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Accumulation points of sublattice densities of lattice varieties

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Abstract: The *sublattice density* $\mathrm{sd}(L)$ of a finite lattice L is defined to be $|\mathrm{Sub}(L)|/2^{|L|}$. For a nontrivial variety $\mathcal{W}$ of lattices, let $\mathrm{SSD}(\mathcal{W})$ denote the set of all sublattice densities of finite lattices belonging to $\mathcal{W}$. We prove that $\mathrm{SSD}(\mathcal{W})$ is a countably infinite dually well-ordered monoid. Furthermore, $\mathrm{SSD}(\mathcal{W})$ has either countably infinitely many accumulation points or exactly one. We show that $\mathrm{SSD}(\mathcal{W})$ has exactly one accumulation point if and only if $\mathcal{W}$ consists of modular lattices but does not contain an infinite modular lattice of length two. As a corollary, it follows that the set $\{\mathrm{SSD}(\mathcal{W}) : \mathcal{W}$ is a variety of modular lattices$\}$ is infinite.

Keywords: Sublattice density, subalgebra density, accumulation point, dually well-ordered set, modular lattice.

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Научная статья

Точки накопления плотностей подрешёток многообразий решёток

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Аннотация: Подрешёточная плотность $\mathrm{sd}(L)$ конечной решётки L определяется как $|\mathrm{Sub}(L)|/2^{|L|}$. Для нетривиального многообразия $\mathcal{W}$ решёток обозначим через $\mathrm{SSD}(\mathcal{W})$ множество всех подрешёточных плотностей конечных решёток, принадлежащих $\mathcal{W}$. Мы доказываем, что $\mathrm{SSD}(\mathcal{W})$ является счётно бесконечным двояко вполне упорядоченным моноидом. Более того, $\mathrm{SSD}(\mathcal{W})$ имеет либо счётно бесконечно много точек накопления, либо ровно одну. Мы показываем, что $\mathrm{SSD}(\mathcal{W})$ имеет ровно одну точку накопления тогда и только тогда, когда $\mathcal{W}$ состоит из модульных решёток, но не содержит бесконечной модульной решётки длины два. В качестве следствия получаем, что множество $\{\mathrm{SSD}(\mathcal{W}) : \mathcal{W}$ — многообразие модульных решёток$\}$ является бесконечным.

Ключевые слова: Плотность подрешёток, плотность подалгебр, точка накопления, двояко вполне упорядоченное множество, модульная решётка

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Dedication

This paper is dedicated to George Grätzer on the occasion of his ninetieth birthday, with appreciation, admiration, and in friendship.

1. Introduction and results

For a lattice $L = (L; \vee, \wedge)$, the *sublattice lattice* of L is denoted by $\mathrm{Sub}(L)$. It consists of the sublattices of L and $\emptyset$. The *sublattice density* $\mathrm{sd}(L)$ of a finite lattice L is $\mathrm{sd}(L) := |\mathrm{Sub}(L)|/2^{|L|}$; it belongs to the real interval $(0,1]_{\mathbb{R}} := \{x \in \mathbb{R} : 0 < x \le 1\}$. Studying $|\mathrm{Sub}(L)| = 2^{|L|} \cdot \mathrm{sd}(L)$ is essentially equivalent to studying $\mathrm{sd}(L)$, but $\mathrm{sd}(L)$ has better properties. A variety $\mathcal{W}$ of lattices is *nontrivial* if it contains a *nontrivial* (non-singleton) lattice. The *set of sublattice densities* of such a $\mathcal{W}$ is defined by

$$\mathrm{SSD}(\mathcal{W}) := \{\mathrm{sd}(L) : L \text{ is a finite lattice belonging to } \mathcal{W}\}. \tag{1.1}$$

The largest element of $\mathrm{SSD}(\mathcal{W})$ is $1 = \mathrm{sd}(C)$, where C is a finite chain. For the variety $\mathcal{L}_{\mathrm{all}}$ of all lattices and for $k \in \{2, 3, \dots, 7\}$, the kth largest value in $\mathrm{SSD}(\mathcal{L}_{\mathrm{all}})$ and the finite lattices witnessing these values were determined by Czédli and Horváth [6], Ahmed and Horváth [1], and Zaja, Haje, and Ahmed [12]. Czédli [3] proves that $\mathrm{sd}(L) > 83/256$ implies the planarity of a finite lattice L, but $\mathrm{sd}(L) \ge 83/256$ does not imply it.

For a subset X of $\mathbb{R}$, the order and multiplication are always the order and multiplication of the real numbers. In particular, a subset X of $\mathbb{R}$ is *dually well-ordered* if there is no strictly increasing infinite sequence of elements of X; equivalently, each nonempty subset of X has a greatest element. With the notation (1.1), we have the following theorem.

Theorem 1. *For every nontrivial variety $\mathcal{W}$ of lattices, $\mathrm{SSD}(\mathcal{W})$ is a dually well-ordered monoid.*

Let $\mathbb{N} := \{1, 2, 3, \dots\}$ denote the set of positive integers, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a cardinal number $\kappa \geq 2$, let M_κ stand for the $(\kappa+2)$-element lattice that consists of κ atoms, a bottom element 0, and a top element 1. In other words, M_κ is the $(\kappa+2)$-element modular lattice of length 2. Mostly, $\kappa = k \in \mathbb{N}$. For $\kappa = \aleph_0$, the least infinite cardinal, we usually write M_ω instead of $M_{\aleph_0}$; here ω is the least infinite ordinal number. For $3 \leq k \leq \omega$, the variety generated by M_k is denoted by $\mathcal{M}_k$, and $\mathcal{N}_5$ stands for the variety generated by the five-element nonmodular lattice N_5. A real number r is an *accumulation point* of a subset X of $\mathbb{R}$ if, for every positive $\varepsilon \in \mathbb{R}$, $(r-\varepsilon, r+\varepsilon)_{\mathbb{R}} \setminus \{r\} \cap X \neq \emptyset$. For $X \subseteq \mathbb{R}$, $\mathrm{Acc}(X)$ denotes the *set of accumulation points* of X. The *topological closure* of X is $X \cup \mathrm{Acc}(X)$.

