i(B) = \delta_i(j, B)$. Since μ_j maps $\{x \in A_j\colon\ x \geq B\}$ into $\{x \in D_i\colon\ x \geq B\}$ and (7.5) with the j chosen turns into an equality, we conclude that $|\{x \in C_i\colon\ x \geq B\}| \leq |\{x \in D_i\colon\ x \geq B\}|$. Further, for all $y \in \{x \in C_i\colon\ x > B\} = \{x \in C_i\colon\ x \geq B\} \setminus B$, $\tau_{i-1}(y)$ is defined and belongs to $\{x \in D_i\colon\ x > B\}$. Therefore there exists an injective map κ_B: $\{x \in C_i\colon\ x \geq B\} \to \{x \in D_i\colon\ x \geq B\}$ such that $\kappa_B(x) = \tau_{i-1}(x)$ if $x \notin B$. Clearly, κ_B is an extensive map. Now let τ_i be the union of τ_{i-1} and the κ_B, $B \in \overline{P}_i$. By the UBF property, Q_i and the sets $\{x \in D_i\colon\ x \geq B\}$, $B \in \overline{P}_i$, are pairwise disjoint, implying the injectivity of τ_i. Hence τ_i is an extensive map.

We have seen that finitely many (but more than zero) coalitions of $\mathcal{L}(P)$ have a supremum. By finiteness and $\emptyset \in \mathcal{L}(P)$ we infer that $\mathcal{L}(P)$ is a quasilattice. □

Proof of Proposition 7.3. With the notation $\overline{X} = P \setminus X$, it suffices to show that, for $A, B \in \mathcal{L}(P)$, $A \leq B \Longrightarrow \overline{B} \leq \overline{A}$, for the reverse implication then also follows. First we show that

$$\text{(7.6)} \qquad \text{For } |A| = |B|, \quad A \leq B \iff A \setminus B \leq B \setminus A.$$

For later reference we will also show that

(7.7) If $|A| = |B|$ and $A \leq B$ then there is an extensive map $A \to B$ which acts identically on $A \cap B$.

To show (7.6), suppose $A \leq B$, and choose an extensive map φ: $A \to B$ with a maximum number of fixed points. Suppose that $u \in A \cap B$ is not a fixed point of φ. By the assumptions φ is surjective; let a be a preimage of u. We have $a \leq \varphi(a) = u \leq \varphi(u)$ and $|\{a, u, \varphi(u)\}| = 3$. Clearly, the map

$$\varphi'\colon\ A \to B, \quad x \mapsto \begin{cases} u, & \text{if } x = u, \\ \varphi(u), & \text{if } x = a, \\ \varphi(x), & \text{otherwise} \end{cases}$$

has one more fixed point than φ, a contradiction. Therefore φ acts identically on $A \cap B$, which already shows (7.7), and its restriction to $A \setminus B$ is an extensive map $A \setminus B \to B \setminus A$, yielding $A \setminus B \leq B \setminus A$. The converse being evident we have verified (7.6).

Now suppose $A \leq B$. Then necessarily $|A| \leq |B|$. If $|A| = |B|$ then, using (7.6) twice, $\overline{B} \setminus \overline{A} = A \setminus B \leq B \setminus A = \overline{A} \setminus \overline{B}$ implies $\overline{B} \leq \overline{A}$. If $|A| < |B|$ and φ: $A \to B$ is an extensive map then $A \leq \varphi(A)$ yields $\overline{\varphi(A)} \leq \overline{A}$ by the previous case, $\overline{B} \subseteq \overline{\varphi(A)}$ gives $\overline{B} \leq \overline{\varphi(A)}$, and $\overline{B} \leq \overline{A}$ follows by transitivity. □

In the sequel we will disregard from quasilattices and deal only with coalition lattices. Therefore, according to Theorem 7.2, P will always assumed to be a finite forest. While those forests P for which $\mathcal{L}(P)$ is distributive or modular are characterized in Proposition 7.C, we do not know whether the class of coalition lattices satisfies any nontrivial identity. Our knowledge is much better about Horn sentences. Developing a constructive way to build an arbitrary coalition lattice from smaller ones, a nontrivial lattice Horn sentence will be constructed which holds in every coalition lattice. In other words, this means that the quasivariety generated by coalition lattices does not include all lattices.

