(1+1+2)-generated equivalence lattices

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Abstract. Strietz [6, 7] and Zádori [10] have shown that Equ(A), the lattice of all equivalences of a finite set A with $|A| \geq 7$, has a four-element generating set such that exactly two of the generators are comparable. In other words, these lattices are $(1+1+2)$-generated. We extend this result for many infinite sets A; even for all sets if there are no inaccessible cardinals. Namely, we prove that if A is a set consisting of at least seven elements and there is no inaccessible cardinal $\leq |A|$, then the *complete* lattice Equ(A) is $(1+1+2)$-generated. This result is sharp in the sense that Equ(A) has neither a three element generating set nor a four-element generating set with more than one pair of comparable generators.

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I. The main result

Given a set A, let Equ(A) denote the (complete) lattice of all equivalences of A. If A is finite, then Equ(A) can be generated by four elements, but three elements are insufficient for $|A| \geq 4$, cf. Strietz [6, 7] and Zádori [10]. Zádori [10] has also shown that Equ(A) is $(1+1+2)$-generated if $|A| \geq 7$ (cf. also Strietz [7] for $|A| \geq 10$), i.e., we can assume that exactly two of the four generators are comparable.

Our aim is to extend the above result on $(1+1+2)$-generation for some infinite sets A. We will consider Equ(A) as a *complete* lattice, for otherwise it would not be finitely generated. A subset Q of Equ(A) is said to generate Equ(A) if no proper complete sublattice of Equ(A) includes Q.

The investigations of this and similar kind started with [1], where the "border" between finite and countable sets was relatively easy to step over. The lattice Equ(A) was shown to be four-generated for "large" sets A in [2], and it was shown to be $(1+1+2)$-generated for countable sets A in [3]; the techniques of these papers will intensively be used. Without Takách's idea of defining boxes by means of semiboxes, cf. [8], the present proof would be at least twice as long.

As usual, $\aleph_0$ denotes the smallest infinite cardinal. A cardinal m is called *inaccessible* if it satisfies the following three conditions: (i) $m > \aleph_0$; (ii) $n < m$ implies $2^n < m$; and (iii) if I is a set of cardinals such that $|I| < m$ and $n < m$ for all $n \in I$, then $\sup\{n\colon n \in I\} < m$. Note that $\sup\{n\colon n \in I\}$ in (iii) can be replaced by $\sum_{n\in I} n$. For details on inaccessible cardinals the reader can resort to standard textbooks, e.g., to Levy [5, pages 138–141]. By Kuratowski's result [4] (cf. also [5]), ZFC has a model without inaccessible cardinals. Hence the existence of inaccessible cardinals cannot be proved from ZFC, and the scope of the following theorem includes all sets in an appropriate model of set theory.

Theorem 1. *Let A be a set with at least seven elements, and suppose that there is no inaccessible cardinal m such that $m \leq |A|$. Then the complete lattice* $\mathrm{Equ}(A)$ *of all equivalences of A has a four-element generating set of order type* $1{+}1{+}2$ *Moreover,* $\mathrm{Equ}(A)$ *can be generated by a subset* $\{\alpha, \beta, \gamma, \delta\} \subseteq \mathrm{Equ}(A)$ *such that* $\{\alpha, \beta, \gamma\}$ *and* $\{\delta, \beta, \gamma\}$ *are antichains,* $\delta < \alpha$*, and* α *does not cover* δ.

Corollary 2. *If $|A| \geq 7$ and there is no inaccessible cardinal m such that $m \leq |A|$ then* $\mathrm{Equ}(A)$ *can be generated by a four-element antichain.*

It is worth mentioning that, for $|A| > 3$, every at most four-element generating set of $\mathrm{Equ}(A)$ is either a four-element antichain or a four-element subset of type 1+1+2. For finite sets A this was proved by Strietz [7]. Parsing Strietz's argument, it is easy to observe that it works for all sets A. (For example, $\mathrm{Equ}(A)$ is simple for any set A, and Wille's D_2-Lemma in [9] remains valid for generating sets of *complete* lattices.)

While Corollary 2 follows from Theorem 1 and from the fact that $\mathrm{Equ}(A)$ is relatively complemented, the rest of the paper is devoted to the proof of the theorem.

II. Semiboxes and their extensions

By Zádori [10], it is sufficient to prove the result only for infinite sets. So even if we start the proof at some finite sets, we do not have to deal (and will not deal) with all finite sets. Basically, the proof is an induction on $|A|$. However, the mere assumption of the statement for a given cardinal is far from being a suitable induction hypothesis. Therefore we have to build a structure on A and study these structures to the necessary extent. Before developing the necessary terminology, we give an example.

Let $L = \{a_0, a_1, \ldots, a_{29}, b_0, b_1, \ldots, b_{28}\}$. For $p, q \in L$ (or p, q in any set), let $\langle p, q\rangle$ denote the smallest equivalence collapsing p and q. Note that $\langle p, q\rangle = \langle q, p\rangle$ is an atom in $\mathrm{Equ}(L)$ if $p \neq q$, and $\langle p, p\rangle = 0 \in \mathrm{Equ}(L)$. If $x \in L$ and $\Theta \in \mathrm{Equ}(L)$, then the Θ-class containing x will be denoted by $[x]\Theta$. Denoting the lattice operations by $\sum$ or $+$ (join) and $\prod$ or $\cdot$ (meet), we let

$$\alpha = \sum_{i=0}^{28} \langle a_i, a_{i+1}\rangle + \sum_{i=0}^{27} \langle b_i, b_{i+1}\rangle, \qquad \beta = \sum_{i=0}^{28} \langle a_i, b_i\rangle$$

$$\gamma = \sum_{i=0}^{28} \langle b_i, a_{i+1}\rangle, \qquad \delta = \langle a_0, a_{29}\rangle + \langle b_0, b_{28}\rangle.$$

These equivalences are represented by horizontal, vertical and oblique lines, and (dotted) arcs in Figure 1 and in the rest of the figures, respectively. Zádori has shown that $\{\alpha, \beta, \gamma, \delta\}$ generates $\mathrm{Equ}(L)$; this will also follow from our proof. The elements $u = a_0$ and $v = b_0$ will be treated as constants. For $i = 0, \ldots, 4$, the 10-tuple

$$s^{(i)} = (a_{5i+2}, a_{5i+3}, \ldots, a_{5i+6}, b_{5i+2}, b_{5i+3}, \ldots, b_{5i+6})$$

is called a *switch*. Let $E = \{s^{(i)} : 0 \le i \le 4\}$ denote the set of switches. From now on L is considered as the structure $L = (L, u, v, E, \alpha, \beta, \gamma, \delta)$. This structure is outlined in Figure 2, where the switches are represented by shaded rectangles.

L is just a particular case of a more general structure, which we introduce under the name "semibox".

Definition 3. By a *semibox* we mean a structure

$$A = (A, u, v, E, \alpha, \beta, \gamma, \delta)$$

provided A is a set, $u, v \in A$ are distinct constants, $\alpha, \beta, \gamma, \delta \in \mathrm{Equ}(A)$, $E \subseteq (A \setminus \{u, v\})^{10}$, each $s \in E$ has ten distinct components, and for any two distinct $s, f \in E$ the set of components of s is disjoint from the set of components of f.

For a semibox $A = (A, u, v, E, \alpha, \beta, \gamma, \delta)$ the elements of E are called switches. For $s \in E$ we will use the notation

$$s = \big(a(s), b(s), c(s), d(s), e(s), a'(s), b'(s), c'(s), d'(s), e'(s)\big),$$

or shortly $s = (a, b, c, d, e, a', b', c', d', e')$. Although the notion of semiboxes does not require any connection between switches and α, β, γ, δ, the switch depicted in Figure 3, which has already occurred in L, will be typical. The idea behind the notion of a semibox is that, under suitable additional conditions, $\{\alpha, \beta, \gamma, \delta\}$ will generate $\mathrm{Equ}(A)$. Let $A_i = (A_i, u_i, v_i, E_i, \alpha_i, \beta_i, \gamma_i, \delta_i)$ be semiboxes for $i = 1, 2$. A bijective map $\varphi\colon A_1 \to A_2$ is called an *isomorphism* if $\varphi(u_1) = u_2$, $\varphi(v_1) = v_2$, $\varphi(\alpha_1) = \{(\varphi(x), \varphi(y))\colon (x, y) \in \alpha_1\} = \alpha_2$, $\varphi(\beta_1) = \beta_2$, $\varphi(\gamma_1) = \gamma_2$, $\varphi(\delta_1) = \delta_2$, and $\varphi(E_1) = \{\big(\varphi(x_1), \ldots, \varphi(x_{10})\big) : (x_1, \ldots, x_{10}) \in E_1\} = E_2$.

Since we want to create large semiboxes from smaller ones, we introduce a concept that expresses how the small semiboxes can be put together. First we give a rigourous definition, then an intuitive one, and finally an example; the reader may want to read them simultaneously.

Definition 4. Suppose $A_0 = (A_0, u_0, v_0, E_0, \alpha_0, \beta_0, \gamma_0, \delta_0)$ and $B = (B, u, v, E, \alpha, \beta, \gamma, \delta)$ are semiboxes and Γ is a partition on the set B. Let A_i $(i \in I)$ denote the classes of this partition. We assume that $0 \in I$, so the support of the semibox A_0 is one of the classes. For $i \in I \setminus \{0\}$, let $\varphi_i\colon A_0 \to A_i$ be a bijection, and define $u_i = \varphi_i(u_0)$ and $v_i = \varphi_i(v_0)$. For $s \in E_0$ let

$$s_i = \varphi_i(s) = \Big(\varphi_i\big(a(s)\big), \varphi_i\big(b(s)\big), \ldots, \varphi_i\big(d'(s)\big), \varphi_i\big(e'(s)\big)\Big),$$

define $E_i = \varphi_i(E_0) = \{s_i\colon s \in E_0\}$, and let $\alpha_i = \varphi_i(\alpha_0)$, $\beta_i = \varphi_i(\beta_0)$, $\gamma_i = \varphi_i(\gamma_0)$, and $\delta_i = \varphi_i(\delta_0)$. Let φ_0 denote the identity automorphism of A_0. Then $A_i = (A_i, u_i, v_i, E_i, \alpha_i, \beta_i, \gamma_i, \delta_i)$ is a semibox isomorphic to A_0.