Theorem 2. *For every nontrivial variety $\mathcal{W}$ of lattices, $|\mathrm{SSD}(\mathcal{W})| = \aleph_0$, $0 \in \mathrm{Acc}(\mathrm{SSD}(\mathcal{W}))$, and $|\mathrm{Acc}(\mathrm{SSD}(\mathcal{W}))| \in \{1, \aleph_0\}$. Furthermore, we have $|\mathrm{Acc}(\mathrm{SSD}(\mathcal{W}))| = \aleph_0$ if and only if $\mathcal{N}_5 \subseteq \mathcal{W}$ or $\mathcal{M}_\omega \subseteq \mathcal{W}$.*

Theorem 1 has the following corollary.

Corollary 1. *For each nontrivial variety $\mathcal{W}$ of lattices, the topological closure of $\mathrm{SSD}(\mathcal{W})$ is a countably infinite dually well-ordered monoid, and the accumulation points of $\mathrm{SSD}(\mathcal{W})$ form a subsemigroup of this monoid.*

The set $\mathrm{SCD}(\mathcal{W})$ of *congruence* densities of finite members of a lattice variety $\mathcal{W}$ is also interesting. Czédli [5] proves only that there exist at least two distinct sets $\mathrm{SCD}(\mathcal{W})$. In contrast, here we prove that the family $\{\mathrm{SSD}(\mathcal{M}_k) : 3 \leq k \in \mathbb{N}\}$ is infinite. Interestingly, our tools seem insufficient to find any concrete $k, n \in \mathbb{N}$ with $3 \leq k < n$ and $\mathrm{SSD}(\mathcal{M}_k) \neq \mathrm{SSD}(\mathcal{M}_n)$.

Corollary 2. *Among the sets $\mathrm{SSD}(\mathcal{M}_k)$, $3 \leq k \in \mathbb{N}$, there are infinitely many distinct ones.*

2. Lemmas and proofs

For finite lattices L_1 and L_2, the ordinal sum $L_1 \oplus L_2$ is obtained by taking the disjoint union of L_1 and L_2 and adding the relations $x_1 < x_2$ for all $x_1 \in L_1$ and $x_2 \in L_2$. Visually, this corresponds to placing the diagram of L_2 atop the diagram of L_1. Sometimes, we write $\emptyset \oplus L$ or $L \oplus \emptyset$, each of which is L. The following four lemmas are easy.

Lemma 1. *For finite lattices L_1 and L_2, $\mathrm{sd}(L_1 \oplus L_2) = \mathrm{sd}(L_1) \cdot \mathrm{sd}(L_2)$.*

Proof. Since the functions

$$f_1 \colon \mathrm{Sub}(L_1 \oplus L_2) \to \mathrm{Sub}(L_1) \times \mathrm{Sub}(L_2), \quad X \mapsto (X \cap L_1, X \cap L_2),$$
$$f_2 \colon \mathrm{Sub}(L_1) \times \mathrm{Sub}(L_2) \to \mathrm{Sub}(L_1 \oplus L_2), \quad (X_1, X_2) \mapsto X_1 \cup X_2$$

are reciprocal bijections, $|\mathrm{Sub}(L_1 \oplus L_2)| = |\mathrm{Sub}(L_1)| \cdot |\mathrm{Sub}(L_2)|$. Hence

$$\mathrm{sd}(L_1 \oplus L_2) = \frac{|\mathrm{Sub}(L_1 \oplus L_2)|}{2^{|L_1 \oplus L_2|}} = \frac{|\mathrm{Sub}(L_1)| \cdot |\mathrm{Sub}(L_2)|}{2^{|L_1|} \cdot 2^{|L_2|}} = \mathrm{sd}(L_1) \cdot \mathrm{sd}(L_2).$$

□

Lemma 2. *If S is a sublattice of a finite lattice L, then* $\mathrm{sd}(L) \le \mathrm{sd}(S)$.

Proof. Each $X \in \mathrm{Sub}(L)$ is determined by $X_1 := X \cap S \in \mathrm{Sub}(S)$ and $X_2 := X \cap (L \setminus S)$. Since X_2 can be chosen in at most $2^{|L \setminus S|} = 2^{|L|-|S|}$ ways and X_1 in at most $|\mathrm{Sub}(S)|$ ways, $|\mathrm{Sub}(L)| \le 2^{|L|-|S|} \cdot |\mathrm{Sub}(S)|$. Thus

$$\mathrm{sd}(L) = \frac{|\mathrm{Sub}(L)|}{2^{|L|}} \le \frac{|\mathrm{Sub}(S)| \cdot 2^{|L|-|S|}}{2^{|L|}} = \frac{|\mathrm{Sub}(S)|}{2^{|S|}} = \mathrm{sd}(S). \qquad \square$$