To achieve our goal we need the following

Lattice construction. Let L_i be a complete lattice with bounds 0_i and 1_i, $i = 1, 2$, and let $\emptyset \neq S_i \subseteq L_i$ such that $1_1 \in S_1$, $0_2 \in S_2$, S_1 is closed under arbitrary meets and S_2 is closed under arbitrary joins. Note that the S_i are necessarily complete lattices under the ordering inherited from L_i but they need not be sublattices. Let ψ: $S_1 \to S_2$ be a lattice isomorphism. Associated with the quintuplet $\langle L_1, L_2, S_1, S_2, \psi\rangle$, we intend to define a lattice $L = L(L_1, L_2, S_1, S_2, \psi)$ as follows. Let L be the disjoint union of L_1 and L_2. For $x, y \in L$ we put $x \leq y$ iff one of the following three possibilities holds:

- $x, y \in L_1$ and $x \leq y$ in L_1;
- $x, y \in L_2$ and $x \leq y$ in L_2;
- $x \in L_1$, $y \in L_2$ and there exists a $z \in S_1$ such that $x \leq z$ in L_1 and $\psi(z) \leq y$ in L_2.

Proposition 7.8. *([Cz14])* $L = L(L_1, L_2, S_1, S_2, \psi) = \langle L(L_1, L_2, S_1, S_2, \psi), \leq\rangle$ *defined above is a complete lattice.*

Proof. It is straightforward to check that $\langle L, \leq\rangle$ is a partially ordered set with least element $0 = 0_1$ and greatest element $1 = 1_2$. To avoid confusion, $\bigwedge$, $\leq_1$, $\wedge_2$, $\bigvee_{S_1}$, etc. will denote the meet in L, the relation in L_1, the binary meet in L_2, the join in S_1, etc., respectively. Of course, $\bigwedge_{S_1} = \bigwedge_1$ and $\bigvee_{S_2} = \bigvee_2$.

Now we intend to show that any nonempty subset of L has a supremum. We start with a particular case. Let $\emptyset \neq A \subseteq L_1$ and $b = \bigvee_1 A$. We claim that b is a supremum of A in L as well. Clearly, $b \in L_1$ is an upper bound of A. Assume that $c \in L$ is another upper bound of A in L. We may suppose that $c \in L_2$, for otherwise $b \leq_1 c$ yields $b \leq c$ prompt. Then for each $a \in A$ there is a $z_a \in S_1$ such that $a \leq_1 z_a$ and $\psi(z_a) \leq c$. We have $b = \bigvee_1\{a\colon a \in A\} \leq_1 \bigvee_1\{z_a\colon a \in A\} \leq_1 \bigvee_{S_1}\{z_a\colon a \in A\}$ and $\psi(\bigvee_{S_1}\{z_a\colon a \in A\}) = \bigvee_{S_2}\{\psi(z_a)\colon a \in A\} = \bigvee_2\{\psi(z_a)\colon a \in A\} \leq_2 c$, whence $b \leq c$. Therefore b is the join of A in L.

Now let $\emptyset \neq C \subseteq L$. Then $C = A_1 \cup A_2$ with $A_i \subseteq L_i$. We claim that C has a supremum in L. The case $A_2 = \emptyset$ has just been settled. If $A_1 = \emptyset$ then $\bigvee_2 A_2$ is clearly the supremum of C in L. Therefore we assume that $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Let $b_i = \bigvee_i A_i$. By the previous arguments we have $b_i = \bigvee A_i$. Consider the element $t = \bigwedge_1\{z \in S_1\colon b_1 \leq_1 z\} = \bigwedge_{S_1}\{z \in S_1\colon b_1 \leq_1 z\} \in S_1$ and let $c = \psi(t) \vee_2 b_2$. Since $b_1 \leq_1 t$, c is an upper bound of b_1 and b_2, whence it is an upper bound of C in L. Suppose $d \in L$ is another upper bound of C. Then d is an upper bound of the A_i and therefore also of the b_i, $i = 1, 2$. Hence $b_2 \leq_2 d$ and there is a $u \in S_1$ such that $b_1 \leq_1 u$ and $\psi(u) \leq_2 d$. The choice of t yields $t \leq_1 u$, whence $\psi(t) \leq_2 \psi(u)$. Consequently, $c = \psi(t) \vee_2 b_2 \leq_2 \psi(u) \vee_2 b_2 \leq_2 d$, implying $c \leq d$. I.e., $c = \bigvee C$. We have shown that each nonempty subset of L has a supremum. Since L has a least element, or using duality, it follows that L is a complete lattice. $\square$

When S_1 is a principal dual ideal of L_1 and S_2 is a principal ideal of L_2 then our construction resembles the Hall – Dilworth gluing (cf. [DH1] or [Gr1, page 31]) with the difference that we do not identify S_1 and S_2. Now we are in the position to formulate

