QUASIORDERS OF LATTICES VERSUS PAIRS OF CONGRUENCES

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ABSTRACT. Given a lattice L, the lattice, in fact the involution lattice, $\mathrm{Quord}(L)$ of quasiorders of L is shown to be isomorphic with $\mathrm{Con}^2(L)$, the direct square of the congruence lattice of L. The isomorphism given is natural in category theoretic sense. As a corollary, a description of compatible partial orderings of a lattice is obtained.

Dedicated to Professors László Leindler on his 60th and Károly Tandori on his 70th birthday

INTRODUCTION

A triplet $L = \langle L; \vee, \wedge, {}^* \rangle$ is called an *involution lattice* if $L = \langle L; \vee, \wedge \rangle$ is a lattice and ${}^*\colon L \to L$ is a lattice automorphism such that $(x^*)^* = x$ holds for all $x \in L$. To present a natural example, let us consider an algebra A. A binary relation $\rho \subseteq A^2$ is called a *quasiorder* of A if ρ is reflexive, transitive and compatible. Defining $\rho^* = \{\langle x, y\rangle \colon \langle y, x\rangle \in \rho\}$, the set $\mathrm{Quord}(A)$ of quasiorders of A becomes an involution lattice $\mathrm{Quord}(A) = \langle \mathrm{Quord}(A); \vee, \wedge, {}^*\rangle$, where $\wedge$ is the intersection and $\vee$ is the transitive closure of the union. These involution lattices were studied in [1,3] and Chajda and Pinus [2]. The sublattice $\{\rho \in \mathrm{Quord}(A) \colon \rho^* = \rho\} = \mathrm{Con}(A)$ is just the congruence lattice of A. Let $\mathrm{Con}^2(A)$ denote the direct square of the lattice $\mathrm{Con}(A)$ equipped with the involution defined by $\langle \alpha, \beta\rangle^* = \langle \beta, \alpha\rangle$. Then $\mathrm{Con}^2(A)$ is an involution lattice.

For an arbitrary algebra A, $\mathrm{Quord}(A)$ determines $\mathrm{Con}^2(A)$ up to isomorphism but not conversely. The aim of this short note is to show that if A is a lattice then the two involution lattices associated with A are isomorphic in a "natural" way. As corollaries, a description of compatible partial orderings of lattices and an abstract characterization of the involution lattices $\mathrm{Quord}(L)$ of lattices L will be obtained.

Let $\mathcal{L}$ denote the category of all lattices in which the morphisms are the surjective lattice homomorphisms. For lattices A, B and a morphism $\varphi\colon A \to B$, let

$$\mathrm{Quord}(\varphi)\colon \mathrm{Quord}(B) \to \mathrm{Quord}(A), \quad \gamma \mapsto \{\langle x, y\rangle \in A^2 \colon \langle \varphi(x), \varphi(y)\rangle \in \gamma\}$$

and

$$\mathrm{Con}^2(\varphi)\colon \mathrm{Con}^2(B) \to \mathrm{Con}^2(A) \quad \langle \alpha, \beta\rangle \mapsto \langle \varphi^{-1}(\alpha), \varphi^{-1}(\beta)\rangle,$$

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where $\varphi^{-1}(\delta) = \{\langle x, y\rangle \in A^2\colon \langle \varphi(x), \varphi(y)\rangle \in \delta\}$. The category of all involution lattices with all homomorphisms will be denoted by $\mathcal{V}$. (Note that we could consider the injective homomorphisms only). It is straightforward to check that Quord and Con^2 are functors from $\mathcal{L}$ to $\mathcal{V}$.

Results and proofs

For a lattice A, let $\rho = \{\langle x, y\rangle\colon x = x \wedge y\}$ denote the usual lattice order $\le$, and consider the maps

$$\tau_A\colon \mathrm{Quord}(A) \to \mathrm{Con}^2(A), \quad \gamma \mapsto \langle(\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*), (\gamma \wedge \rho^*) \vee (\gamma^* \wedge \rho)\rangle$$

and

$$\nu_A\colon \mathrm{Con}^2(A) \to \mathrm{Quord}(A), \quad \langle\alpha, \beta\rangle \mapsto (\alpha \wedge \rho) \vee (\beta \wedge \rho^*).$$

In the above formulas, $\wedge$, $\vee$ and * are taken in the domain of τ_A resp. ν_A.