For a finite lattice L, the *set of join-reducible elements* (that is, the elements with at least two lower covers) is denoted by $\mathrm{Jr}(L)$. Dually, $\mathrm{Mr}(L)$ stands for the *set of meet-reducible elements*. We define the *reducible part* $\mathrm{Rp}(L)$ and the *irreducible part* $\mathrm{Ip}(L)$ of L by $\mathrm{Rp}(L) := \mathrm{Jr}(L) \cup \mathrm{Mr}(L)$ and $\mathrm{Ip}(L) := L \setminus \mathrm{Rp}(L)$. Observe that

$$\mathrm{Rp}(L) \in \mathrm{Sub}(L). \tag{2.1}$$

Indeed, for $x, y \in \mathrm{Rp}(L)$, if x and y are *comparable* (in notation, $x \nparallel y$), then $x \vee y, x \wedge y \in \{x, y\} \subseteq \mathrm{Rp}(L)$; if they are *incomparable* (in notation, $x \parallel y$), then $x \vee y \in \mathrm{Jr}(L) \subseteq \mathrm{Rp}(L)$ and $x \wedge y \in \mathrm{Mr}(L) \subseteq \mathrm{Rp}(L)$.

Lemma 3. *Let L be a nonsingleton finite lattice, let $v \in \mathrm{Ip}(L)$, and denote the sublattice $L \setminus \{v\}$ by L'. If there is a $w \in L$ with $v \parallel w$, then* $\mathrm{sd}(L) < \mathrm{sd}(L')$. *If v is comparable with all elements of L, then* $\mathrm{sd}(L) = \mathrm{sd}(L')$.

Proof. Let $n := |L|$. First, assume that $v \parallel w$ for some $w \in L$. There are exactly two kinds of members of $\mathrm{Sub}(L)$: (1) sublattices $X \in \mathrm{Sub}(L')$, and (2) sublattices of the form $Y \cup \{v\}$ with $Y \in \mathrm{Sub}(L')$. We can choose X in $|\mathrm{Sub}(L')| = 2^{n-1}\mathrm{sd}(L')$ ways, but Y can be chosen in fewer ways, since Y cannot be $\{w\}$. Hence $\mathrm{sd}(L) < 2 \cdot 2^{n-1}\mathrm{sd}(L')/2^n = \mathrm{sd}(L')$, as required.

Second, if v is comparable with all elements of L, then any $Y \in \mathrm{Sub}(L')$ is allowed. Hence '$<$' above turns into '$=$'. □

For a lattice K and $u \in K$, let $\mathrm{idl}(u)$ (or $\mathrm{idl}_K(u)$) denote the *ideal* $\{x \in K : x \le u\}$, and let $\mathrm{fil}(u)$ (or $\mathrm{fil}_K(u)$) stand for the *filter* $\{x \in K : u \le x\}$.

Lemma 4. *If a finite lattice L is not a chain, then it has a sublattice S such that $0_S \in \mathrm{Mr}(S)$, $1_S \in \mathrm{Jr}(S)$, and* $\mathrm{sd}(S) = \mathrm{sd}(L)$.

Proof. Since L is not a chain, $\mathrm{Rp}(L) \ne \emptyset$. Hence $\mathrm{Rp}(L)$ is a sublattice by (2.1). By its finiteness, $u := 0_{\mathrm{Rp}(L)}$ and $v := 1_{\mathrm{Rp}(L)}$ exist. Let S be the interval $[u, v]_L$ of L. If $x \in \mathrm{idl}_L(u)$, then there is no $y \in L$ with $x \parallel y$, since otherwise $u < x \wedge y \in \mathrm{Mr}(L) \subseteq \mathrm{Rp}(L)$ would contradict $u = 0_{\mathrm{Rp}(L)}$. Thus $C := \mathrm{idl}_L(u) \setminus \{u\}$ is a chain or $\emptyset$, and $L = C \oplus \mathrm{fil}_L(u)$. Dually, $C' := \mathrm{fil}_L(v) \setminus \{v\}$ is a chain or empty, and $L = \mathrm{idl}_L(v) \oplus C'$. Hence $L = C \oplus [u, v]_L \oplus C'$. Therefore, since the sublattice density of any chain is 1, Lemma 1 implies $\mathrm{sd}(S) = \mathrm{sd}(L)$, completing the proof. □

Next, assume that L is a finite lattice *with reducible top and bottom*, that is, $0_L \in \mathrm{Mr}(L)$ and $1_L \in \mathrm{Jr}(L)$. Then $0_L, 1_L \in \mathrm{Rp}(L)$. Hence, for each $x \in L$, $\mathrm{fil}_L(x) \cap \mathrm{Rp}(L)$ has a smallest element and $\mathrm{idl}_L(x) \cap \mathrm{Rp}(L)$ a largest element; these elements are defined and denoted as follows:

$$x^{\mathrm{up}} := \bigwedge\bigl(\mathrm{fil}_L(x) \cap \mathrm{Rp}(L)\bigr) \quad\text{and}\quad x_{\mathrm{dn}} := \bigvee\bigl(\mathrm{idl}_L(x) \cap \mathrm{Rp}(L)\bigr).$$

Note that for $x \in \mathrm{Rp}(L)$ and $y \in L \setminus \mathrm{Rp}(L)$,

$$x_{\mathrm{dn}} = x = x^{\mathrm{up}} \quad\text{and}\quad y_{\mathrm{dn}} < y < y^{\mathrm{up}}. \tag{2.2}$$

Let us agree that, for $a < b$ in L and $Y \subseteq L$, we use the notations

$$(a, b)_Y := \{x \in Y : a < x < b\} \text{ and } (a, b]_Y := \{x \in Y : a < x \le b\},$$

and similarly for $[a, b)_Y$. To avoid confusion with the pair (a, b), we never drop the subscript from $(a, b)_Y$. Of course, Y can be L or $\mathrm{Ip}(L)$.