Theorem 7.9. *([Cz14]) Let P be a finite forest, v a maximal element of P, $u \in P$, and suppose that v covers u in P. Let $L_1 := \{X \in \mathcal{L}(P)\colon\ v \notin X\}$, $L_2 := \{X \in \mathcal{L}(P)\colon\ v \in X\}$, $S_1 := \{X \in L_1\colon\ u \in X\}$, $S_2 := \{X \in L_2\colon\ u \notin X\}$, and $\psi\colon S_1 \to S_2$, $X \mapsto (X \setminus \{u\}) \cup \{v\}$. Then L_1 is a prime ideal and L_2 is a dual prime ideal of $\mathcal{L}(P)$, both L_1 and L_2 are isomorphic to $\mathcal{L}(P \setminus \{v\})$, the conditions of our lattice construction (right before Proposition 7.8) are fulfilled, and $\mathcal{L}(P)$ is exactly the lattice $L(L_1, L_2, S_1, S_2, \psi)$.*

Note that a rather special case of Theorem 7.9, when P is a chain and the lattice construction resembles the Hall – Dilworth gluing in the sense mentioned right before Theorem 7.9, implicitly occurs in [CP1]. The first conspicuous use of Theorem 7.9 is that we can easily draw the diagram of $\mathcal{L}(P)$ for a tree P, provided it has not too many elements. A more serious consequence is

Corollary 7.10. *([Cz14]) Each coalition lattice can be obtained from the two-element lattice by our lattice construction (i.e. forming lattices $L(L_1, L_2, S_1, S_2, \psi)$ from L_1 and L_2 with appropriate S_1, S_2 and ψ) and forming direct products of finitely many lattices in a finite number of steps.*

Consider the lattice Horn sentence

$$(x \wedge y = x \wedge z = y \wedge z \ \&\ x \vee y = x \vee z = y \vee z) \Longrightarrow x = y,$$

which we denote by χ. Note that χ is a nontrivial Horn sentence. Indeed, it is very easy to see that χ holds in a lattice L iff M_3, the five-element nondistributive modular lattice, cannot be embedded in L.

Corollary 7.11. *([Cz14]) χ holds in every coalition lattice. In other words, M_3 cannot be embedded in a coalition lattice.*

Now let us prove the above-mentioned results.

Proof of Theorem 7.9. It is easy to see that $L_1 = (P \setminus \{v\}]$ and $L_2 = [\{v\})$. Thus, being complementary subsets of $\mathcal{L}(P)$, L_1 is a prime ideal and L_2 is a dual prime ideal. $L_1 \cong \mathcal{L}(P \setminus \{v\})$ hardly needs any proof. To show $L_2 \cong \mathcal{L}(P \setminus \{v\})$ let us consider the map $\alpha\colon \mathcal{L}(P \setminus \{v\}) \to L_2$, $X \mapsto X \cup \{v\}$. Then α is bijective and $X \leq Y$ implies $\alpha(X) \leq \alpha(Y)$. Conversely, if $\alpha(X) \leq \alpha(Y)$ then take an extensive map β: $\alpha(X) \to \alpha(Y)$. Since v is a maximal element, $\beta(v) = v$. So the restriction of β to $X = \alpha(X) \setminus \{v\}$ is an $X \to Y$ map and $X \leq Y$ follows. Thus, $L_2 \cong \mathcal{L}(P \setminus \{v\})$.

Now we claim that, for any coalitions $A_1, \ldots, A_k \in \mathcal{L}(P)$,

$$\bigwedge_{i=1}^{k} A_i \supseteq \bigcap_{i=1}^{k} A_i. \tag{7.12}$$

Using Proposition 7.A[1], this will be shown via an induction on $|A_1| + \ldots + |A_n|$. If $\bigcap_{i=1}^{k} A_i$ is empty then there is nothing to prove. Suppose that $d \in \bigcap_{i=1}^{k} A_i$. If d is a maximal element of M given in Proposition 7.A then choosing $c = d$ we obtain $d \in \bigwedge_{i=1}^{k} A_i$. If d is not maximal in M then choose a maximal element $c \in M$ such that $d < c$. Then $d < c \leq a_i$ for the a_i occurring in Proposition 7.A. So $d \in A_i'$ and $d \in P'$. By the induction hypothesis we obtain $d \in \bigcap_{i=1}^{k} A_i' \subseteq \bigwedge_{i=1}^{k} A_i'$ and $d \in \bigwedge_{i=1}^{k} A_i$ follows from Proposition 7.A. (7.12) has been proved.

It follows instantly from (7.12) that $S_1 \subseteq L_1$ is closed under meets. Clearly, $1_{L_1} = P \setminus \{v\} \in S_1$ and $0_{L_2} = \{v\} \in S_2$. Combining (7.4) with (7.12) (or analysing the description of joins given in the proof of Theorem 7.1) we easily obtain

$$\bigvee_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{k} A_i.$$

Hence it follows that $S_2 \subseteq L_2$ is closed with respect to joins.