Let us assume that

$$E \subseteq \bigcup_{i \in I} E_i,$$

and, further, there exist $F, G \subseteq E_0 \times I$ and $H \subseteq E_0 \times I \times I$ such that $(s,i) \in F \cup G$ implies $\{(s,i,j),\ (s,j,i)\} \cap H = \emptyset$ for all $j \in I$, $F \cap G = \emptyset$, $(s,i,j) \in H$ implies $i \neq j$, $(E \times I) \cap (F \cup G) = \emptyset$, $(E \times I \times I) \cap H = \emptyset$, and

$$\alpha = \bigcup_{i \in I} \alpha_i,$$
$$\beta = \bigcup_{i \in I} \beta_i + \sum_{(s,i,j) \in H} \langle b(s_i), b(s_j) \rangle,$$
$$\gamma = \bigcup_{i \in I} \gamma_i + \sum_{(s,i,j) \in H} \langle d(s_i), d(s_j) \rangle$$
$$\delta = \bigcup_{i \in I} \delta_i + \sum_{(s,i) \in F} \langle b(s_i), d(s_i) \rangle + \sum_{(s,i) \in G} \langle b'(s_i), d'(s_i) \rangle.$$

Further, let us assume that $u = u_0$, $v = v_0$. Then B is called an *extension* of A_0, in notation $B \mid A_0$. The A_i $(i \in I)$ are sometimes called the *canonical copies* of A_0 in the extension. Let $\Phi = (\Gamma, \{\varphi_i\colon\ i \in I\})$; it is called the *way of the extension*. Sometimes, when Φ is relevant, we say that B *is an extension of* A_0 *by* Φ, in notation $B \mid_\Phi A_0$. For $E_0' \subseteq E_0$, $\bigcup_{i \in I} E_i' = \{s_i\colon\ i \in I,\ s \in E_0'\}$ is called the *extension* of E_0' to B (by Φ). If the extension of E_0' $(\subseteq E_0)$ is included in E, then the semibox extension $B \mid A_0$ is called E_0'*-preserving*. By the *degree* of the semibox extension $B \mid_\Phi A_0$ we mean $|I|$; the degree is denoted by $[B : A_0]$. (We will use this notation only when the meaning of Φ — at least implicitly — is already given. Note that $|B| = [B : A_0] \cdot |A_0|$.)

It is reasonable to describe less formally what a semibox extension means.

Definition 5. Given a switch s, the atoms $\langle b(s), d(s) \rangle$ and $\langle b'(s), d'(s) \rangle$ are called the upper atom and the lower atom associated with s, respectively. These atoms will be used to enlarge δ. On figures we use small arcs attached to shaded rectangles to indicate that the upper or lower atom of a switch is used this way, cf. Figure 4. Now, we obtain an extension B of the semibox A_0 as follows.

- We take disjoint copies of A_0. These copies are denoted by A_i $(i \in I)$, B is defined to be their union, and, at the beginning, α, β, γ and δ are defined as the union of the α_i, β_i, γ_i and δ_i $(i \in I)$, respectively. We assume that $0 \in I$, and we define $u = u_0$, $v = v_0$.
- We use some switches to make a distinction among these (originally isomorphic) copies such that either the upper or the lower atom (but not both) associated with a given "distinguishing" switch is joined to δ. The switches we use here cannot be used in the sequel.
- We use pairs of switches to connect distinct copies of A_i. If $s_i = \varphi_i(s) \in E_i$ and $s_j = \varphi_j(s) \in E_j$ are canonical copies of the same switch $s \in E_0$ and $i \neq j$, then we may connect these two switches by joining the atom $\langle b(s_i), b(s_j) \rangle$ to β and joining the atom $\langle d(s_i), d(s_j) \rangle$ to γ.
- E is a subset of $\bigcup_{i \in I} E_i$, but E cannot contain switches that we previously used to differentiate or connect.

The connection of two switches is depicted in Figure 5. Notice that, instead of $\langle d(s_i), d(s_j)\rangle$, Figures 5 and 6 indicate the atom $\langle c'(s_i), d(s_j)\rangle$. Joining this atom to γ gives the same as joining $\langle d(s_i), d(s_j)\rangle$. The advantage to draw the edge representing $\langle c'(s_i), d(s_j)\rangle$ is that it is oblique, as all γ-edges have to be, and it makes the figure less crowded.

Now we present an example. If $A_0 = L$ is the semibox defined by Figures 1 and 2, $I = \{0 = U_0 = \emptyset, U_1, U_2, U_3\}$, $H = \{(s^{(0)}, U_i, U_{i+1}) : 0 \le i \le 2\}$, $E = \{s_i^{(3)} : i \in I\} \cup \{s_i^{(4)} : i \in I\}$, and F and G are appropriately chosen, then an extension B of A_0 is depicted in Figure 6. Notice that $A_0 \subseteq B$ but $E_0 \not\subseteq E$.

Now we formulate some kind of transitivity for extensions. Let $B_0 = (B_0, \bar u_0, \bar v_0, \bar E_0, \bar\alpha_0, \bar\beta_0, \bar\gamma_0, \bar\delta_0)$ be an extension of A_0 by $\Phi = (\Gamma, \{\varphi_i\colon\ i \in I\})$. Further, let $C = (C, u, v, E, \alpha, \beta, \gamma, \delta)$ be an extension of B_0 by $\Psi = (\Delta, \{\psi_j\colon\ j \in J\})$. I.e., $\Delta = \{B_j\colon\ j \in J\}$ is a partition of C, $\psi_j\colon\ B_0 \to B_j$ is a bijection ($j \in J$), ψ_0 is the identical map of B_0, $\bar u_j := \psi_j(\bar u_0)$, $\bar v_j := \psi_j(\bar v_0)$, $\bar\alpha_j := \psi_j(\bar\alpha_0)$, etc. For $i \in I$ and $j \in J$, let $A_{j,i} = \psi_j(\varphi_i(A_0)) = (\psi_j \circ \varphi_i)(A_0)$, and define $\varrho_{j,i} = \psi_j \circ \varphi_i$, cf. Figure 7. Then the $A_{j,i}$ $\quad((j,i) \in J \times I)$ form a partition of C, which we denote by $\Delta \circ \Gamma$. Let us identify $(0,0)$ with 0. Then we obtain $\Psi \circ \Phi := (\Delta \circ \Gamma, \{\varrho_{j,i}\colon\ (j,i) \in J \times I\})$, which we call the *composition of* Ψ *and* Φ. With the above notations we have

Claim 6. *Suppose A_0, B_0 and C are semiboxes, $B_0 \mid A_0$ and $C \mid B_0$. Then*

(i) if $B_0 \mid_\Phi A_0$ and $C \mid_\Psi B_0$, then $C \mid_{\Psi\circ\Phi} A_0$;

(ii) $[C : A_0] = [C : B_0] \cdot [B_0 : A_0]$;

(iii) if $E_0' \subseteq E_0$, $B_0 \mid_\Phi A_0$ is an E_0'-preserving extension, the extension of E_0' to B_0 is denoted by $\bar E_0'$, and $C \mid_\Psi B_0$ is an $\bar E_0'$-preserving extension, then $C \mid_{\Psi\circ\Phi} A_0$ is an E_0'-preserving extension.

We do not know if $\Psi \circ \Phi$ is the only way of extension $C \mid A_0$, but permitting other ways would cause trouble in the sequel.

Proof. It suffices to show (i); the rest will follow as an evident consequence. Fortunately, (i) is more or less clear even without heavy formalism. Indeed, C is the disjoint union of $A_{j,i}$, $(j,i) \in J \times I$, and $\alpha = \bigcup_{(j,i)\in J\times I} \alpha_{j,i}$. The distinguishing and connecting switches that we used to obtain B_j do not belong to $\bar E_j$, whence no switch is used twice. This ensures that C is obtained from A_0 according to Definition 5. $\diamond$

Definition 7. Extensions of the semibox L given by Figures 1 and 2 are called *boxes*.

For example, L, being a trivial extension of itself, is a box. Every box is a semibox, but not conversely. It is clear from Claim 6 that if A is a box then all semibox extensions of A are boxes. When dealing with boxes, the geometrical arrangement of their elements, as suggested by Figures 1 and 2, will be useful. Suppose A is a box, say $A \mid_\Phi L$, where $\Phi = (\Gamma, \{\varphi_i\colon\ i \in I\})$. The canonical copies $\varphi_i(L)$ of L ($i \in I$) will be called the *ladders* of A. Each ladder $\varphi_i(L)$ consists of two *rows*; $\varphi_i(\{a_0, \dots, a_{29}\})$ is the *upper row*, while $\varphi_i(\{b_0, \dots, b_{28}\})$ is the *lower row*. For $0 \le k \le 28$, $\{\varphi_i(a_k) : i \in I\} \cup \{\varphi_i(b_k) : i \in I\}$ is called the k-th *column*, and $\{\varphi_i(a_{29}) : i \in I\}$ is the 29th column. We write $\mathrm{col}(x) = k$ if $x \in A$ belongs to the k-th column. Given a fixed switch s of L, the a-column (of s) or, to be more precise, the $a(s)$-column is defined to be $\{\varphi_i(a(s)) : i \in I\} \cup \{\varphi_i(a'(s)) : i \in I\}$; the meaning of b-columns, ..., e-columns are analogous. An equivalence $\Theta \in \mathrm{Equ}(A)$ is said to be *ladder preserving*, *row preserving*, *row changing*, *column preserving*, *column changing*, *column k-changing*, and *column 1-preserving* if for all $(x, y) \in \Theta$ with $x \neq y$ the elements x and y belong to the same ladder, they belong to the same row, they belong to distinct rows, they belong to the same column, they belong to distinct columns, $|\mathrm{col}(x) - \mathrm{col}(y)| = k$, and $|\mathrm{col}(x) - \mathrm{col}(y)| \leq 1$, respectively. For $X \subseteq A$, X is said to be *closed* with respect to Θ if $[x]\Theta \subseteq X$ holds for every $x \in X$. For example, in case of L, γ is row changing and column 1-changing, while α is row preserving, i.e., both rows are closed with respect to α. Many geometric properties of L are inherited by all boxes.

Lemma 8. *Let $A = (A, u, v, E, \alpha, \beta, \gamma, \delta)$ be a box.*

(1) α and δ are row preserving and column changing, β is column preserving, γ is column 1-preserving, and β and γ are row changing.

(2) $\alpha\beta = \alpha\gamma = \beta\gamma = 0$, $\delta < \alpha$, $\{\alpha, \beta, \gamma\}$ and $\{\delta, \beta, \gamma\}$ are antichains, and α does not cover δ.

(3) If $x \in A$ and $[x]\beta$ is a singleton, then $\mathrm{col}(x) = 29$. If $|[x]\beta| > 2$ or $[x]\beta$ is not included in a ladder then there is an s in L with $\mathrm{col}(x) = \mathrm{col}(b(s))$ and $s \notin E$.