For a lattice K, we denote $\{(a, b) \in K^2 : a < b\}$ by $\mathrm{Sr}(K)$; the acronym comes from '*strict relation*'. Letting $S := \mathrm{Rp}(L)$, (2.2) implies that

$$L = S \,\dot\cup \dot{\bigcup_{(a,b)\in \mathrm{Sr}(S)}} (a, b)_{\mathrm{Ip}(L)}, \tag{2.3}$$

where the dots above $\cup$ and $\bigcup$ indicate disjoint unions. We say that a finite *poset* (partially ordered set) P is a *forest of chains* if it is the union $C_1 \cup \dots \cup C_k$ of chains for some $k \in \mathbb{N}$ such that for all $x \in C_i$ and $y \in C_j$ with $1 \le i < j \le k$, $x \parallel y$. Here $C_1, \dots, C_k$ are the *maximal chains* of P, and we always assume that $|C_1| \ge |C_2| \ge \dots \ge |C_k|$. Define

$$\text{the sequence } \Phi(P) := (|C_1|, \dots, |C_k|) \in \mathbb{N}^k; \tag{2.4}$$

we treat sequences as vectors. For P, a forest of chains, we have that

$$P \text{ is determined, up to isomorphism, by } \Phi(P). \tag{2.5}$$

We define the following two sets of sequences:

$$\begin{aligned} &{\mathbb{N}_0}^* := \{(a_1, \dots, a_k) \in {\mathbb{N}_0}^k : \ k \in \mathbb{N}_0\} \ \text{ and} \\ &\mathbb{N}^{*,\downarrow} := \{(a_1, \dots, a_k) \in \mathbb{N}^k : \ k \in \mathbb{N}_0 \text{ and } a_1 \ge a_2 \ge \dots \ge a_k\}. \end{aligned} \tag{2.6}$$

Note that $\mathbb{N}^{*,\downarrow} \subseteq {\mathbb{N}_0}^*$, and that both ${\mathbb{N}_0}^*$ and $\mathbb{N}^{*,\downarrow}$ contain the *empty sequence* (). Based on (2.3), (2.4), and (2.6), we define the function

$$F_L\colon \mathrm{Sr}(S) \to \mathbb{N}^{*,\downarrow}, \quad (a,b) \mapsto \Phi((a,b)_{\mathrm{Ip}(L)}). \tag{2.7}$$

For $x = (x_1, \dots, x_k) \in \mathbb{N}^{*,\downarrow}$, let $\mathrm{len}(x) := k$ denote the *length* of x. We define a partial order on $\mathbb{N}^{*,\downarrow}$ by stipulating that for $x, y \in \mathbb{N}^{*,\downarrow}$,

$$x \le y \overset{\text{def}}{\iff} \big(\mathrm{len}(x) \le \mathrm{len}(y) \text{ and } x_i \le y_i \text{ for } \forall i \in \{1, \dots, \mathrm{len}(x)\}\big). \tag{2.8}$$

For functions f and g with a common domain, $f \le g$ means that $f(x) \le g(x)$ for all x in that domain. In particular, if K and L are finite lattices and $S = \mathrm{Rp}(K) = \mathrm{Rp}(L)$, then $F_K \le F_L$ means that $F_K(a,b) \le F_L(a,b)$ for all $(a,b) \in \mathrm{Sr}(S)$. Note that we write $F_L(a,b)$ rather than of $F_L((a,b))$.

Lemma 5. *Each finite lattice L with reducible top and bottom is determined, up to isomorphism, by $S := \mathrm{Rp}(L)$ and the function $F_L\colon \mathrm{Sr}(S) \to \mathbb{N}^{*,\downarrow}$. Furthermore, if K and L are finite lattices with reducible top and bottom such that $\mathrm{Rp}(K) = \mathrm{Rp}(L) = S$, then $F_K \le F_L$ implies that K is isomorphic to a sublattice of L.*

Proof. It follows from (2.5) that F_L determines the (Hasse) diagram of L. Therefore F_L determines L up to isomorphism.

Next, assume that $S = \mathrm{Rp}(K) = \mathrm{Rp}(L)$ and $F_K \le F_L$. Observe that if we omit one or more elements of $\mathrm{Ip}(L)$, then what remains is a sublattice. Now, let $(a,b) \in \mathrm{Sr}(S)$. Since $F_K \le F_L$,

$$x = (x_1, \dots, x_n) := F_K(a,b) \le F_L(a,b) =: y = (y_1, \dots, y_m),$$

By definition, $n \le m$ and, for $i \in \{1, \dots, n\}$, $x_i \le y_i$. Choose notation so that $C_1, \dots, C_n$ are the maximal chains of $(a,b)_{\mathrm{Ip}(K)}$, $|C_i| = x_i$ for $i \in \{1, \dots, n\}$, $D_1, \dots, D_m$ are the maximal chains of $(a,b)_{\mathrm{Ip}(L)}$, and $|D_j| = y_j$ for $j \in \{1, \dots, m\}$. For each $i \in \{1, \dots, n\}$, let us omit $y_i - x_i$ elements from D_i, and let us omit all elements of D_{n+1}, …, D_m. Then we obtain a sublattice L_1 of L, and $F_{L_1}(a,b) = x = F_K(a,b)$. Repeating the procedure for each $(c,d) \in \mathrm{Sr}(S) \setminus \{(a,b)\}$ in place of (a,b), finally we obtain a sublattice L' with $F_{L'} = F_K$. Since F_K determines K up to isomorphism, it follows that K is isomorphic to L', completing the proof. □