Now we intend to show that ψ is a lattice isomorphism. ψ is clearly bijective. First let us assume that $X \leq Y$ in S_1 and $|X| = |Y|$. By (7.7) there is an extensive α: $X \to Y$ with $\alpha(u) = u$. Clearly, $(\alpha \setminus \{\langle u, u\rangle\}) \cup \{\langle v, v\rangle\}$ is an extensive $\psi(X) \to \psi(Y)$ map, yielding $\psi(X) \leq \psi(Y)$. Now let $X, Y \in S_1$ be arbitrary with $X \leq Y$. If $u \notin \alpha(X)$ then we can replace α by $(\alpha \setminus \{\langle u, \alpha(u)\rangle\}) \cup \{\langle u, u\rangle\}$, which is also an extensive $X \to Y$ map. This way we can assume that $Y_1 = \alpha(X)$ contains u. Since $X \leq Y_1$ and $|X| = |Y_1|$, the previous argument gives $\psi(X) \leq \psi(Y_1)$ and we conclude the desired $\psi(X) \leq \psi(Y)$ from $\psi(Y_1) \leq \psi(Y)$. Hence ψ is monotone. Suppose now that $\psi(X) \leq \psi(Y)$ and let β: $\psi(X) \to \psi(Y)$ be an extensive map. Since $v \in \psi(X)$ is maximal in P, $\beta(v) = v$. Hence $(\beta \setminus \{\langle v, v\rangle\}) \cup \{\langle u, u\rangle\}$ is an extensive $X \to Y$ map, whence $X \leq Y$. Thus, ψ is an isomorphism.

What we have shown so far says that the lattice construction introduced right before Proposition 7.8 makes sense in our case. The base set of $L = L(L_1, L_2, S_1, S_2, \psi)$ and that of $\mathcal{L}(P)$ are identical, but we have to show that they possess the same partial order. Since $Z \leq \psi(Z)$ (in $\mathcal{L}(P)$) for every $Z \in S_1$, it follows easily that if $X \leq Y$ in L then $X \leq Y$ in $\mathcal{L}(P)$. The converse implication will be derived less easily.

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[1]Note that the description of joins given in the proof of Theorem 7.1 together with the dual of (7.4) could also be used.

Suppose $X \le Y$ in $\mathcal{L}(P)$; we have to show the same relation in L. Since v is a fixed point of any extensive map, $X \in L_2$ and $Y \in L_1$ is impossible. The cases $\{X, Y\} \subseteq L_1$ and $\{X, Y\} \subseteq L_2$ are trivial.

Consequently, we can assume that $X \in L_1$ and $Y \in L_2$. Let us fix an extensive map α: $X \to Y$; we have to show the existence of a $Z \in S_1$ such that $X \le Z$ and $\psi(Z) \le Y$. (Here and from now on the " $\le$ " is understood in $\mathcal{L}(P)$.)

First we deal with the case $v \notin \alpha(X)$. If $u \notin X$ then let $Z = X \cup \{u\} \ge X$ and the extensive map $\alpha \cup \{\langle v, v\rangle\}$: $\psi(Z) \to Y$ yields $\psi(Z) \le Y$. If $u \in X$ then put $Z = X$ and consider the extensive map $(\alpha \setminus \{\langle u, \alpha(u)\rangle) \cup \{\langle v, v\rangle\}$: $\psi(Z) \to Y$, which gives $\psi(Z) \le Y$.

From now on we assume that $v \in \alpha(X)$, say $\alpha(b) = v$. Since $u \prec v$, $b \ne v$, and b and u are comparable by $b, u \in (v]$, we conclude $b \le u$. If $u \notin X$ then let $Z = (X \setminus \{b\}) \cup \{u\}$; clearly $X \le Z$ and the extensive map $(\alpha \setminus \{\langle b, v\rangle\}) \cup \{\langle v, v\rangle\}$: $\psi(Z) \to Y$ yields $\psi(Z) \le Y$. Thus, we suppose that $u \in X$. We can also assume that $b = u$, for otherwise, by $b < u < v$, we could consider the extensive map

$$X \to Y, \quad x \mapsto \begin{cases} v = \alpha(b), & \text{if } x = u, \\ \alpha(u), & \text{if } x = b, \\ \alpha(x), & \text{otherwise} \end{cases}$$

instead of α. Now we put $Z = X$ and the map $(\alpha \setminus \{\langle u, v\rangle\}) \cup \{\langle v, v\rangle\}$: $\psi(Z) \to Y$ yields $\psi(Z) \le Y$. $\square$