Proof. (1) and (3) follow from definitions. (2) is a straightforward consequence of (1), (3) and the definitions. ◇

Now let $B \mid_\Phi A_0$, and let us use the notations of Definition 4. For $p, q \in A_0$ we introduce the notation

$$\langle p, q\rangle^{(A_0,B)} = \langle p, q\rangle^{(A_0,B,\Phi)} = \sum_{i \in I} \langle \varphi_i(p), \varphi_i(q)\rangle \in \mathrm{Equ}(B).$$

For $i \in I$, we will also use the notations $\langle \varphi_i(p), \varphi_i(q)\rangle^{(A_0,B,\Phi)} := \langle p, q\rangle^{(A_0,B,\Phi)}$ and $\langle \varphi_i(p), \varphi_i(q)\rangle^{(A_0,B)} := \langle p, q\rangle^{(A_0,B)}$. I.e., for $x, y \in A_i$ we define $\langle x, y\rangle^{(A_0,B,\Phi)}$ as

$$\langle x, y\rangle^{(A_0,B,\Phi)} = \langle \varphi_i^{-1}(x), \varphi_i^{-1}(y)\rangle^{(A_0,B,\Phi)}.$$

Usually we drop Φ from these notations but we must be careful: always a fixed Φ should be understood when it is not indicated.

Definition 9. A box A is called a *good box* if, for every extension $B = (B, u, v, E, \alpha, \beta, \gamma, \delta)$ of A, say $B \mid_\Phi A$, and every complete sublattice Q of $\mathrm{Equ}(B)$ with $\{\alpha, \beta, \gamma, \delta\} \subseteq Q$, $\langle p, q\rangle^{(A,B,\Phi)} \in Q$ for all $p, q \in A$.

Notice that A is an extension of itself, and $\langle p, q\rangle^{(A,A)} = \langle p, q\rangle$ in this case. Since $\mathrm{Equ}(A)$ is clearly generated by its atoms $\langle p, q\rangle$ $(p \neq q)$, we conclude that $\mathrm{Equ}(A)$ is $(1 + 1 + 2)$-generated, provided A is a good box. This is why we want to find good boxes of any cardinality below the first inaccessible cardinal. To accomplish this task, first we show that L (given in Figures 1 and 2) is a good box, then we give two methods to obtain larger good boxes from given good boxes, and finally we show that we can reach all infinite cardinals m (such that no inaccessible cardinal is $\leq m$) this way.

Claim 10. *L, the box defined in Figures 1 and 2, is a good box.*

Proof. Let L be denoted by $A = A_0 = (A_0, u_0, v_0, E_0, \alpha_0, \beta_0, \gamma_0, \delta_0)$, and let us consider an extension B of A; the notations from Definition 4 will be in effect. Let Q be a complete sublattice of $\mathrm{Equ}(B)$ such that $\{\alpha, \beta, \gamma, \delta\} \subseteq Q$. The key step is to show

$$\langle u, v\rangle^{(L,B)} = \langle a_0, b_0\rangle^{(A,B)} = \beta(\gamma + \delta). \tag{1}$$

The $\subseteq$ inclusion hardly needs any proof. In order to show the converse inclusion, suppose $(x, y) \in \beta(\gamma + \delta)$ and $x \neq y$. Let $x \in A_i$ and $y \in A_j$. Since β is column preserving, $\mathrm{col}(x) = \mathrm{col}(y)$. By Lemma 8(3), $\mathrm{col}(x) \le 28$. If $\mathrm{col}(x) = \mathrm{col}(y) = 0$ then Lemma 8(3) applies again, and $\{x, y\} = \{\varphi_i(a_0), \varphi_i(b_0)\} = \{\varphi_i(u), \varphi_i(v)\}$ yields $(x, y) \in \langle u, v\rangle^{(L,B)}$.

If $\mathrm{col}(x) = \mathrm{col}(y) = 28$ and $k \in \{i, j\}$ then we have $[\varphi_k(a_{28})](\gamma + \delta) = \{\varphi_k(a_{28}), \varphi_k(b_{27})\}$ and $[\varphi_k(b_{28})](\gamma + \delta) = \{\varphi_k(b_{28}), \varphi_k(a_{29}), u_k, v_k, \varphi_k(a_1)\}$. Hence $[x](\gamma_+\delta) \cap [y](\gamma_+\delta) = \emptyset$, which is a contradiction.

A similar contradiction can easily been achieved from the assumption $\mathrm{col}(x) = 1$ or $\mathrm{col}(x) = 27$. Hence $2 \le \mathrm{col}(x) = \mathrm{col}(y) \le 26$ can be assumed. Therefore x and y belong to the a-column or b-column or ... or e-column of some switch s of L. If s_i or s_j belongs to E, the set of switches of B, i.e., if at least one of s_i and s_j was not used to build B from L, then we easily obtain a contradiction like in case $\mathrm{col}(x) = \mathrm{col}(y) = 28$. Hence we assume that both s_i and s_j were used to build B from L.

Suppose at least one of these switches, say s_i, was used to enlarge δ during the construction of B. Since s_i was used only once, the $(\gamma + \delta)$-class of every component of s_i is included in the i-th ladder A_i. If the lower atom, $\langle b'(s_i), d'(s_i)\rangle$ of s_i was joined to δ then the $(\gamma + \delta)$-classes $\{b'(s_i), c(s_i), d'(s_i), e(s_i)\}$, $\{a'(s_i), b(s_i)\}$, $\{c'(s_i), d(s_i)\}$, and the two-element classes $[a(s_i)](\gamma + \delta)$ and $[e'(s_i)](\gamma + \delta)$ cover (the set of components of) s_i but none of then has two distinct component from the same column, which contradicts $\mathrm{col}(x) = \mathrm{col}(y)$. The case of the upper atom is similar.

Suppose now that the switches s_i and s_j were connected during the construction of B from L. Of course, $i \neq j$ and these switches have never been used to enlarge δ. From Lemma 8(3) we conclude that x and y belong to the b-column. But this is impossible, for $[b(s_i)](\gamma + \delta) = \{b(s_i), a'(s_i)\}$ and $[b'(s_i)](\gamma + \delta) = \{b'(s_i), c(s_i)\}$. We have proved (1), and therefore $\langle u, v\rangle^{(L,B)} \in Q$.

Now let $\Theta = \langle u, v\rangle^{(L,B)} + \alpha$. Clearly, Θ is ladder preserving and Θ collapses each A_i $(i \in I)$. Let

$$\beta' = \beta\Theta, \qquad \gamma' = \gamma\Theta, \qquad \delta' = \delta\Theta;$$

they belong to Q. Let

$$g_0 = \langle u, v\rangle^{(L,B)} = \langle a_0, b_0\rangle^{(A,B)} \qquad \text{and} \qquad H_0 = \langle b_{28}, a_{29}\rangle^{(A,B)}.$$

Using the fact that β', γ' and δ' are ladder preserving, it is easy to list the $(\beta' + \delta')$-classes of any ladder $A_i = \varphi_i(L)$ $(i \in I)$. (There is a five element class; the number of four element classes is the number of switches of A_i that were used as distinguishing switches; all other classes consist of two elements.) Armed with this list we obtain $H_0 := \gamma'(\beta' + \delta')$, so g_0

and H_0 belong to Q. Now, similarly to Zádori [10], we can define some further members of Q inductively:

$$\begin{aligned} h_{i+1} &= \big((g_i+\gamma')\alpha+g_i\big)\gamma' \qquad (i=0,1,\dots,28),\\ g_{i+1} &= \big((h_{i+1}+\beta')\alpha+h_{i+1}\big)\beta' \qquad (i=0,1,\dots,27),\\ G_{i+1} &= \big((H_i+\beta')\alpha+H_i\big)\beta' \qquad (i=0,1,\dots,28), \quad \text{and}\\ H_{i+1} &= \big((G_{i+1}+\gamma')\alpha+G_{i+1}\big)\gamma' \qquad (i=0,1,\dots,27). \end{aligned}$$

Since all the A_j ($j\in J$) are closed with respect to α, β', γ', g_0 and H_0, they are also closed with respect to g_i, h_i, G_i and H_i. Now an easy induction shows that

$$\begin{aligned} g_j &= \sum_{i=0}^{j}\langle a_i,b_i\rangle^{(A,B)} \qquad (j=0,1,\dots,28),\\ h_j &= \sum_{i=1}^{j}\langle b_{i-1},a_i\rangle^{(A,B)} \qquad (j=1,2,\dots,29),\\ H_j &= \sum_{i=0}^{j}\langle a_{29-i},b_{28-i}\rangle^{(A,B)} \qquad (j=0,1,\dots,28), \quad \text{and}\\ G_j &= \sum_{i=1}^{j}\langle a_{29-i},b_{29-i}\rangle^{(A,B)} \qquad (j=1,2,\dots,29). \end{aligned}$$

(Note that, for $B=A=L$, these formulas with notational changes occur in Zádori [10, p. 582].) Therefore the following elements

$$\begin{aligned} \langle a_j,b_j\rangle^{(A,B)} &= g_j\cdot G_{29-j} \quad (j=0,1,\dots,28),\\ \langle b_{j-1},a_j\rangle^{(A,B)} &= h_j\cdot H_{29-j} \quad (j=1,2,\dots,29),\\ \langle a_{j-1},a_j\rangle^{(A,B)} &= \big(\langle a_{j-1},b_{j-1}\rangle^{(A,B)}+\langle b_{j-1},a_j\rangle^{(A,B)}\big)\alpha \quad (j=1,2,\dots,29),\\ \langle b_{j-1},b_j\rangle^{(A,B)} &= \big(\langle b_{j-1},a_j\rangle^{(A,B)}+\langle a_j,b_j\rangle^{(A,B)}\big)\alpha \quad (j=1,2,\dots,28) \end{aligned}$$

all belong to Q. Now let $p,q\in A=A_0=L$ be distinct elements. Then there is a circle $p=z_0$, $z_1,\dots,z_i=q$, $z_{i+1},\dots,z_{i+j-1}$, $z_{i+j}=p$ in the graph depicted in Figure 1 such that $|\{z_0,z_1,\dots,z_{i+j-1}\}|=i+j$. Since the $\langle z_{\ell-1},z_\ell\rangle^{(A,B)}$ already belong to Q,

$$\langle p,q\rangle^{(A,B)} = \left(\sum_{\ell=1}^{i}\langle z_{\ell-1},z_\ell\rangle^{(A,B)}\right)\cdot\left(\sum_{\ell=i+1}^{i+j}\langle z_{\ell-1},z_\ell\rangle^{(A,B)}\right)\in Q.$$

This proves Claim 10. ◇

III. Successors and limits of boxes

Given a cardinal m, the smallest cardinal that is greater than m will be denoted by m^+. For a finite set X, let $R(X)=P(X)$, the set of all subsets of X. When X is infinite, $R(X)$ will always denote a fixed subset of $P(X)$ such that $\emptyset\in R(X)$ and, unless explicitly otherwise stated, $|R(X)|=|X|^+$. $R^+(X)$ will always stand for $R(X)\setminus\{\emptyset\}$.