A quasiorder ρ on a set U is a reflexive and transitive relation on U. In this case, the structure $(U; \rho)$ is a *quasiordered set*. If ρ is also antisymmetric, then $(U; \rho)$ is a *poset*. We usually write $\le$ instead of ρ, even when it is not antisymmetric, and we often write U instead of $(U; \rho)$. For $x, y \in U$, $x < y$ denotes that $x \le y$ and $y \not\le x$. The elements $x, y \in U$ are *incomparable* (in notation, $x \parallel y$) if $x \not\le y$ and $y \not\le x$. A sequence $(x_1, x_2, x_3, \dots)$ of elements of U is *strictly decreasing* if $x_i > x_{i+1}$ for all $i \in \mathbb{N}$. An *antichain* in U is a subset Y of U such that $x \parallel y$ for all distinct $x, y \in Y$. A quasiordered set U is *well-quasiordered* (*wqo*, for short), if it has no infinite strictly decreasing sequence and every antichain in U is finite. The following statement is known from several publications including Gallier [7, Lemma 2.4], Harzheim [9, Theorem 2.3] and Higman [10, Theorem 2.1(v)–(vi)]; at the time of writing, [7] is freely available online.

Lemma 6. *A quasiordered set U is wqo if and only if for every infinite sequence $(x_1, x_2, x_3, \dots)$ in U, there exist $i, k \in \mathbb{N}$ such that $i < k$ and $x_i \le x_k$.*

Lemma 7. $\mathbb{N}^{*,\downarrow}$*, equipped with the order defined in* (2.8)*, is a wqo.*

Proof. For a quasiordered set $A = (A; \le)$, let A^* denote the set of finite sequences formed from elements of A. Define an order $\le'$ on A^* as follows: for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ in A^*, $x \le' y$ if there are subscripts $j_1, \dots, j_n$ such that $1 \le j_1 < j_2 < \dots < j_n \le m$, $x_1 \le y_{j_1}$, $x_2 \le y_{j_2}$, $\dots$, $x_n \le y_{j_n}$. Theorem 4.3 of Higman [10] (proved again in Gallier [7, Theorem 3.2] and Harzheim [9, Theorem 6.3]) asserts that

$$\text{if } (A; \le) \text{ is wqo, then so is } (A^*; \le'). \tag{2.9}$$

Let $(x^{(1)}, x^{(2)}, x^{(3)}, \dots)$ be an infinite sequence in $\mathbb{N}^{*,\downarrow}$. For $i \in \mathbb{N}$, we may write $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)})$, where $m_i := \operatorname{len}(x^{(i)})$. Since $\mathbb{N}_0$ with its natural order is a wqo, so is $({\mathbb{N}_0}^*; \le')$ by Higman's (2.9). Thus, by Lemma 6, there exist $i, k \in \mathbb{N}$ with $i < k$ and $x^{(i)} \le' x^{(k)}$. Hence there are subscripts $j_1, \dots, j_{m_i}$ such that $1 \le j_1 < j_2 < \dots < j_{m_i} \le m_k$ and $x_t^{(i)} \le x_{j_t}^{(k)}$ for all $t \in \{1, \dots, m_i\}$. Therefore, since $t \le j_t$ and $x^{(k)} \in \mathbb{N}^{*,\downarrow}$, we obtain $x_t^{(i)} \le x_{j_t}^{(k)} \le x_t^{(k)}$ for all $t \in \{1, 2, \dots, m_i\}$. Thus $x^{(i)} \le x^{(k)}$. Hence $(\mathbb{N}^{*,\downarrow}; \le)$ is wqo by Lemma 6. □

For a set X and a quasiordered set $Y = (Y; \le)$, let $\operatorname{Hom}(X, Y)$ denote the quasiordered set consisting of all functions $X \to Y$ and quasiordered as follows: for $f, g \in \operatorname{Hom}(X, Y)$, $f \le g$ means that $f(z) \le g(z)$ for all $z \in X$.

Lemma 8. *For any finite set X,* $\operatorname{Hom}(X, \mathbb{N}^{*,\downarrow})$ *is wqo.*

Proof. For $k \in \mathbb{N}$, Higman [10, Theorem 2.3] (see also Gallier [7, Lemma 2.6] or Harzheim [9, Theorem 2.10 in Chapter 8]) asserts that if $U_1, \dots, U_k$ are wqo sets, then so is their direct product. Since $\operatorname{Hom}(X, \mathbb{N}^{*,\downarrow})$ is (isomorphic to) the direct product of $|X|$ copies of $\mathbb{N}^{*,\downarrow}$, Lemma 7 and Higman's result just mentioned imply Lemma 8. □

Lemma 9. *Let $\mathcal{W}$ be a nontrivial variety of lattices. Then $\mathcal{W}$ is closed under forming ordinal sums of finite lattices belonging to $\mathcal{W}$.*

Proof. Since $L_1 \oplus L_2$ is isomorphic to $S := \{(x, 0, 0) : x \in L_1\} \cup \{(1, 1, y) : y \in L_2\}$, and S is a sublattice of $L_1 \times L_1 \times L_2$, the lemma follows. □

Lemma 10. *Let $\mathcal{W}$ be a variety of lattices. Then $M_\omega \in \mathcal{W}$ if and only if $M_k \in \mathcal{W}$ for all $k \in \mathbb{N}$ with $k \geq 3$.*

Proof. Since M_k is a sublattice of M_ω for all $k \in \mathbb{N}$ with $k \geq 3$, the 'only if' part is clear. Conversely, M_ω is a directed union (a special case of directed limits) of the lattices M_k, $3 \leq k \in \mathbb{N}$. This gives the 'if' part, since varieties are closed under direct limits; see, e.g., Grätzer [8, Exercise 3.34]. □

For $k \in \mathbb{N}_0$, a lattice L is of *length* k if L contains a $(k+1)$-element chain but no $(k+2)$-element chain. For a variety $\mathcal{W}$ of lattices, let $\mathrm{Fin}(\mathcal{W})$ denote the *class of finite members* of $\mathcal{W}$.