Proof of Corollary 7.10. If $|P| = 1$ then $|\mathcal{L}(P)| = 2$ and the statement holds. Suppose $|P| > 1$ and the corollary holds for all forests with less than $|P|$ elements. If there is a pair $\langle u, v\rangle$ of elements in P such that v is a maximal element and $u \prec v$ then Theorem 7.9 applies. Otherwise P is an antichain, $X \le Y$ in $\mathcal{L}(P)$ is equivalent to $X \subseteq Y$, and $\mathcal{L}(P)$ is the $|P|$-th direct power of the two-element lattice. $\square$

Proof of Corollary 7.11. Since M_3 cannot be embedded in the two-element lattice, in virtue of Corollary 7.10 it suffices to show that this property is preserved under the lattice construction (defined before Proposition 7.8) and direct products. Suppose M_3 is embedded in a direct product $\prod_{i\in I} L_i$ but it cannot be embedded in the direct components L_i. Let π_j: $\prod_{i\in I} L_i \to L_j$ denote the jth projection. Since $\pi_j(M_3) \ncong M_3$ and M_3 is a simple lattice, $\pi_j(M_3)$ is a singleton for every $j \in I$, a contradiction. Thus, direct products preserve χ. Now suppose that M_3 is embedded in $L = L(L_1, L_2, S_1, S_2, \psi)$. Since $L = L_1 \cup L_2$ and M_3 has three atoms, there is an $i \in \{1, 2\}$ such that L_i contains at least two atoms of M_3. Since L_i is an ideal or a dual ideal of L, $M_3 \subseteq L_i$. Thus, if χ holds in L_1 and L_2 then it also holds in L. $\square$

Given a coalition lattice $\mathcal{L}(P)$, an $X \in \mathcal{L}(P)$ is called a *winning coalition* if $P \setminus X \le X$. To give a more complex picture about coalition lattices, let us cite a surprising result of Pollák even if we do not plan to use it in the sequel:

Theorem 7.D. *(Pollák [CP2]) Given a coalition lattice $\mathcal{L}(P)$, the winning coalitions form a dual ideal of $\mathcal{L}(P)$. Equivalently, there exists a winning coalition $W \in \mathcal{L}(P)$ such that, for any $X \in \mathcal{L}(P)$, X is a winning coalition iff $W \leq X$.*

We do not give a proof for this theorem here. What we will prove to close this chapter is the following two assertions.

Theorem 7.13. *(Czédli [CP2]) Every coalition lattice $\mathcal{L}(P)$ satisfies the Jordan – Hölder chain condition. I.e., any two maximal chains of $\mathcal{L}(P)$ have the same number of elements.*

Theorem 7.14. *(Czédli [CP2]) The coalition lattice $\mathcal{L}(P)$ determines the forest P up to isomorphism. In other words, if $\mathcal{L}(P) \cong \mathcal{L}(P')$ then $P \cong P'$.*

The proof of Theorem 7.13 relies on the following lemma. For $a \in P$ let $\mu(a)$ denote the cardinality of the chain $(a]$, i.e. $\mu(a) = |(a]|$. For $A \in \mathcal{L}(P)$ we define $\mu(a) = \sum_{a \in A} \mu(a)$. To avoid confusion, the elements of P resp. $\mathcal{L}(P)$ will be denoted by lower-case resp. capital letters.

Lemma 7.15. *Let $A, B \in \mathcal{L}(P)$. Then*

$$A < B \iff (A \leq B \;\&\; \mu(A) < \mu(B)). \tag{7.16}$$

and

$$A \prec B \iff (A \leq B \;\&\; \mu(A) + 1 = \mu(B)). \tag{7.17}$$

Proof. Suppose $A < B$ and choose an extensive map α: $A \to B$. Then

$$\mu(A) = \sum_{a \in A} \mu(a) \leq \sum_{a \in A} \mu(\alpha(a)) \leq \sum_{b \in B} \mu(b) = \mu(B).$$

If both inequalities in the above formula were equations then $(\forall a)(a \leq \alpha(a))$ and $\alpha(A) = B$ would imply $A = B$, a contradiction. Hence $\mu(A) < \mu(B)$. The converse direction of (7.16) is evident. The "$\Longleftarrow$" direction of (7.17) follows from (7.16). To show the "$\Longrightarrow$" direction of (7.17) let us assume that $A \prec B$. We have to distinguish two cases.

Case (i): $|A| < |B|$. Let $\{b_1, b_2, \dots, b_k\} = B \setminus A$. Since $A < A \cup \{b_1\} < A \cup \{b_1, b_2\} < \dots < A \cup \{b_1, b_2, \dots, b_k\} = B$, we conclude $k = 1$. Let z denote the smallest element in the chain $(b_1]$. If z belonged to A then $A < (A \setminus \{z\}) \cup \{b_1\} < A \cup \{b_1\} = B$ would contradict $A \prec B$. Hence $z \notin A$. The assumption $z < b_1$ would lead to $A < A \cup \{z\} < A \cup \{b_1\} = B$, another contradiction. Thus, $b_1 = z$ and $\mu(B) = \mu(A) + \mu(z) = \mu(A) + 1$, indeed.