Definition 11. Suppose $A = A_0 = (A_0, u_0, v_0, E_0, \alpha_0, \beta_0, \gamma_0, \delta_0)$ is a box. Choose $r \in E_0$, $D \subseteq E_0$ and $F \subseteq E_0$ such that $E_0 = \{r\} \cup D \cup F$ and, further, the sets $\{r\}$, D and F are pairwise disjoint. For each $U \in R(D)$ we take an isomorphic copy $A(U) = (A(U), u(U), v(U), E(U), \alpha(U), \beta(U), \gamma(U), \delta(U))$ of A such that these copies are pairwise disjoint, and $A(\emptyset)$ coincides with $A = A_0$. Let $\varphi_U \colon A \to A(U)$ be a fixed isomorphism for $U \in R^+(D)$; $\varphi_\emptyset = \varphi_0$ will stand for the identity map on $A = A(\emptyset)$. Let

$$\begin{aligned}
B &= \bigcup_{U \in R(D)} A(U), \qquad \alpha = \bigcup_{U \in R(D)} \alpha(U),\\
\beta &= \bigcup_{U \in R(D)} \beta(U) + \sum_{U \in R^+(D)} \langle b(r(\emptyset)), b(r(U))\rangle,\\
\gamma &= \bigcup_{U \in R(D)} \gamma(U) + \sum_{U \in R^+(D)} \langle d(r(\emptyset)), d(r(U))\rangle,\\
\delta &= \bigcup_{U \in R(D)} \delta(U) + \sum_{U \in R(D)} \Big(\sum_{s \in U} \langle b(s(U)), d(s(U))\rangle + \\
&\qquad \sum_{s \in D \setminus U} \langle b'(s(U)), d'(s(U))\rangle \Big).
\end{aligned}$$

Define $E = \bigcup_{U \in R(D)} \varphi_U(F)$, $u = u_0 = u(\emptyset)$, and $v = v_0 = v(\emptyset)$. This way we obtain $B = (B, u, v, E, \alpha, \beta, \gamma, \delta)$, which we call a *successor* of A.

For example, if $A = L$, the box defined by Figures 1 and 2, $D = \{s^{(1)}, s^{(2)}\}$, $F = \{s^{(3)}, s^{(4)}\}$ and $r = s^{(0)}$, then the corresponding successor of A is depicted in Figure 6, where $U_1 = \{s^{(1)}\}$, $U_2 = \{s^{(2)}\}$, and $U_3 = \{s^{(1)}, s^{(2)}\}$.

Clearly, the successor of a box A is a semibox that extends A. Hence the successor of a box is a box by the remark after Definition 7. Therefore the following statement is obvious.

Claim 12. *Let B be a successor of a box A. Then B is a box. Moreover, $B \mid_\Phi A$ for the "canonical" $\Phi = (\Gamma, \{\varphi_U \colon U \in R(D)\})$, where the classes of Γ are the $A(U)$, $U \in R(D)$.*

Claim 13. *Let B be a successor of A. If A is a good box, then so is B.*

Proof. Let $A_0 = A$ and use the notations of Definition 11. We know that $B \mid_\Phi A$ with the canonical Φ. Let us consider an extension $C = (C, \bar{u}, \bar{v}, \bar{E}, \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta})$ of B, say $C \mid_\Psi B$. Then C is an extension of A_0 by $\Psi \circ \Phi$, cf. Claim 6. So we can use the following self-explaining notations: $B_0 = B$, $\Psi = (\Delta, \{\psi_j \colon j \in J\})$, $B_j := \psi_j(B)$, $E_j = \psi_j(E)$, $A(j, U) := \psi_j(A(U))$ for $U \in R(D)$, and $A(0, U) = A(U)$. Then $B_j := \bigcup_{U \in R(D)} A(j, U)$ and $C = \bigcup_{(j,U) \in J \times R(D)} A(j, U)$; both of these unions are disjoint ones. Similar notations (with obvious meaning) will be used for $s \in E_0$, u, α, etc. The smallest complete sublattice of $\mathrm{Equ}(C)$ that contains $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, and $\bar{\delta}$ will be denoted by Q.

First we deal with the case $p = u(0,U) \in B = B_0$ and $q = v(0,U) \in B$. Then we assert

$$\begin{aligned}\langle p, q\rangle^{(B,C)} =&\langle p, q\rangle^{(A,C)} \cdot \prod_{s\in U}\Big(\langle p, b(s(0,U))\rangle^{(A,C)} + \bar\delta + \langle d(s(0,U)), q\rangle^{(A,C)}\Big)\cdot \\ &\cdot \prod_{s\in D\setminus U}\Big(\langle p, b'(s(0,U))\rangle^{(A,C)} + \bar\delta + \langle d'(s(0,U)), q\rangle^{(A,C)}\Big).\end{aligned} \tag{2}$$

Before proving (2) let us point out that it easily implies $\langle p, q\rangle^{(B,C)} \in Q$. Indeed, this follows from Claim 6 and the goodness of A. Similarly, all the subsequent equations will automatically imply that their left-hand sides belong to Q; we will rely on this fact implicitly.

The "$\subseteq$" inclusion in (2) is an obvious consequence of the definitions. To show the reverse inclusion, let us assume that (x, y) belongs to the right-hand side of (2), $x, y \in C$ and $x \neq y$. From $(x, y) \in \langle p, q\rangle^{(A,C)}$ we infer that x and y are in the same copy of A; say they are in $A(j, V) \subseteq B_j$. It also follows from $(x, y) \in \langle p, q\rangle^{(A,C)} = \langle u(j, V), v(j, V)\rangle^{(A,C)}$ that $\{x, y\} = \{u(j, V), v(j, V)\}$. Hence, for any $s \in U$, (2) yields

$$\begin{aligned}(u(j,V), v(j,V)) \in&\langle u(j,V), b(s(j,V))\rangle^{(A,C)}+ \\ &+ \bar\delta + \langle d(s(j,V)), v(j,V)\rangle^{(A,C)}.\end{aligned} \tag{3}$$

Since $s(j, V)$ is not a switch of B_j ($\cong B$), $s(j, V)$ was not used to construct C from the B_k ($k \in J$). Therefore the restriction of $\bar\delta$ to (the set of components of) $s(j, V)$ coincides with the restriction of δ_j. Now, combining (3) with the fact that the $\bar\delta$ classes of $u(j, V)$ and $v(j, V)$ are singletons, we obtain $\big(b(s(j, V)), d(s(j, V))\big) \in \bar\delta$. Hence $\big(b(s(j, V)), d(s(j, V))\big)$ belongs to δ_j, and we infer $s \in V$. This shows $U \subseteq V$. We obtain $D \setminus U \subseteq D \setminus V$ analogously. Thus $U = V$, and we conclude

$$\begin{aligned}(x, y) \in&\langle u(j,V), v(j,V)\rangle^{(B,C)} = \langle u(j,U), v(j,U)\rangle^{(B,C)} = \\ &\langle u(0,U), v(0,U)\rangle^{(B,C)} = \langle p, q\rangle^{(B,C)}.\end{aligned}$$

This proves (2).

If $p, q \in A(0,U) \subseteq B = B_0$ such that $|\{p, q, u(0,U), v(0,U)\}| = 4$, then we easily obtain

$$\begin{aligned}\langle p, q\rangle^{(B,C)} = \langle p, q\rangle^{(A,C)} \cdot \Big(&\langle p, u(0,U)\rangle^{(A,C)} + \langle u(0,U), v(0,U)\rangle^{(B,C)}+ \\ &+ \langle v(0,U), q\rangle^{(A,C)}\Big).\end{aligned}$$

Now let $p, q \in A(0,U) \subseteq B_0$ be arbitrary distinct elements. Since $|A(0,U)| \geq |L| = 59$, we can choose distinct $p_1, q_1 \in A(0,U) \setminus \{u(0,U),\ v(0,U), p, q\}$. The previous formula applies for (p_1, q_1), and we obtain

$$\langle p, q\rangle^{(B,C)} = \langle p, q\rangle^{(A,C)} \cdot \Big(\langle p, p_1\rangle^{(A,C)} + \langle p_1, q_1\rangle^{(B,C)} + \langle q_1, q\rangle^{(A,C)}\Big). \tag{4}$$

The next step is to deal with the case $p = b(r(0,\emptyset)) \in A(0,\emptyset)$ and $q = b(r(0,U)) \in A(0,U)$ for $U \in R^+(D)$. (Remember, r was the switch connecting the copies of A when we constructed B.) We assert that

$$\langle b(r(0,\emptyset)), b(r(0,U))\rangle^{(B,C)} = \bar\beta \cdot \Big(\langle b(r(0,\emptyset)), d(r(0,\emptyset))\rangle^{(B,C)} + \bar\gamma + \langle d(r(0,U)), b(r(0,U))\rangle^{(B,C)}\Big). \tag{5}$$

The "$\subseteq$" part is evident. Suppose now that $x \neq y \in C$, and (x,y) belongs to the right-hand side of (5). Since $\bar\beta$ is column-preserving by Lemma 8, $\mathrm{col}(x) = \mathrm{col}(y)$. Let, say, $x \in B_j$. Then there is a shortest sequence $z_0 = x$, z_1, …, z_{t-1}, $z_t = y$ such that every (z_{i-1}, z_i) belongs to

$$\langle b(r(0,\emptyset)), d(r(0,\emptyset))\rangle^{(B,C)} \cup \bar\gamma \cup \langle d(r(0,U)), b(r(0,U))\rangle^{(B,C)} = \langle b(r(j,\emptyset)), d(r(j,\emptyset))\rangle^{(B,C)} \cup \bar\gamma \cup \langle d(r(j,U)), b(r(j,U))\rangle^{(B,C)}.$$

Let $a(r(j,V),\swarrow)$ denote the element in the row of $a'(r(j,V))$ such that $\mathrm{col}(a(r(j,V),\swarrow)) = \mathrm{col}(a(r(j,V))) - 1$. Similarly, $e'(r(j,V),\nearrow)$ denotes the element in the row of $e(r(j,V))$ such that $\mathrm{col}(e'(r(j,V),\nearrow)) = \mathrm{col}(e'(r(j,V))) + 1$. The set

$$H_j = \bigcup_{V \in R(D)} \big\{a(r(j,V)), \ldots, e(r(j,V)), a'(r(j,V)), \ldots, e'(r(j,V)), a(r(j,V),\swarrow), e'(r(j,V),\nearrow)\big\}$$

is clearly closed with respect to $\langle b(r(j,\emptyset)), d(r(j,\emptyset))\rangle^{(B,C)}$ and $\langle d(r(j,U)), b(r(j,U))\rangle^{(B,C)}$. It is also closed with respect to $\gamma_j = \psi_j(\gamma)$ and $\bar\gamma$, for B_j was obtained from the $A(j,V)$, $V \in R(D)$, by connecting the switches $r(j,V)$, $V \in R(D)$. Hence all the z_i, $0 \le i \le t$, belong to $H_j \subseteq B_j$. Since $\bar\beta\bar\gamma = 0$ by Lemma 8, not all the (z_{i-1}, z_i) belong to $\bar\gamma$. Hence there is an ℓ with

$$(z_{\ell-1}, z_\ell) \in \langle b(r(j,\emptyset)), d(r(j,\emptyset))\rangle^{(B,C)} \tag{6}$$

or there is an m with

$$(z_{m-1}, z_m) \in \langle d(r(j,U)), b(r(j,U))\rangle^{(B,C)}. \tag{7}$$

Since $\langle b(r(j,\emptyset)), d(r(j,\emptyset))\rangle^{(B,C)}$ and $\langle d(r(j,U)), b(r(j,U))\rangle^{(B,C)}$ are column 2-changing, $\bar\gamma$ is column 1-preserving and $\mathrm{col}(x) = \mathrm{col}(y)$, both ℓ and m exist. By the minimality of t, (6) resp. (7) can hold only for one ℓ resp. m, so ℓ and m are uniquely determined. Since $\bar\beta$ is column preserving but no element of $\big[d(r(j,\emptyset))\big]\bar\gamma = \big[d(r(j,U))\big]\bar\gamma$ belongs to a b-column, it is easy to derive that $\{x,y\} = \{b(r(j,\emptyset)), b(r(j,U))\}$. Thus $(x,y) \in \langle b(r(0,\emptyset)), b(r(0,U))\rangle^{(B,C)}$, proving (5).

Now, using the fact that $\bar\gamma$ is column 1-preserving, we conclude easily that

$$\langle d(r(0,\emptyset)), d(r(0,U))\rangle^{(B,C)} = \bar\gamma \cdot \Big(\langle d(r(0,\emptyset)), b(r(0,\emptyset))\rangle^{(B,C)} + \langle b(r(0,\emptyset)), b(r(0,U))\rangle^{(B,C)} + \langle b(r(0,U)), d(r(0,U))\rangle^{(B,C)}\Big). \tag{8}$$

Finally, for any two distinct elements x and y in B_0 there is a circle $x = z_0, z_1, \ldots, z_i = y, z_{i+1}, \ldots, z_{i+j-1}, z_{i+j} = x$ of elements in B_0 such that $|\{z_0, z_1, \ldots, z_{i+j-1}\}| = i + j$, and all the $\langle z_{\ell-1}, z_\ell\rangle^{(B,C)}$ already belong to Q by (4), (5) and (8). Hence

$$\langle p, q\rangle^{(B,C)} = \left(\sum_{\ell=1}^{i} \langle z_{\ell-1}, z_\ell\rangle^{(B,C)}\right) \cdot \left(\sum_{\ell=i+1}^{i+j} \langle z_{\ell-1}, z_\ell\rangle^{(B,C)}\right) \in Q,$$

proving Claim 13. ◇

Definition 14. Let μ be an ordinal number. For $\nu < \mu$ let $A_\nu = (A_\nu, u_\nu, v_\nu, E_\nu, \alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu)$ be a box. Suppose that $A_\nu \mid_{\Phi_{\lambda\nu}} A_\lambda$ for $\lambda < \nu < \mu$ such that $\Phi_{\kappa\nu} = \Phi_{\lambda\nu} \circ \Phi_{\kappa\lambda}$ for all $\kappa < \lambda < \nu < \mu$. (This condition will be referred to as "the ways of extensions are compatible".) Then we say that the A_ν ($\nu < \mu$) together with the $\Phi_{\lambda\nu}$ ($\lambda < \nu < \mu$) form a *directed system of boxes*. Associated with this directed system we define

$$A = \bigcup_{\nu<\mu} A_\nu, \quad \alpha = \bigcup_{\nu<\mu} \alpha_\nu, \quad \beta = \bigcup_{\nu<\mu} \beta_\nu, \quad \gamma = \bigcup_{\nu<\mu} \gamma_\nu, \quad \text{and } \delta = \bigcup_{\nu<\mu} \delta_\nu.$$

We let $u = u_0$ and $v = v_0$; note that $u = u_\nu$ and $v = v_\nu$ for all $\nu < \mu$. Let us choose a subset $E \subseteq \bigcup_{\nu<\mu} E_\nu$ such that

$$E \subseteq \bigcup_{\nu<\mu} \bigcap_{\substack{\lambda \\ \nu\le\lambda<\mu}} E_\lambda \,. \tag{9}$$

Then $A = (A, u, v, E, \alpha, \beta, \gamma, \delta)$ is called a *limit* of the A_ν ($\nu < \mu$).

Note that A, α, β, γ and δ are unions of ascending chains. If μ is a successor ordinal, say $\mu = \varrho + 1$, then the limit is A_ϱ (with less switches, perhaps). Hence the limit of boxes is interesting (and will be used) for limit ordinals μ only. Unfortunately, the union of the right-hand side of (9) (and therefore E) can be empty. This phenomenon is responsible for a lot of work put in the rest of the paper.

Claim 15. *The limit A defined above is a box. There are canonical $\Phi_{\nu\mu}$ such that $A \mid_{\Phi_{\nu\mu}} A_\nu$ for all $\nu < \mu$. Moreover, denoting A by A_μ, the A_ν ($\nu < \mu + 1$) with the $\Phi_{\nu\lambda}$ ($\nu < \lambda < \mu + 1$) form a directed system of boxes.*

Proof. It is evident that A is a semibox. We can assume that μ is a limit ordinal, for otherwise the statement is trivial. Let us fix a $\nu < \mu$. For $\nu \le \lambda < \mu$, let $\Phi_{\nu\lambda} = (\Gamma_{\nu\lambda}, \{\varphi_i\colon\ i \in I_{\nu\lambda}\})$ where the classes of $\Gamma_{\nu\lambda}$ are denoted by $A_{\nu,i}$ ($i \in I_{\nu\lambda}$). By the compatibility of the $\Phi_{\kappa\varrho}$ we may assume that $I_{\nu\lambda} \subseteq I_{\nu\kappa}$ for $\lambda \le \kappa$, and φ_i and $A_{\nu,i}$ are the same for $i \in I_{\nu\lambda}$ as they are for $i \in I_{\nu\kappa}$. (This is clear from definition if we choose $I_{\varrho\lambda} = \Gamma_{\varrho\lambda}$, i.e. the set of classes, for all $\varrho < \lambda < \mu$.) Let $I = I_{\nu\mu} = \bigcup_{\nu\le\lambda<\mu} I_{\nu\lambda}$. Then $\{A_{\nu,i}\colon\ i \in I\}$ is a partition $\Gamma_{\nu\mu}$ on A, and (the collection of these) $\Phi_{\nu\mu} = (\Gamma_{\nu\mu}, \{\varphi_i\colon\ i \in I\})$ is compatible with all $\Phi_{\kappa\varrho}$ in the (original) directed system.

So all we have to show is that $\Phi_{\nu\mu}$ establishes an extension. Since $\alpha_\lambda = \bigcup_{i\in I_{\nu\lambda}} \alpha_{\nu,i}$ (where $\alpha_{\nu,i} = \varphi_i(\alpha_\nu)$), we obtain $\alpha = \bigcup_{\nu\le\lambda<\mu} \alpha_\lambda = \bigcup_{i\in I} \alpha_{\nu,i}$. For $\nu \le \lambda < \mu$, $A_\lambda \mid A_\nu$ gives that

$$\delta_\lambda = \bigcup_{i\in I_{\nu\lambda}} \delta_{\nu,i} + \sum_{(s,i)\in F_\lambda} \langle b(s_i), d(s_i)\rangle + \sum_{(s,i)\in G_\lambda} \langle b'(s_i), d'(s_i)\rangle \tag{10}$$

It is easy to observe that

$$\begin{aligned} F_\lambda &= \{(s,i) \in E \times I_{\nu\lambda}\colon\ (b(s_i), d(s_i)) \in \delta_\lambda\},\\ G_\lambda &= \{(s,i) \in E \times I_{\nu\lambda}\colon\ (b'(s_i), d'(s_i)) \in \delta_\lambda\}. \end{aligned} \tag{11}$$

Since $\delta_\lambda \subseteq \delta_\kappa$ for $\nu \le \lambda \le \kappa < \mu$, we infer $F_\lambda \subseteq F_\kappa$ and $G_\lambda \subseteq G_\kappa$. Let $F = \bigcup_{\nu\le\lambda<\mu} F_\lambda$ and $G = \bigcup_{\nu\le\lambda<\mu} G_\lambda$. Forming the union of (10) for all permitted λ we obtain the "$\subseteq$" part of

$$\delta = \bigcup_{i\in I} \delta_{\nu,i} + \sum_{(s,i)\in F} \langle b(s_i), d(s_i)\rangle + \sum_{(s,i)\in G} \langle b'(s_i), d'(s_i)\rangle, \tag{12}$$

while the "$\supseteq$" part is clear from the fact that each of the $\delta_{\nu,i}$ and the upper and lower atoms occurring in the sum on the right-hand side is smaller than some $\delta_\lambda \subseteq \delta$. For $\nu \le \lambda < \mu$, $A_\lambda \mid A_\nu$ gives that

$$\beta_\lambda = \bigcup_{i\in I_{\nu\lambda}} \beta_{\nu,i} + \sum_{(s,i,j)\in H_\lambda} \langle b(s_i), b(s_j)\rangle \tag{13}$$

(Notice that here and in the sequel, β resp. $b(\ )$ could be replaced by γ resp. $d(\ )$.) We may assume that

$$H_\lambda = \{(s,i,j) \in E \times I_{\nu\lambda} \times I_{\nu\lambda}\colon\ i \ne j, \quad (b(s_i), b(s_j)) \in \beta_\lambda\}. \tag{14}$$

Since $\beta_\lambda \subseteq \beta_\kappa$ for $\nu \le \lambda \le \kappa < \mu$, we infer $H_\lambda \subseteq H_\kappa$. Let $H = \bigcup_{\nu\le\lambda<\mu} H_\lambda$; then

$$\beta = \bigcup_{i\in I_{\nu\lambda}} \beta_{\nu,i} + \sum_{(s,i,j)\in H} \langle b(s_i), b(s_j)\rangle.$$

follows similarly to (12). Since the necessary disjointness conditions on F, G and H are implied by these conditions on F_λ, G_λ and H_λ, we have shown that $\Phi_{\nu\mu}$ establishes an extension. Finally, being an extension of boxes, the semibox A is a box. $\diamond$

Claim 16. *With the notations of Definition 14, if all the A_ν ($\nu < \mu$) are good boxes, then their limit, A, is a good box as well.*

Proof. Let $B = (B,\ \bar u,\ \bar v,\ \bar E,\ \bar\alpha,\ \bar\beta,\ \bar\gamma,\ \bar\delta)$ be an extension of $A = A_\mu$ by Ψ. We know from Claim 15 that $A \mid_{\Phi_{\nu\mu}} A_\nu$ for $\nu < \mu$. Claim 6 yields that B is an extension of A_ν by $\Psi \circ \Phi_{\nu\mu}$. Now all the necessary ways of extensions are fixed and compatible, so the notations $\langle p,q\rangle^{(A,B)}$ and $\langle p,q\rangle^{(A_\nu,B)}$ will make sense later in the proof.

Let Ψ be of the form $(\Gamma, \{\varphi_i\colon\ i \in I\})$ where $\Gamma = \{A^{(i)}\colon\ i \in I\}$ and $A = A^{(0)}$. Let Q denote the smallest complete sublattice of $\mathrm{Equ}(B)$ that includes $\{\bar\alpha, \bar\beta, \bar\gamma, \bar\delta\}$. Denote $\varphi_i(A_\nu)$ by $A_\nu^{(i)}$. Suppose that $p, q \in A$ are distinct elements. Then there is a smallest λ such that $p, q \in A_\lambda$. The goodness of A_ν yields $\langle p,q\rangle^{(A_\nu,B)} \in Q$ for all $\lambda \le \nu < \mu$. We assert that

$$\langle p,q\rangle^{(A,B)} = \prod_{\lambda\le\nu<\mu} \langle p,q\rangle^{(A_\nu,B)} \in Q; \tag{15}$$

only the equality has to be checked. The "$\subseteq$" part follows from the fact that the ways of extensions are compatible. For the converse inclusion, suppose (x,y) belongs to the right-hand side of (15), $x \ne y \in B$. From $(x,y) \in \langle p,q\rangle^{(A_\lambda,B)}$ we obtain that x and y are in the same copy of A_λ. Hence, by compatibility, they are in the same $A^{(i)}$. Choose a (sufficiently large) $\lambda \le \nu < \mu$ such that $\{\varphi_i(p), \varphi_i(q), x, y\} \subseteq A_\nu^{(i)}$. Then $(x,y) \in \langle p,q\rangle^{(A_\nu,B)} = \langle \varphi_i(p), \varphi_i(q)\rangle^{(A_\nu,B)}$ gives $\{x,y\} = \{\varphi_i(p), \varphi_i(q)\}$. Consequently, $(x,y) \in \langle \varphi_i(p), \varphi_i(q)\rangle^{(A,B)} = \langle p,q\rangle^{(A,B)}$, proving (15). $\diamond$

IV. Pursuing boxes at infinity

Starting from $L = A_0 = (A_0,\ u_0,\ v_0,\ E_0,\ \alpha_0,\ \beta_0,\ \gamma_0,\ \delta_0)$ (cf. Figures 1 and 2) we intend to define a directed system $A_i = (A_i,\ u_i,\ v_i,\ E_i,\ \alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i)$ of boxes, $i \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$, together with ways Φ_{ij} of extension $(i < j)$ such that, for all $i \in \mathbf{N}_0$, A_{i+1} is a successor of A_i and $A_{i+1} \mid_{\Phi_{i,i+1}} A_i$ is the canonical extension associated with the successor construction. Denoting by r_i, D_i and F_i the parameters establishing that A_{i+1} is a successor of A_i (cf. Definition 11, note that $r_i \in E_i$ and $D_i, F_i \subseteq E_i$), we also want that $F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$. Let A_1 be the box defined by Figure 6; the meaning of Φ_{01} is obvious. Now suppose that $i \ge 1$ and A_0, A_1, …, A_i are already defined together with compatible Φ_{jk} $(j < k \le i)$. By the construction of A_i from A_{i-1} and $A_i \mid_{\Phi_{i-1,i}} A_{i-1}$ we obtain $F_{i-1} \subseteq E_i$. Choose $F_i \subseteq E_i$ such that $F_{i-1} \subseteq F_i$ and $|F_i| = \frac{1}{2}|E_i|$ (we will see that this is possible); let $r_i \in E_i \setminus F_i$ and let $D_i = E_i \setminus (\{r_i\} \cup F_i)$. These parameters determine a unique successor A_{i+1} of A_i and a unique (canonical) $\Phi_{i,i+1}$ with $A_{i+1} \mid_{\Phi_{i,i+1}} A_i$; for $j < i$ we set $\Phi_{j,i+1} = \Phi_{i,i+1} \circ \Phi_{j,i}$. The sequence $t_i = (|E_i|, |F_i|, |D_i|)$, $i = 1, 2, 3, \ldots$, clearly obeys the following rule:

$$t_1 = (8,4,3), \quad t_2 = (32,16,15), \quad t_3 = (16\cdot 2^{15}, 16\cdot 2^{14}, 16\cdot 2^{14} - 1), \quad \ldots,$$
$$t_{i+1} = (|F_i|\cdot 2^{|D_i|}, |F_i|\cdot 2^{|D_i|-1}, |F_i|\cdot 2^{|D_i|-1} - 1), \quad \ldots$$

It is easy to see that $2\cdot|F_{i-1}| \le |E_i|$ for $i = 1, 2, 3, \ldots$, so the choice of F_i is always possible. Since $F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$ and $F_i \subseteq E_i$, for all natural numbers n we obtain $\bigcap_{n\le\ell} E_\ell \supseteq \bigcap_{n\le\ell} F_\ell = F_n$. Hence, the choice $E = \bigcup_{n\in\mathbf{N}_0} F_n$ is in accordance with (9). Now let $A = (A,\ u,\ v,\ E,\ \alpha,\ \beta,\ \gamma,\ \delta)$ be the limit of the directed system we have just defined. The fast growing of the sequence $(t_i)_{i\in\mathbf{N}_0}$ makes it clear that $|E| = \aleph_0$. Therefore $|A| = |E|$, and this property will be so important in the sequel that it deserves a separate name.

Definition 17. A box $A = (A,\, u,\, v,\, E,\, \alpha,\, \beta,\, \gamma,\, \delta)$ is called a *perfect box* if it is a good box and $|A| = |E|$. A cardinal m will be called *small* if $\aleph_0 \le m$ and there is no inaccessible cardinal $\le m$.

In virtue of Claim 16, the box we have defined before Definition 17 is a countable perfect box. Clearly, every perfect box is necessarily infinite. We want to show that for each small cardinal m there is a perfect box of power m. However, we need an even stronger induction hypothesis.

Definition 18. Given a small cardinal m, we say that the condition H(m) holds if

(i) for each small cardinal $n \le m$ there is a perfect box of cardinality n; and

(ii) for any two small cardinals $n < k \le m$, for every perfect box $A = (A,\, u,\, v,\, E,\, \alpha,\, \beta,\, \gamma,\, \delta)$ with $|A| = n$, and for each $E' \subset E$ with $|E \setminus E'| = n$ there is a perfect box B of power k such that B is an E'-preserving extension of A.

Clearly, if a cardinal m is small, then so is m^+, the least cardinal that is greater than m. The existence of a countable perfect box trivially implies that H($\aleph_0$) holds.

Claim 19. *Suppose m is a small cardinal and H(m) holds. Then H(m^+) holds as well.*

Proof. It suffices to show H(m^+)(ii) for $n \le m$ and $k = m^+$. Let us take a perfect box $A = (A,\, u,\, v,\, E,\, \alpha,\, \beta,\, \gamma,\, \delta)$ of power n (such a box exists by H(m)), and a subset $E' \subset E$ with $|E \setminus E'| = |E| = n$. We want to construct an E'-preserving extension B of A such that $|B| = m^+$. Since $n = n + n$ by the cardinal arithmetics, we can chose an E'' such that $E' \subset E'' \subset E$ and $|E \setminus E''| = |E'' \setminus E'| = n$. In virtue of H($m$) we obtain a perfect box $C = (C,\, \hat{u},\, \hat{v},\, \hat{E},\, \hat{\alpha},\, \hat{\beta},\, \hat{\gamma},\, \hat{\delta})$ of power m such that C is an E''-preserving extension of A. Let $\hat{E}'$ and $\hat{E}''$ denote the extensions of E' and E'' to C, respectively. Then $m = |\hat{E}| \ge |\hat{E} \setminus \hat{E}'| \ge |\hat{E}'' \setminus \hat{E}'| = |E'' \setminus E'| \cdot [C : A] = |A| \cdot [C : A] = |C| = m$, i.e., $|\hat{E} \setminus \hat{E}'| = m$. Hence we can partition $\hat{E}$ into $\{\hat{r}\}$, $\hat{D}$ and $\hat{F}$ such that $\hat{E}' \subseteq \hat{F}$ and $|\hat{F}| = |\hat{D}| = m$. These parameters determine a successor B of C. By Claim 12 and Definition 11, B is an $\hat{E}'$-preserving extension of C. Therefore, by Claim 6, B is an E'-preserving extension of A. ◇

Claim 20. *Suppose that k is a small limit cardinal (i.e., $k = m^+$ holds for no m) and H(m) holds for all $m < k$. Then H(k) holds as well.*

Proof. We can assume that $k > \aleph_0$. Since k is small, either

($*$) $2^m \ge k$ for some $m < k$, or

($**$) there is a set M of cardinals such that $|M| < k$, $m < k$ for all $m \in M$, and $\sup\{m\colon m \in M\} = k$.

The treatment of ($*$) is very similar to that of Claim 19; the only difference is that to obtain B from C we use the successor construction with $|R(\hat{D})| = k$ instead of $|R(\hat{D})| = |\hat{D}|^+ = m^+$.

From now on we deal with ($**$). Again, we have to prove only (ii) and only in a particular case. I.e., let $A = (A,\, u,\, v,\, E,\, \alpha,\, \beta,\, \gamma,\, \delta)$ be a perfect box of power $n < k$, and let $E' \subset E$ with $|E \setminus E'| = |A| = n$. By H($n$), A and E' exist. Our task is to give an E'-preserving extension of A to a perfect box of power k. We have to distinguish two cases.

Case (A): $|M| \leq n$. Then we may assume that $n \in M$ and $(\forall m \in M)$ $(n \leq m)$, for otherwise we can replace M by $\{n\} \cup \{m \in M : n \leq m\}$. Since any set of cardinals is well-ordered, M is of the form $M = \{m_\xi \colon \xi < \mu\}$ where μ is a limit ordinal, $|\mu| = |M| \leq n < k$, $m_0 = n$, and $m_\xi < m_\eta$ for $\xi < \eta < \mu$. For convenience, define $m_\mu = k$; then $m_\mu = \sup\{m_\xi \colon \xi < \mu\}$.

Like in the previous proof, we can choose an E'' such that $E' \subset E'' \subset E$ and $|E \setminus E''| = |E'' \setminus E'| = n = m_0$. Since $n = 2n \cdot |\mu+1|$, we can choose a partition $\{X_\xi \colon \xi \leq \mu\} \cup \{Y_\xi \colon \xi \leq \mu\}$ of $E'' \setminus E'$ such that $|X_\xi| = |Y_\xi| = n$ for all $\xi \leq \mu$. We define

$$E^{(\xi)} = E'' \setminus \Big(\bigcup_{\eta \leq \xi} X_\eta \cup \bigcup_{\eta < \xi} Y_\eta \Big) \quad \text{and}$$
$$T^{(\xi)} = E'' \setminus \Big(\bigcup_{\eta \leq \xi} X_\eta \cup \bigcup_{\eta \leq \xi} Y_\eta \Big) = E^{(\xi)} \setminus Y^{(\xi)}.$$

Then $T^{(\xi)}, E^{(\xi)} \subseteq E''$ for all $\xi \leq \mu$, $T^{(\mu)} = E'$, and, for all $\xi < \eta \leq \mu$,

$$E^{(\xi)} \supset T^{(\xi)} \supset E^{(\eta)} \supset T^{(\eta)} \quad \text{and}$$
$$|E^{(\xi)} \setminus T^{(\xi)}| = |T^{(\xi)} \setminus E^{(\eta)}| = |E^{(\eta)} \setminus T^{(\eta)}| = n.$$

Via induction on ν, for each $\nu \leq \mu$ we want to define a directed system S_ν of perfect boxes $A_\xi = (A_\xi,\ u_\xi,\ v_\xi,\ E_\xi,\ \alpha_\xi,\ \beta_\xi,\ \gamma_\xi,\ \delta_\xi)$ $\quad (\xi \leq \nu)$ together with compatible $\Phi_{\xi\eta}$ $(\xi < \eta \leq \nu)$ such that $|A_\xi| = m_\xi$, $A_0 = A$, $A_\xi \mid_{\Phi_{0\xi}} A_0$ is a $T^{(\xi)}$-preserving extension, and $S_\lambda \subseteq S_\nu$ for all $\lambda \leq \nu$. Let $I(\nu)$ denote this collection of conditions that we expect from S_ν.

Let S_0 consist only of $A_0 = A$; $I(0)$ is evident.

Now suppose that S_ν satisfying $I(\nu)$ is already constructed; we want to construct $S_{\nu+1}$. Since $A_\nu \mid_{\Phi_{0\nu}} A_0$ is $T^{(\nu)}$-preserving, we can extend $T^{(\nu)}$ and $T^{(\nu+1)}$ to A_ν; let $T_\nu^{(\nu)}$ and $T_\nu^{(\nu+1)}$ denote their extension, respectively. We have $m_\nu = |E_\nu| \geq |E_\nu \setminus T_\nu^{(\nu+1)}| \geq |T_\nu^{(\nu)} \setminus T_\nu^{(\nu+1)}| = |T^{(\nu)} \setminus T^{(\nu+1)}| \cdot [A_\nu : A_0] = |A_0| \cdot [A_\nu : A_0] = |A_\nu| = m_\nu$. I.e., $|E_\nu \setminus T_\nu^{(\nu+1)}| = m_\nu$. Hence, by $\mathrm{H}(m_{\nu+1})$, there exists a perfect box $A_{\nu+1} = (A_{\nu+1},\ u_{\nu+1},\ v_{\nu+1},\ E_{\nu+1},\ \alpha_{\nu+1},\ \beta_{\nu+1},\ \gamma_{\nu+1},\ \delta_{\nu+1})$ of power $m_{\nu+1}$ such that $A_{\nu+1} | A_\nu$ is a $T_\nu^{(\nu+1)}$-preserving extension. From Claim 6 we infer that $A_{\nu+1}$ is a $T^{(\nu+1)}$-preserving extension of A_0. Let $\Phi_{\nu,\nu+1}$ denote the way of $A_{\nu+1} | A_\nu$, and define $\Phi_{\xi,\nu+1} = \Phi_{\nu,\nu+1} \circ \Phi_{\xi\nu}$ for $\xi < \nu$. Hence, augmenting S_ν by $A_{\nu+1}$ and by the $\Phi_{\xi,\nu+1}$ $(\xi < \nu + 1)$, we obtain a directed system $S_{\nu+1}$. $S_{\nu+1}$ clearly satisfies $\mathrm{I}(\nu + 1)$.

Now let ν $(\nu \leq \mu)$ be a limit ordinal, and suppose that the S_ξ satisfying $I(\xi)$ are already defined for all $\xi < \nu$. The union $\bigcup_{\xi<\nu} S_\xi$ is clearly a directed system again; let $B_\nu = (B_\nu,\ \hat{u}_\nu,\ \hat{v}_\nu,\ \hat{E}_\nu,\ \hat{\alpha}_\nu,\ \hat{\beta}_\nu,\ \hat{\gamma}_\nu,\ \hat{\delta}_\nu)$ be its limit. ($\hat{E}_\nu$ will be given soon.) By Claim 15, $B_\nu \mid_{\Psi_{\xi\nu}} A_\xi$ such that the $\Psi_{\xi\nu}$ $(\xi < \nu)$ are compatible with the $\Phi_{\xi\varrho}$ $(\xi < \varrho < \nu)$. For

$\xi < \nu$, $A_\xi|A_0$ is a $T^{(\xi)}$-preserving extension, and $E^{(\nu)} \subset T^{(\xi)}$. So $E^{(\nu)}$ extends to A_ξ; let $E^{(\nu)}_\xi$ denote its extension. For $\xi < \varrho < \nu$ we have $E^{(\nu)}_\xi \subseteq E^{(\nu)}_\varrho$, for $\Phi_{0\varrho} = \Phi_{\xi\varrho} \circ \Phi_{0\xi}$. So we obtain

$$\bigcup_{\xi<\nu} \bigcap_{\xi\le\varrho<\nu} E_\varrho \supseteq \bigcup_{\xi<\nu} \bigcap_{\xi\le\varrho<\nu} E^{(\nu)}_\varrho = \bigcup_{\xi<\nu} E^{(\nu)}_\xi .$$

Hence, according to (9), we can let $\hat{E}_\nu = \bigcup_{\xi<\nu} E^{(\nu)}_\xi$, and B_ν becomes an $E^{(\nu)}$-preserving extension of A_0. Since $m_\xi = |E_\xi| \ge |E^{(\nu)}_\xi| = |E^{(\nu)}|\cdot[A_\xi : A_0] = |A_0|\cdot[A_\xi : A_0] = |A_\xi| = m_\xi$, we obtain $|\hat{E}_\nu| = \sup\{m_\xi\colon\ \xi < \nu\} = |B_\nu|$. Hence B_ν is a perfect box, and it is an $E^{(\nu)}$-preserving extension of A_0.

Now we have to distinguish two cases. First assume that $\nu < \mu$. Since $T^{(\nu)} \subset E^{(\nu)}$, $T^{(\nu)}$ extends to B_ν. Let $\hat{T}^{(\nu)}_\nu$ and $\hat{E}^{(\nu)}_\nu$ denote the extension of $T^{(\nu)}$ and $E^{(\nu)}$ to B_ν, respectively. Then $|\hat{E}_\nu \setminus \hat{T}^{(\nu)}_\nu| \ge |\hat{E}^{(\nu)}_\nu \setminus \hat{T}^{(\nu)}_\nu| = |E^{(\nu)} \setminus T^{(\nu)}| \cdot [B_\nu : A_0] = |B_\nu|$. On the other hand, $|B_\nu| = \sup\{m_\xi\colon\ \xi < \nu\} \le m_\nu$. Thus $\mathrm{H}(m_\nu)$ applies and yields a perfect box $A_\nu = (A_\nu,\, u_\nu,\, v_\nu,\, E_\nu,\, \alpha_\nu,\, \beta_\nu,\, \gamma_\nu,\, \delta_\nu)$ of power m_ν such that $A_\nu\ |_{\hat{\Psi}}\ B_\nu$ is a $\hat{T}^{(\nu)}_\nu$-preserving extension. Then A_ν is an extension of A_ξ by $\Phi_{\xi\nu} := \hat{\Psi} \circ \Psi_{\xi\nu}\quad (\xi < \nu)$, and A_ν is a $T^{(\nu)}$-preserving extension of A_0, cf. Claim 6. Further, for $\xi < \varrho < \nu$,

$$\Phi_{\varrho\nu} \circ \Phi_{\xi\varrho} = (\hat{\Psi} \circ \Psi_{\varrho\nu}) \circ \Phi_{\xi\varrho} = \hat{\Psi} \circ (\Psi_{\varrho\nu} \circ \Phi_{\xi\varrho}) = \hat{\Psi} \circ \Psi_{\xi\nu} = \Phi_{\xi\nu}.$$

This shows $\mathrm{I}(\nu)$.

The other case is $\nu = \mu$. Then $|B_\mu| = |B_\nu| = \sup\{m_\xi\colon\ \xi < \mu\} = k = m_\mu$. Hence we do not have to (and we are even not allowed to) apply $\mathrm{H}(m_\nu)$ to extend this B_ν to A_ν. We simply let $A_\mu := B_\mu$, $\Phi_{\xi\mu} := \Psi_{\xi\mu}\ \ (\xi < \mu)$, and $\mathrm{I}(\mu)$ clearly holds.

From $\mathrm{I}(\mu)$ we obtain that A_μ is a $T^{(\mu)} = E'$-preserving extension of $A = A_0$ and $|A_\mu| = m_\mu = k$.

Case (B): $n < |M|$. Now we can choose a subset $E'' \subset E$ such that $E' \subset E'' \subset E$ and $|E \setminus E''| = |E'' \setminus E'| = n$, and an element $m \in M$ with $|M| \le m < k$. By $H(m)$, A has an E''-preserving extension to a perfect box $C = (C,\, \hat{u},\, \hat{v},\, \hat{E},\, \hat{\alpha},\, \hat{\beta},\, \hat{\gamma},\, \hat{\delta})$ with $|C| = m$. Let $\hat{E}'$ and $\hat{E}''$ denote the extensions of E' and E'' to C, respectively. Then $m = |\hat{E}| \ge |\hat{E} \setminus \hat{E}'| \ge |\hat{E}'' \setminus \hat{E}'| = |E'' \setminus E'| \cdot [C : A] = |A| \cdot [C : A] = m$, i.e. $|\hat{E} \setminus \hat{E}'| = m$. By Case (A), C has an $\hat{E}'$-preserving extension to a perfect box B with $|B| = k$. In virtue of Claim 6, B is an E'-preserving extension of A. This proves Claim 20. ◇

Finally, from the existence of a countable perfect box (i.e., $\mathrm{H}(\aleph_0)$) and Claims 19 and 20 we derive $\mathrm{H}(m)$ for all small cardinals m via induction. According to the remark after Definition 9, this proves Theorem 1.