Lemma 11. $\mathrm{Fin}(\mathcal{M}_\omega) = \bigcup_{3 \leq k \in \mathbb{N}} \mathrm{Fin}(\mathcal{M}_k)$.

Proof. The '$\supseteq$' part is trivial. To show the converse inclusion, note that there exists a first-order formula Φ such that, for any lattice L, L satisfies Φ if and only if L is a modular lattice of length 2.

Assume that $L \in \mathrm{Fin}(\mathcal{M}_\omega)$ and $|L| > 1$. Then L is a subdirect product of finitely many subdirectly irreducible finite lattices $L_1, \ldots, L_t$. For each $i \in \{1, \ldots, t\}$, L_i is a homomorphic image of a sublattice S_i of an ultrapower K_i of M_ω by Jónsson [11, Corollary 3.2]. Using Φ and Łoś's theorem, it follows that K_i is a modular lattice of length 2. Thus $K_i = M_{\kappa_i}$ for some cardinal κ_i. Therefore S_i is either a distributive lattice of size at most four, or $S_i = M_{\lambda_i}$ for some cardinal λ_i with $3 \leq \lambda_i \leq \kappa_i$. There are two cases. If S_i is distributive, then so is L_i, whereby $L_i \in \mathrm{Fin}(\mathcal{M}_k)$ for all $k \in \mathbb{N} \setminus \{1, 2\}$. Otherwise, $S_i = M_{\lambda_i}$ with $3 \leq \lambda_i \leq \kappa_i$. Then take a surjective homomorphism $\varphi \colon S_i \to L_i$. Since $|L_i| > 1$, the *congruence kernel* $\ker(\varphi) := \{(x, y) \in S_i^2 : \varphi(x) = \varphi(y)\}$ of φ is not $1_{\mathrm{Con}(S_i)}$. Using that $S_i = M_{\lambda_i}$ is a simple lattice and thereby $\mathrm{Con}(S_i) = \{0_{\mathrm{Con}(S_i)}, 1_{\mathrm{Con}(S_i)}\}$, we obtain that $\ker(\varphi) = 0_{\mathrm{Con}(S_i)}$. Hence φ is an isomorphism, and $L_i \cong S_i \cong M_{\lambda_i}$. By finiteness, $L_i \cong M_{k_i}$ for some $k_i := \lambda_i \in \mathbb{N} \setminus \{1, 2\}$. Therefore, in both cases, $L_i \in \mathcal{M}_{k_i}$ for some $k_i \in \mathbb{N} \setminus \{1, 2\}$. Let $k := \max\{k_1, \ldots, k_t\}$. Then $L_i \subseteq \mathcal{M}_{k_i} \subseteq \mathcal{M}_k$ for $i \in \{1, \ldots, t\}$. Thus $L \in \mathcal{M}_k$. In fact, $L \in \mathrm{Fin}(\mathcal{M}_k)$, since L is finite, completing the proof of Lemma 11. □

For $5 \leq k \in \mathbb{N}$, let N_k be the k-element lattice with base set $\{0, 1, a_1, \ldots, a_{k-3}, b\}$ such that the edges (covering pairs) are exactly $0 \prec a_1$, $a_{k-3} \prec 1$, $a_i \prec a_{i+1}$ for $i \in \{1, \ldots, k-4\}$, $0 \prec b$, and $b \prec 1$.

Lemma 12. *If a variety $\mathcal{W}$ of lattices contains N_5, then it contains N_k for all $k \in \mathbb{N}$ with $k \geq 5$.*

Proof. For $6 \leq k \in \mathbb{N}$ and $i \in \{1, \ldots, k-4\}$, let β_i be the congruence of N_k with blocks $\{a_1, \ldots, a_i\}$, $\{a_{i+1}, \ldots, a_{k-3}\}$, $\{0\}$, $\{b\}$, and $\{1\}$. As $N_k/\beta_i \cong N_5$ and $\bigwedge_{i=1}^{k-4} \beta_i = 0_{\mathrm{Con}(N_k)}$, N_k is a subdirect power of N_5, and $N_k \in \mathcal{W}$. □

Lemma 13. *For each positive $r \in \mathbb{R}$, there is a $k = k(r) \in \mathbb{N}$ such that whenever L is a finite lattice with $|\mathrm{Rp}(L)| \geq k$, then $\mathrm{sd}(L) < r$.*

Proof. Let L be a finite lattice, $t \in \mathbb{N}$, and assume that $|\mathrm{Rp}(L)| \geq 6t$. Since $\mathrm{Rp}(L) = \mathrm{Jr}(L) \cup \mathrm{Mr}(L)$, we have $|\mathrm{Jr}(L)| \geq 3t$ or $|\mathrm{Mr}(L)| \geq 3t$. By duality, we may assume $|\mathrm{Jr}(L)| \geq 3t$. Pick $a_1 \in \mathrm{Jr}(L)$, and let $b_1, c_1 \in L$ be distinct lower covers of a_1. For $i \in \{2, \dots, t\}$ we pick $a_i \in \mathrm{Jr}(L) \setminus \{a_1, b_1, c_1, \dots, a_{i-1}, b_{i-1}, c_{i-1}\}$, and let b_i and c_i be distinct lower covers of a_i. Thus we obtain a $3t$-element subset $\{a_1, b_1, c_1, \dots, a_t, b_t, c_t\}$ of L such that $b_i \vee c_i = a_i \notin \{b_i, c_i\}$ for all $i \in \{1, \dots, t\}$. For $i \in \{1, \dots, t\}$, let $B_i := \{a_i, b_i, c_i\}$, and let $B_0 := L \setminus (B_1 \cup \dots \cup B_t)$.