Case (ii): $|A| = |B|$. Then we have an extensive bijection α: $A \to B$. The set $H = \{x \in A\colon x < \alpha(x)\}$ cannot be empty, for otherwise $A = \alpha(A) = B$ would follow. Let u be a minimal element of H and denote $\alpha(u)$ by v. We claim $u \notin B$. Indeed, otherwise $u = \alpha(y)$ would hold for some $y \in A$, the minimality of u would imply $y = u$, and $u = \alpha(y) = \alpha(u) = v$ would contradict $u < v$. Let $A_1 = A \setminus \{u\}$ and $B_1 = B \setminus \{v\}$. Since $u \notin B = \alpha(A)$, $(\alpha \setminus \{\langle u, v\rangle\}) \cup \{\langle u, u\rangle\}$: $A_1 \cup \{u\} \to B_1 \cup \{u\}$ is an extensive map. Hence $A = A_1 \cup \{u\} \leq B_1 \cup \{u\} < B_1 \cup \{v\} = B$ yields $A_1 \cup \{u\} = B_1 \cup \{u\}$, whence $A_1 = B_1$. The extensive map $\alpha_1 = \alpha \setminus \{\langle u, v\rangle\}$: $A_1 \to B_1$ must be the identical map, for otherwise

$$\mu(A_1) = \sum_{a \in A_1} \mu(a) < \sum_{a \in A_1} \mu(\alpha(a)) = \sum_{b \in B_1} \mu(b) = \mu(B_1)$$

would contradict $A_1 = B_1$. Since $\mu(B) - \mu(A) = \mu(v) - \mu(u)$, it suffices to show that $u \prec v$. Suppose this is not the case, i.e. $u < c < v$ holds for some $c \in P$. If $c \notin A_1$ then $A = A_1 \cup \{u\} < A_1 \cup \{c\} < A_1 \cup \{v\} = B_1 \cup \{v\} = B$ is a contradiction, so $c \in A_1$. Denoting $A_1 \setminus \{c\} = B_1 \setminus \{c\}$ by D we have $A = D \cup \{u, c\}$, $B = D \cup \{c, v\}$, and $A < D \cup \{u, v\} < B$ is a contradiction again. □

Proof of Theorem 7.13. Let $\emptyset = C_0 \prec C_1 \prec C_2 \prec \ldots \prec C_t = P$ be a maximal chain in $\mathcal{L}(P)$. We infer from Lemma 7.15 that $\mu(P) = \mu(C_t) = \mu(C_{t-1}) + 1 = \mu(C_{t-2}) + 2 = \ldots = \mu(C_0) + t = t$, whence every maximal chain has $\mu(P) + 1$ elements. □

Proof of Theorem 7.14. Let $\mathcal{S} = \mathcal{S}(\mathcal{L}(P))$ denote the set of singleton coalitions in $\mathcal{L}(P)$, i.e., $\mathcal{S} = \{X \in \mathcal{L}(P)\colon |X| = 1\}$. For $a, b \in P$, $a \leq b$ in P iff $\{a\} \leq \{b\}$ in $\mathcal{L}(P)$. Therefore it suffices to describe $\mathcal{S}$ in a lattice theoretic language, i.e. in a way which is invariant under lattice isomorphisms; the theorem then will follow. Unfortunately, this description is not always possible. For example, if P is the three-element chain $\{0 < a < b\}$ then $\mathcal{L}(P)$ has an automorphism interchanging $\{a, 0\}$ and $\{b\}$, and the same can be said when one of the tree components of P is a three-element chain. That is why we deal with trees before settling the general case.

From now on let P be a tree. This property of P can be recognized from $\mathcal{L}(P)$ since it is easy to derive from Proposition 7.B that P is a tree iff $\mathcal{L}(P)$ has exactly one atom. Note that the only atom of $\mathcal{L}(P)$ is $\{0\}$ where 0 is the smallest element of the tree P. A coalition $X \in \mathcal{L}(P)$ is called a *cycle* if the principal ideal $(X]$ is a chain in $\mathcal{L}(P)$. All singleton coalitions are cycles but not conversely. For a cycle X, distinct from the empty coalition, let X^- denote the unique coalition covered by X in $\mathcal{L}(P)$. Let $\mathcal{C}$ denote the set of cycles in $\mathcal{L}(P)$. For a coalition $X \in \mathcal{L}(P)$ let $h(X)$ denote the *height* of X, i.e. the length of any maximal chain from $\emptyset$ to X. Note that X is a cycle iff $|(X]| = h(X) + 1$. Now we define several subsets of