Theorem 1. *τ is a natural equivalence from the functor Quord to the functor Con^2. The inverse of τ is ν: $Con^2 \to Quord$.*

Proof. In $\mathrm{Quord}(A)$, ρ^* is a complement of ρ, and $\mathrm{Quord}(A)$ is a distributive lattice by [5, Cor. 5.2 and pages 53–54] (cf. also Chajda and Pinus [2, p. 315] for a simpler proof). Therefore the map $\kappa\colon \mathrm{Quord}(A) \to [0, \rho] \times [0, \rho^*]$, $x \mapsto \langle x \wedge \rho, x \wedge \rho^*\rangle$ is a lattice isomorphism with inverse $\kappa'\colon [0, \rho] \times [0, \rho^*] \to \mathrm{Quord}(A)$, $\langle x, y\rangle \mapsto x \vee y$ by Grätzer [6, Thm. 14 on p. 169 plus the remark after it]. Since $(x \vee x^*)^* = x \vee x^*$, $\lambda\colon x \mapsto x \vee x^*$ is $[0, \rho] \to \mathrm{Con}(A)$ map. Consider the map $\lambda'\colon \mathrm{Con}(A) \to [0, \rho]$, $x \mapsto x \wedge \rho$. By distributivity, for $x \in [0, \rho]$ we have $\lambda'(\lambda(x)) = (x \vee x^*) \wedge \rho = (x \wedge \rho) \vee (x^* \wedge \rho) = x \vee (x \wedge \rho^*)^* = x \vee (x \wedge \rho \wedge \rho^*)^* = x \vee 0 = x$, and for $y \in \mathrm{Con}(A) \subseteq \mathrm{Quord}(A)$ we obtain $\lambda(\lambda'(y)) = (y \wedge \rho) \vee (y \wedge \rho)^* = (y \wedge \rho) \vee (y^* \wedge \rho^*) = (y \wedge \rho) \vee (y \wedge \rho^*) = y \wedge (\rho \vee \rho^*) = y \wedge 1 = y$. Since both λ and λ' are monotone, they are reciprocal lattice isomorphisms. Similarly, $\mu\colon [0, \rho^*] \to \mathrm{Con}(A)$, $x \mapsto x \vee x^*$ and $\mu'\colon \mathrm{Con}(A) \to [0, \rho^*]$, $x \mapsto x \wedge \rho^*$ are reciprocal lattice isomorphisms as well. Thus, $\tau_A = (\lambda \times \mu) \circ \kappa$ is a lattice isomorphism with inverse $\kappa' \circ (\lambda' \times \mu') = \nu_A$. Clearly, both τ_A and ν_A preserve the involution operation *. This proves that τ_A and ν_A are isomorphisms and inverses of each other.

To prove that τ is a natural transformation, let $\varphi\colon A \to B$ a surjective lattice homomorphism, and let φ_q and φ_c denote $\mathrm{Quord}(\varphi)$ and $\mathrm{Con}^2(\varphi)$, respectively. We have to show that the following diagram

$$\begin{array}{ccc} \mathrm{Quord}(B) & \xrightarrow{\varphi_q} & \mathrm{Quord}(A) \\ {\scriptstyle \tau_B}\big\downarrow & & \big\downarrow{\scriptstyle \tau_A} \\ \mathrm{Con}^2(B) & \xrightarrow{\varphi_c} & \mathrm{Con}^2(A) \end{array}$$

commutes. Let $\gamma \in \mathrm{Quord}(B)$ and $\delta := \varphi_q(\gamma)$; we have to show that φ_c sends $\tau_B(\gamma) = \langle(\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*), (\gamma \wedge \rho^*) \vee (\gamma^* \wedge \rho)\rangle$ to $\tau_A(\delta) = \langle(\delta \wedge \rho) \vee (\delta^* \wedge \rho^*), (\delta \wedge \rho^*) \vee (\delta^* \wedge \rho)\rangle$. This means that, for any $x, y \in A$, $\langle x, y\rangle \in (\delta \wedge \rho) \vee (\delta^* \wedge \rho^*)$ iff $\langle \varphi(x), \varphi(y)\rangle \in (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*)$. The details for the second components are analogous and will be omitted. Suppose $\langle \varphi(x), \varphi(y)\rangle \in (\gamma \wedge \rho) \vee (\gamma^* \wedge \rho^*)$. Then there is an $n \ge 1$ and there are elements $b_0 = \varphi(x), b_1, b_2, \dots, b_{2n} = \varphi(y)$ in B such that $\langle b_i, b_{i+1}\rangle \in \gamma \wedge \rho$ for i even and $\langle b_i, b_{i+1}\rangle \in \gamma^* \wedge \rho^*$ for i odd, $i < 2n$.