Denote $|L|$ by n. For any subset X of L, we define $X_i := X \cap B_i$ for $i \in \{0, 1, \dots, t\}$. Call these X_i the *components* of X. Since X is the union of its components, these components determine X. Now let $X \in \mathrm{Sub}(L)$. For $i \in \{1, \dots, t\}$, $X_i \neq \{b_i, c_i\}$, since otherwise $a_i = b_i \vee c_i$ would not belong to X. Hence X_i is one of the seven subsets of B_i that are distinct from $\{b_i, c_i\}$. As $|B_0| = n - 3t$, X_0 is one of the 2^{n-3t} subsets of B_0. Therefore $|\mathrm{Sub}(L)| \leq 7^t \cdot 2^{n-3t}$. Dividing this inequality by 2^n, we obtain that $\mathrm{sd}(L) \leq 7^t/2^{3t} = (7/8)^t$. Since $(7/8)^t \to 0$ as $t \to \infty$, there is a $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$ we have $\mathrm{sd}(L) \leq (7/8)^t < r$. Therefore if $|\mathrm{Rp}(L)| \geq 6t_0 =: k$, then $\mathrm{sd}(L) \leq (7/8)^{t_0} < r$. □

Lemma 14. *Let $X \subseteq [0,1]_\mathbb{R}$ be a dually well-ordered monoid. Then $X \cup \mathrm{Acc}(X)$ is also a dually well-ordered monoid. Furthermore, $\mathrm{Acc}(X)$ is a subsemigroup of $X \cup \mathrm{Acc}(X)$.*

Proof. Suppose, for contradiction, that $(x_1, x_2, x_3, \dots)$ is an infinite strictly increasing sequence of elements of $B := X \cup \mathrm{Acc}(X)$. For $2 \leq n \in \mathbb{N}$, let $\delta_n := \min\{x_n - x_{n-1}, x_{n+1} - x_n\}/3$, and pick an element $y_n \in (x_n - \delta_n, x_n + \delta_n)_\mathbb{R} \cap X$. Then $(y_2, y_3, y_4, \dots)$ is an infinite strictly increasing sequence in X, a contradiction. Thus B is dually well-ordered.

Observe that for each $p \in \mathrm{Acc}(X)$,

$$\text{there exists an infinite sequence in } X \cap (p,1)_\mathbb{R} \text{ converging to } p. \tag{2.10}$$

If no element of X is smaller than p, then (2.10) is trivial. Otherwise, let h be the largest element of $X \cap [0,p)_\mathbb{R}$. As p is an accumulation point, $X \cap \big((h,p)_\mathbb{R} \cup (p,1)_\mathbb{R}\big)$ contains an infinite sequence converging to p. Since $X \cap (h,p)_\mathbb{R} = \emptyset$, this sequence is in $X \cap (p,1)_\mathbb{R}$, proving (2.10).

For $p, q \in \mathrm{Acc}(X)$, choose infinite sequences $(p_1, p_2, p_3, \dots)$ and $(q_1, q_2, q_3, \dots)$ according to (2.10) that converge to p and q, respectively. Since multiplication is continuous, $\lim_{n\to\infty} p_n q_n = pq$. Furthermore, $pq < p_n q_n$ for all $n \in \mathbb{N}$, hence $pq \in \mathrm{Acc}(X)$. Thus $\mathrm{Acc}(X)$ is a semigroup.

Since X is a monoid and $\mathrm{Acc}(X)$ is a semigroup, it suffices to show that $pq \in \mathrm{Acc}(X)$ for all $p \in \mathrm{Acc}(X)$ and $q \in X$ to prove that B is a monoid.

Pick $(p_1, p_2, p_3, \dots)$ according to (2.10). Then $p_n q > pq$ for all $n \in \mathbb{N}$, and $pq = \lim_{n\to\infty} p_n q \in \mathrm{Acc}(X)$, completing the proof of Lemma 14. $\square$

Proof of Theorem 1. Let $\mathcal{W}$ be a nontrivial variety of lattices. Then $\mathcal{W}$ is closed under product by Lemmas 1 and 9, and it contains 1, the sublattice density of every finite chain. Therefore $\mathrm{SSD}(\mathcal{W})$ is a monoid.

Next, for the sake of contradiction, suppose that $\mathrm{SSD}(\mathcal{W})$ fails to be dually well-ordered. Then there is an infinite sequence $x := (x_1, x_2, x_3, \dots)$ with $x_i \in \mathrm{SSD}(\mathcal{W})$ and $x_i < x_{i+1}$ for all $i \in \mathbb{N}$. Clearly, $x_i < 1$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, by Lemma 4 and since $\mathcal{W}$ is closed under taking sublattices, we can pick a lattice $L_i \in \mathcal{W}$ with reducible top and bottom such that $x_i = \mathrm{sd}(L_i)$. With reference to Lemma 13, let $k := k(x_1)$.