$\mathcal{L}(P)$ as follows:

$$\begin{aligned}
\mathcal{A} &= \{X \in \mathcal{C}\colon\ h(X) = 2\},\\
\mathcal{B} &= \{X \in \mathcal{C}\colon\ h(X) \geq 4\},\\
\mathcal{T}_1 &= \{X \in \mathcal{C}\colon\ h(X) = 3 \text{ and } X < Y \text{ for some } Y \in \mathcal{B}\},\\
\mathcal{T}_2 &= \{X \in \mathcal{C}\colon\ h(X) = 3 \text{ and there is a } Z \in \mathcal{A}\\
&\qquad \text{such that } X^- \parallel Z \text{ and } |(X \vee Z]| \geq 8\}, \text{ and}\\
\mathcal{T}_3 &= \{X \in \mathcal{C}\colon\ h(X) = 3 \text{ and there is a } Y \in \mathcal{C}\\
&\qquad \text{such that } X \neq Y,\ X^- = Y^- \text{and } |(X \vee Y]| \geq 8\}.
\end{aligned}$$

Let

$$\mathcal{R} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \{\{0\}\}.$$

Here $\{0\}$ is, of course, the unique atom of $\mathcal{L}(P)$. We claim that

(7.18) If $\mathcal{L}(P)$ is not distributive then $\mathcal{S} = \mathcal{R}$.

First we show $\mathcal{S} \subseteq \mathcal{R}$. Let g denote the height function on P. I.e., with μ defined in the previous proof, $g(a) = |(a]| - 1 = \mu(a) - 1$ for $a \in P$. Clearly, $h(\{a\}) = g(a) + 1$. Therefore $\{a\} \in \mathcal{R}$ for every $a \in P$ with $g(a) \neq 2$. Now assume that $g(a) = 2$. If a is not a maximal element in P then $\{a\} \in \mathcal{T}_1 \subseteq \mathcal{R}$. Therefore we can assume that a is a maximal element of P. Let b be the unique lower cover of a, i.e. $b \prec a$.

Firstly, assume that a is the only element of P which covers b. Since $\mathcal{L}(P)$ is not distributive, P is not a chain by Proposition 7.C. Hence $P \setminus (a] \neq \emptyset$. Let c be a minimal element of $P \setminus (a]$. Denoting $\{a\}$, $\{b\}$ and $\{c\}$ by X, X^- and Z, respectively, we obtain $\{a\} \in \mathcal{T}_2$, for $(X \vee Z] = (\{a, c\}]$ contains $\emptyset$, $\{0\}$, $\{b\}$, $\{c\}$, $\{a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, a\}$, $\{b, c\}$, $\{a, c\}$, i.e. more than eight distinct coalitions.

Secondly, assume that $\{a = a_1, a_2, \dots, a_k\}$ is the set of elements covering b, $k \geq 2$. Putting $X = \{a\}$ and $Y = \{a_2\}$ we see that $\{a\} \in \mathcal{T}_3$, for the coalitions $\emptyset$, $\{0\}$, $\{b\}$, $\{a\}$, $\{a_2\}$, $\{0, b\}$, $\{0, a\}$, $\{0, a_2\}$ all belong to $(X \vee Y] = \{a, a_2\}$. We have shown $\mathcal{S} \subseteq \mathcal{R}$.

As a first step towards the converse inclusion in (7.18) we claim

(7.19) $X \in (\mathcal{L}(P) \setminus\ \mathcal{S}) \cap \mathcal{C} \Longrightarrow X = \{0, b\}$ for some $0 \prec b$.

Let $X \in \mathcal{L}(P) \setminus \mathcal{S}$ be a cycle. If $|X| \geq 3$ then, for any maximal element u of X, $\{u\} \parallel X \setminus \{u\}$, contradicting the fact that $(X]$ is a chain. Therefore $|X| = 2$. Let $X = \{a, b\}$. From $\{a\}, \{b\} \in (X]$ we infer that a and b are comparable, so we assume $0 \leq a < b$. If $0 < a < b$ then $\{0, a\} \parallel \{b\}$ in $(X]$, a contradiction. Hence $X = \{0, b\}$. If $0 < c < b$ for some $c \in P$ then $\{0, c\} \parallel \{b\}$ in $(X]$, a contradiction again. Therefore $0 \prec b$, proving (7.19).

For $0 \prec b$ we have $h(\{0, b\}) = 3$. This fact and (7.19) clearly yield $\mathcal{A} \cup B \cup \{\{0\}\} \subseteq \mathcal{S}$. Hence, by $\mathcal{B} \subseteq \mathcal{S}$, $\mathcal{T}_1 \subseteq \mathcal{S}$ follows immediately. Suppose $X \in \mathcal{T}_2 \setminus \mathcal{S}$.