Let $a'_0 = x$, $a'_{2n} = y$, and for $i = 1, \dots, 2n-1$ let $a'_i \in A$ such that $\varphi(a'_i) = b_i$. Put $a_i = a'_i$ for i even and $a_i = a'_i \vee a'_{i-1} \vee a'_{i+1}$ for i odd. For i odd we obtain $\varphi(a_i)=\varphi(a'_i) \vee \varphi(a'_{i-1}) \vee \varphi(a'_{i+1}) = b_i \vee b_{i-1} \vee b_{i+1} = b_i$, whence $\varphi(a_i) = b_i$ holds for all i. Since $\langle a_i, a_{i+1}\rangle \in \delta \wedge \rho$ for i even and $\langle a_i, a_{i+1}\rangle \in \delta^* \wedge \rho^*$ for i odd, we conclude $\langle x, y\rangle = \langle a_0, a_{2n}\rangle \in (\delta \wedge \rho) \vee (\delta^* \wedge \rho^*)$. Since the proof of the converse implication is straightforward we have shown that τ is a natural transformation. Thus both τ and its inverse ν are natural equivalences. □

Remark 1. The fact that $\mathrm{Quord}(A)$ and $\mathrm{Con}^2(A)$ are isomorphic via τ_A and ν_A was proved in [8]. Although [8] and, up to the authors' best knowledge, $\mathrm{Quord}(A) \cong \mathrm{Con}^2(A)$ have never appeared in print, [8] and the following corollary from it were cited in Rosenberg and Schweigert [7]. Our lattice theoretic proof of $\mathrm{Quord}(A) \cong \mathrm{Con}^2(A)$ is much simpler than [8] and offers an easier approach to Corollary 1.

Corollary 1. *([8], [7], for finite lattices [4]) Every compatible (partial) order γ of a lattice A is induced by a subdirect representation of A as a subdirect product of A_1 and A_2 such that $\langle x, y\rangle \in \gamma$ iff $x_1 \le y_1$ in A_1 and $x_2 \ge y_2$ in A_2. Conversely, any relation derived from a subdirect decomposition this way is a compatible ordering of A.*

Proof. For $\langle \alpha, \beta\rangle \in \mathrm{Con}^2(A)$, suppose $\nu_A(\langle \alpha, \beta\rangle)$ is an ordering. Then $\nu_A(\langle \alpha, \beta\rangle) \wedge \nu_A(\langle \alpha, \beta\rangle)^* = 0$. Computing by distributivity and using $\alpha^* = \alpha$, $\beta^* = \beta$ we obtain $0 = \nu_A(\langle \alpha, \beta\rangle) \wedge \nu_A(\langle \alpha, \beta\rangle)^* = ((\alpha \wedge \rho) \vee (\beta \wedge \rho^*)) \wedge ((\alpha \wedge \rho) \vee (\beta \wedge \rho^*))^* = ((\alpha\wedge\rho)\vee(\beta\wedge\rho^*))\wedge((\alpha\wedge\rho^*)\vee(\beta\wedge\rho)) = (\alpha\wedge\rho\wedge\rho^*)\vee(\alpha\wedge\beta\wedge\rho^*)\vee(\alpha\wedge\beta\wedge\rho)\vee(\beta\wedge\rho\wedge\rho^*) = (\alpha \wedge 0) \vee (\alpha \wedge \beta \wedge (\rho \vee \rho^*)) \vee (\beta \wedge 0) = \alpha \wedge \beta \wedge 1 = \alpha \wedge \beta$. From $\alpha \wedge \beta = 0$ we infer that A is a subdirect product of $A_1 = A/\beta$ and $A_2 = A/\alpha$. Since $\nu_A(\langle \alpha, \beta\rangle) = 0 \vee 0 \vee (\alpha \wedge \rho) \vee (\beta \wedge \rho^*) = (\alpha \wedge \beta) \vee (\rho \wedge \rho^*) \vee (\alpha \wedge \rho) \vee (\beta \wedge \rho^*) = (\alpha \vee \rho^*) \wedge (\beta \vee \rho)$, it is not hard to see that $\nu_A(\langle \alpha, \beta\rangle)$ is induced by this subdirect decomposition. The rest of the corollary is evident. □

Remark 2. We have proved a bit more than stated. Let $\mathcal{L}$ be a variety with two distinguished binary terms $\vee$ and $\wedge$ in its language such that the reduct $\langle A; \vee, \wedge\rangle$ is a lattice for each $A \in \mathcal{L}$ and all basic operations of A are monotone with respect to the natural ordering of this lattice. E.g., $\mathcal{L}$ can be the variety of lattice-ordered semigroups or that of involution lattices. Then Theorem 1 and, for $A \in \mathcal{L}$, Corollary 1 are still valid.

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