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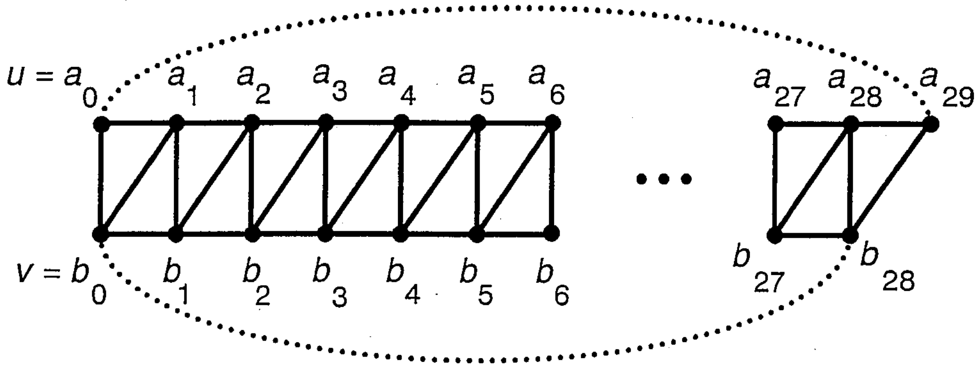


Figure 1

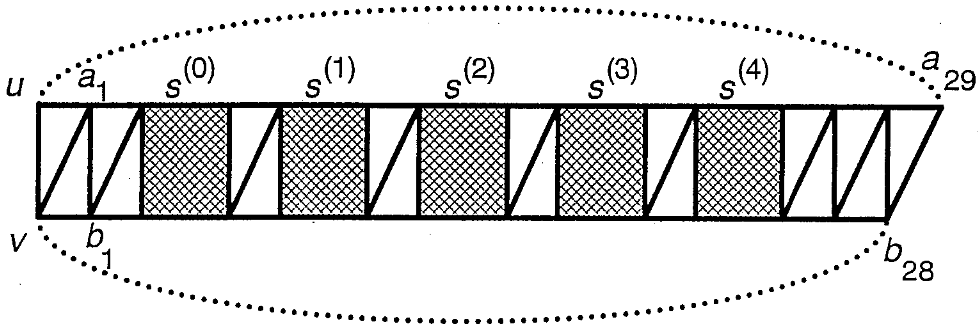


Figure 2

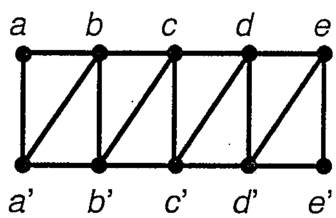
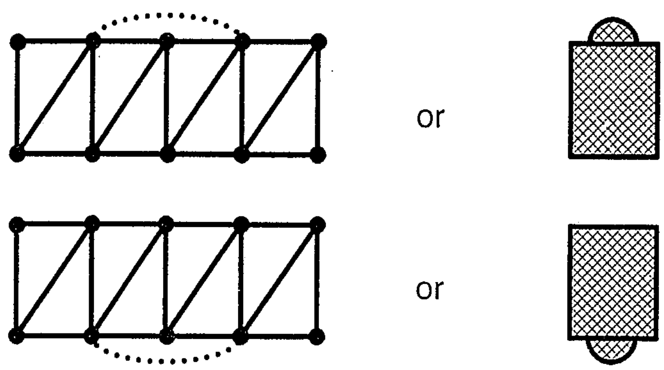


Figure 3



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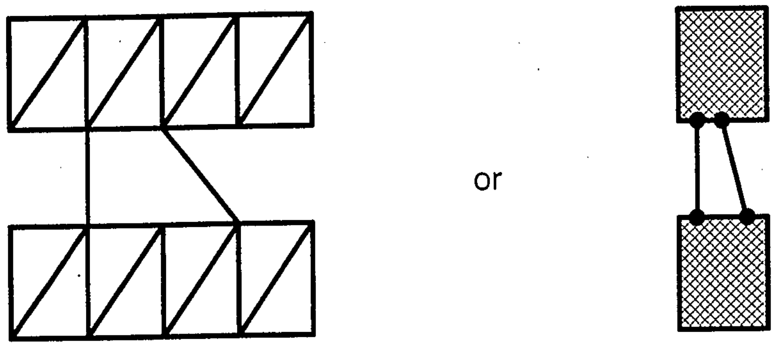

Figure 5

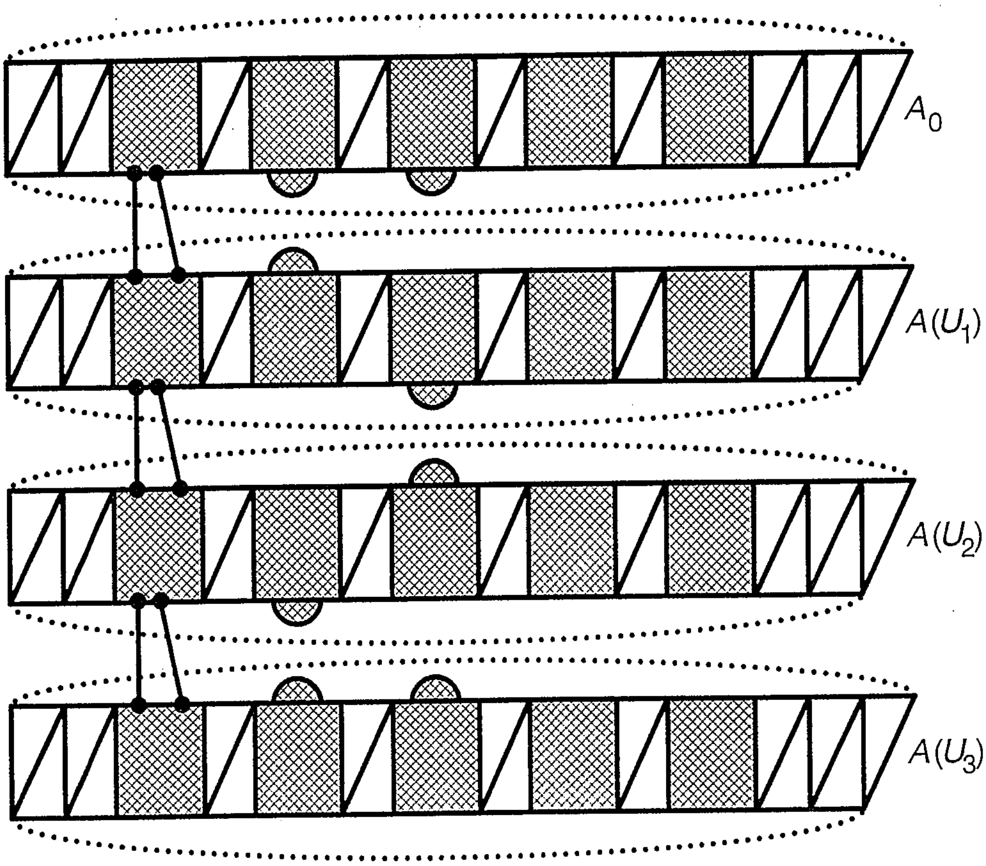

Figure 6

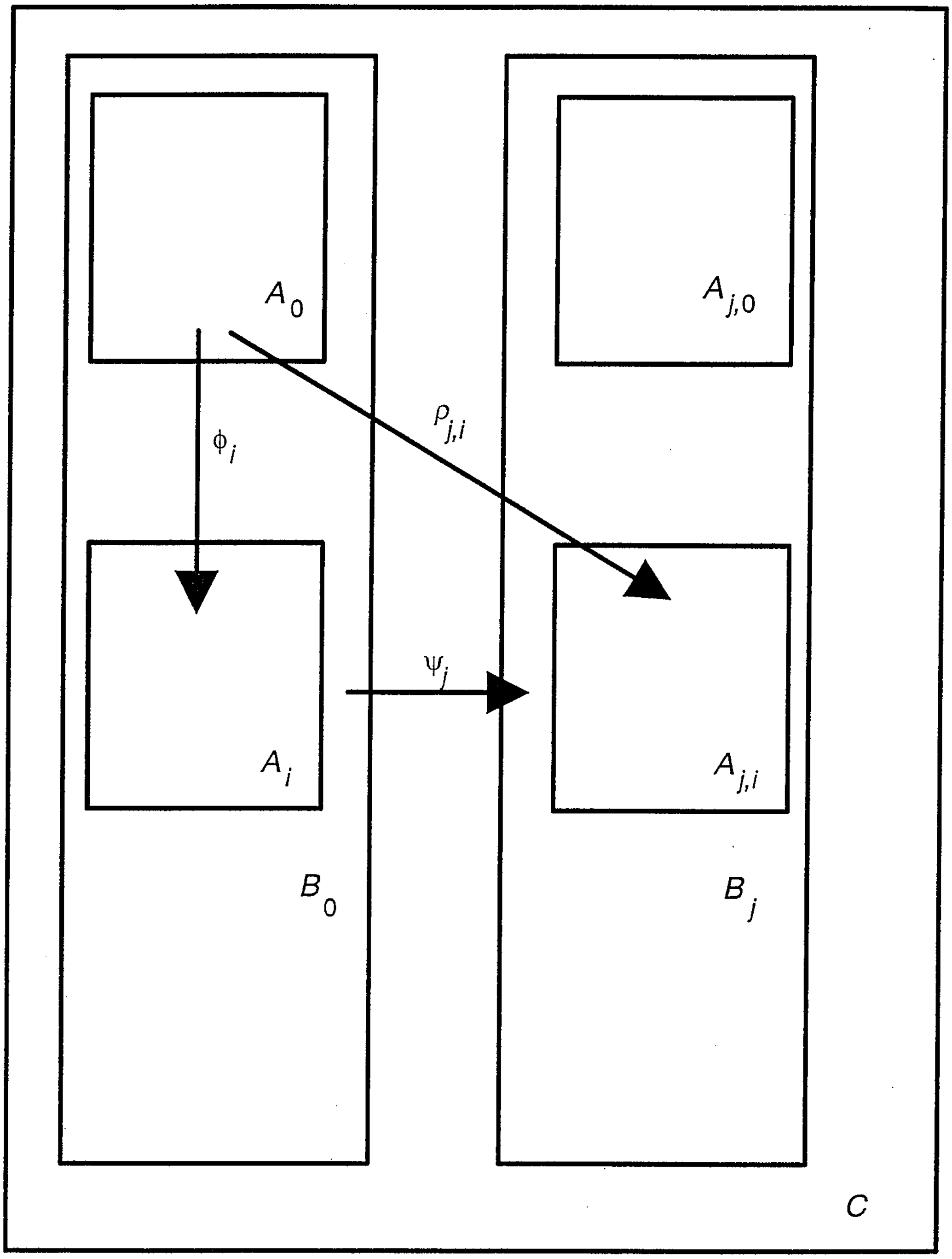

Figure 7