By (7.19), $X = \{0, b\}$ for some $0 \prec b$. We have $X^- = \{b\}$, $Z = \{a\}$ from $\mathcal{A} \subseteq \mathcal{S}$, $a \parallel b$ and, by $h(Z) = 2$, $0 \prec a$. Since $X \vee Z = \{a, b\}$, $(X \vee Z] \cong \mathcal{L}(Q) \setminus \{Q\}$ where Q is $\{0, a, b\}$, as a sub-poset of P. Hence $|(X \vee Z]| = 2^3 - 1 = 7$, contradicting $X \in \mathcal{T}_2$. Thus, $\mathcal{T}_2 \subseteq \mathcal{S}$.

Suppose $X \in \mathcal{T}_3 \setminus \mathcal{S}$. As previously, $X = \{0, b\}$ and $X^- = \{b\} = Y^-$ for some $0 \prec b$. Now Y is a singleton, for otherwise $h(Y) = h(Y^-) + 1 = 3$ and (7.19) would imply $Y = \{0, b\} = X$, a contradiction. Therefore $Y = \{a\}$ for some $b \prec a$. We have $X \vee Y = \{0, a\}$. Using $Q = \{0, a, b\}$ as before we can derive $|(X \vee Y]| = 2^3 - 1 = 7$. This contradiction shows $\mathcal{T}_3 \subseteq \mathcal{S}$. This proves $\mathcal{R} \subseteq \mathcal{S}$ and (7.18).

Now let us assume first that $\mathcal{L}(P)$ has only one atom, i.e. P is a tree. If $\mathcal{L}(P)$ is distributive then P is a chain by Proposition 7.C. Since the chain P is determined by $|P|$ and $|P|$ uniquely comes from $2^{|P|} = |\mathcal{L}(P)|$, this case is settled. If $\mathcal{L}(P)$ is not distributive then $P \cong \mathcal{S}$ is determined up to isomorphism by (7.18).

Secondly let us assume that $\mathcal{L}(P)$ has more than one atom. Then, by Proposition 7.B,

$$\mathcal{L}(P) \cong \prod_{i=1}^{k} \mathcal{L}(T_i), \tag{7.20}$$

where the T_i are the tree component of P. But, as we mentioned before, the $\mathcal{L}(T_i)$ are directly indecomposable. It is known, cf. Grätzer [Gr1, p. 153, Cor. III.4.4] that if we decompose a finite lattice as a direct product of directly indecomposable factors then these factors are uniquely determined up to isomorphism. Applying this to (7.20) we infer that the $\mathcal{L}(T_i)$ are determined up to isomorphism. But any of them has only one atom. Consequently, by the previous part of the proof, they determine the T_i, i.e. the tree components, and therefore the whole P, up to isomorphism. $\square$

VIII. Summary of methods and applicability of results

This brief last chapter is to give an account on the methods used in this work and to survey some expected applications of the results.

The methods have already been detailed in the corresponding chapters; now only some general features of them will be mentioned. Chapters II, III, IV and V are more or less connected with our former research, so some results from [Cz16] and from our papers prior to [Cz16] could serve here as methods. (Of course, results and methods from the literature have also been used.) Another method in this work is the intensive use of computers. Although some grandiose program packages like MAPLE, GAP or MATHEMATICA are occasionally used for algebraic investigations, it is not very frequent that a dissertation in universal algebra and lattice theory contains, on an enclosed floppy diskette, two distinct computer programs developed by the author.

The new topics developed in Chapters VI and VII seem to be promising for future investigations. The interest for them has already started. For example, "three" will hopefully be changed to "four" in the title of [CP2] with adopting a third coauthor in the near future. The methods and, in some sense, a part of the results of Chapter V will perhaps be extended to the case of zero characteristic in the future. Some day, maybe, the stipulation " $\Sigma \models$ modularity " can be removed from Theorem 2.1 even if this does not seem to be easy.

Haiman [Ha1] gives a "procedure" to show that a given lattice identity holds in all linear lattices. But this is not an algorithm, for we cannot prove this way that an identity *fails* in some linear lattice. But even if the identity holds in all linear lattices, it would be impossible to find a Haiman type proof by enumerating all possible manipulations in the practice. However, as it has appeared recently, if one has a proof showing that a lattice identity λ holds in all submodule lattices or all congruence varieties of congruence permutable varieties then this proof sometimes can show the way how to find a Haiman type proof; (4.3) can be useful in this aspect.

The above argument already indicates that the computer programs we have developed can contribute to further investigations.

CHAPTER IX

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CONTENTS

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX