Subdirect representation and semimodularity of weak congruence lattices

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A *weak congruence* on an algebra A is a symmetric and transitive subalgebra of A^2. Weak congruences of A form an algebraic lattice $\mathrm{C_w}(A)$ with respect to inclusion, cf. [4]. The diagonal relation $\Delta = \{(x,x) : x \in A\} \in \mathrm{C_w}(A)$ plays a special role. Δ is a codistributive element in $\mathrm{C_w}(A)$, i.e.,

$$\Delta \wedge (\alpha \vee \beta) = (\Delta \wedge \alpha) \vee (\Delta \wedge \beta) \tag{D}$$

for all $\alpha, \beta \in \mathrm{C_w}(A)$. If the dual of condition (D) holds then A is said to satisfy the *congruence intersection property* (CIP for short), cf. [1]. Notice that the CIP simply means that Δ is a distributive element in $\mathrm{C_w}(A)$. The filter $[\Delta)$ is just $\mathrm{Con}(A)$, the congruence lattice, while the ideal $(\Delta]$ is isomorphic to $\mathrm{Sub}(A)$, the subalgebra lattice.

There are results stating that under reasonable conditions certain lattice properties are inherited from $\mathrm{Con}(A)$ and $\mathrm{Sub}(A)$ to $\mathrm{C_w}(A)$, cf. [1, 2]. Our goal is to strengthen a previous result while radically simplifying its proof and to give a new result.

Recall that A has the *congruence extension property* (CEP for short), if each congruence on every subalgebra of A is a restriction of a congruence on A. Vojvodič and Šešelja [3] have shown that an algebra A satisfies the CEP and the CIP if and only if the mapping $f : \mathrm{C_w}(A) \to (\Delta] \times [\Delta)$, given by

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$f(a) = (\alpha \wedge \Delta, \alpha \vee \Delta)$ is an embedding. Since the composite of f with each projection is clearly surjective, the following follows.

THEOREM 1 *Let A be an algebra satisfying the CIP and the CEP. Then* $\mathrm{C_w}(A)$ *is isomorphic to a subdirect product of lattices* $\mathrm{Sub}(A)$ *and* $\mathrm{Con}(A)$.

Now, since $\mathrm{Con}(A)$ and $\mathrm{Sub}(A)$ are embedded in $\mathrm{C_w}(A)$, we obviously obtain

COROLLARY 2 *Suppose A satisfies the CIP and the CEP. Then an arbitrary lattice quasi identity (i.e., Horn sentence) holds in* $\mathrm{C_w}(A)$ *if and only if it holds in* $\mathrm{Con}(A)$ *and* $\mathrm{Sub}(A)$.

This corollary strengthens Thm. 3 of [1] from identities to quasi identities. A lattice L is called (*upper*) *semimodular* if for all $a, b \in L$, $a \wedge b \preceq a$ implies $b \preceq a \vee b$. Here $\preceq$ stands for "covered by or equal to". Lower semimodularity is defined dually.

PROPOSITION 3 *If L is a subdirect product of semimodular lattices then L is semimodular as well.*

PROOF. For simplicity we prove the statement only for subdirect products of two factors, which will be used in the sequel; the general case is similar. Let $L \subseteq L_1 \times L_2$ be a subdirect product and suppose that L_1 and L_2 are semimodular. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be in L such that $a \wedge b \preceq a$. (For elements in L, $\preceq$ is always understood in L, not in the direct product.) First we show that for $i = 1, 2$, $a_i \wedge b_i \preceq a_i$. Suppose not and, by symmetry, let $i = 1$. Then $a_1 \wedge b_1 < c_1 < a_1$ for some $c_1 \in L_1$. Since L is a subdirect product, there is a $c_2 \in L_2$ with $c = (c_1, c_2) \in L$. Now $(c_1, (a_2 \wedge b_2) \vee (a_2 \wedge c_2)) = (a \wedge b) \vee (a \wedge c)$ is in L and strictly between $a \wedge b$ and a, a contradiction. We have seen that $a_1 \wedge b_1 \preceq a_1$ and $a_2 \wedge b_2 \preceq a_2$. Semimodularity gives $b_i \preceq a_i \vee b_i$ for $i = 1, 2$. Now let $d = (d_1, d_2)$ be an arbitrary element of L with $b \leq d < a \vee b$. Then $d_i \in \{b_i, a_i \vee b_i\}$ for $i = 1, 2$. Since $a \wedge d$ belongs to the prime interval $[a \wedge b, a]$ and $a \not\leq d$, we have $a \wedge d = a \wedge b$. Hence if $d_i = a_i \vee b_i$ then we obtain $a_i = a_i \wedge (a_i \vee b_i) = a_i \wedge d_i = a_i \wedge b_i$, which gives $d_i = a_i \vee b_i = b_i$. Thus $d = b$, $b \preceq a \vee b$, and L is semimodular. ∎

Since $\mathrm{Con}(A)$ and $\mathrm{Sub}(A)$ are convex sublattices of $\mathrm{C_w}(A)$, we conclude the following result from Theorem 1, Proposition 3 and its dual.

THEOREM 4 *Let A satisfy the CIP and the CEP. Then* $\mathrm{C_w}(A)$ *is lower resp. upper semimodular if and only if* $\mathrm{Con}(A)$ *and* $\mathrm{Sub}(A)$ *are lower resp. upper semimodular.*

The "upper" part of Theorem 4 (and in fact, a stronger statement) is known, cf. Thm. 3 in [2]. To show an application, we remark that an arbitrary unary algebra satisfies the CIP and the CEP, and its subalgebra lattice is distributive. Therefore Theorem 4 implies

COROLLARY 5 *The lattice of weak congruences of a unary algebra is lower (upper) semimodular if and only if its congruence lattice is lower (upper) semimodular.*

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