Observe that for every $i \in \mathbb{N}$, we have $|\mathrm{Rp}(L_i)| < k$. Indeed, if $|\mathrm{Rp}(L_i)| \geq k$, then Lemma 13 would yield $x_i = \mathrm{sd}(L_i) < x_1$, a contradiction. Hence, up to isomorphism, $\mathrm{Rp}(L_i)$ is one of the finitely many lattices with fewer than k elements. By the pigeonhole principle, there is a lattice S with $|S| < k$ such that $S \cong \mathrm{Rp}(L_i)$ holds for infinitely many $i \in \mathbb{N}$. After deleting the components x_j with $\mathrm{Rp}(L_j) \ncong S$ from the sequence, we may assume that $\mathrm{Rp}(L_i) = S$ for all $i \in \mathbb{N}$. Since $\mathrm{Hom}(S, \mathbb{N}^{*,\downarrow})$ is wqo (by Lemma 8) and F_{L_i} (defined in (2.7)) belongs to it for every $i \in \mathbb{N}$, Lemma 6 provides $i, j \in \mathbb{N}$ such that $i < j$ and $F_{L_i} \leq F_{L_j}$. Thus, by Lemma 5, L_i is isomorphic to a sublattice T_i of L_j. By this isomorphism, $x_i = \mathrm{sd}(L_i) = \mathrm{sd}(T_i)$. Applying Lemma 2, we obtain $x_j = \mathrm{sd}(L_j) \leq \mathrm{sd}(T_i) = x_i$, a contradiction, proving that $\mathrm{SSD}(\mathcal{W})$ is dually well-ordered. $\square$

Proof of Theorem 2. For brevity, let $A(\mathcal{W}) := \mathrm{Acc}(\mathrm{SSD}(\mathcal{W}))$. Since there are only countably many finite lattices, $|\mathrm{SSD}(\mathcal{W})| \leq \aleph_0$. Alternatively, we know from Theorem 1 that $\mathrm{SSD}(\mathcal{W})$ is dually well-ordered, and it is an easy exercise[1] to derive $|\mathrm{SSD}(\mathcal{W})| \leq \aleph_0$ from this fact. Let B_4 denote the four-element Boolean lattice. By Czédli and Horváth [6, Theorem 1] (or trivially), $\mathrm{sd}(B_4) = 13/16$. Since $B_4 \in \mathcal{W}$ and $\mathrm{SSD}(\mathcal{W})$ is a monoid,

$$(13/16)^k \in \mathrm{SSD}(\mathcal{W}) \text{ for all } k \in \mathbb{N} \text{ and } 0 \in A(\mathcal{W}). \tag{2.11}$$

Since (2.11) implies $|\mathrm{SSD}(\mathcal{W})| \geq \aleph_0$, we obtain $|\mathrm{SSD}(\mathcal{W})| = \aleph_0$.

Next, let r be a positive number in $A(\mathcal{W})$. By (2.11), the set $\mathrm{SSD}(\mathcal{W}) \cap (0, r)_{\mathbb{R}}$ is nonempty. Therefore, since $\mathrm{SSD}(\mathcal{W})$ is dually well-ordered, this set has a largest element, which we denote by q_r. Then q_r belongs to $\mathbb{Q}$, the set of rational numbers, since $\mathrm{SSD}(\mathcal{W}) \subseteq \mathbb{Q}$. Now let $r, s \in A(\mathcal{W})$ with $0 < s < r$. Since $(q_r, r)_{\mathbb{R}}$ is disjoint from $\mathrm{SSD}(\mathcal{W})$, it cannot contain accumulation points. Hence $q_s < s \leq q_r < r$, so $q_s < q_r$. Thus the function $A(\mathcal{W}) \setminus \{0\} \to \mathbb{Q}$ defined by $r \mapsto q_r$ is injective, whereby $|A(\mathcal{W})| \leq \aleph_0$.

[1] At the time of writing, see https://math.stackexchange.com/questions/1983425/there-is-no-well-ordered-uncountable-set-of-real-numbers.

Assume that $\mathrm{SSD}(\mathcal{W}) \neq \{0\}$. Pick a positive $r \in A(\mathcal{W})$. Since the largest element of $\mathrm{SSD}(\mathcal{W})$ is 1 and the second largest is 13/16 by Czédli an Horváth [6], $1 \notin A(\mathcal{W})$. Hence $r < 1$. Since multiplication is continuous and $\mathrm{SSD}(\mathcal{W})$ is a monoid, the powers r^k ($k \in \mathbb{N}$) belong to $A(\mathcal{W})$, and they are distinct, because $0 < r < 1$. Thus $|A(\mathcal{W})| \geq \aleph_0$, proving that

$$\big(|A(\mathcal{W})| \neq 1 \text{ or } A(\mathcal{W}) \neq \{0\}\big) \Rightarrow |A(\mathcal{W})| = \aleph_0. \tag{2.12}$$

Next, assume that $\mathcal{N}_5 \subseteq W$. Then, by Lemma 12, $\mathrm{SSD}(\mathcal{W})$ contains $\mathrm{sd}(N_k)$ for all $5 \leq k \in \mathbb{N}$. Using the notation introduced above Lemma 12, every subset of $\{a_1, \dots, a_{k-3}\}$ belongs to $\mathrm{Sub}(N_k)$. Hence $\mathrm{sd}(N_k) \geq 2^{k-3}/2^k = 1/8$. Furthermore, we obtain from Lemma 3 that $1 \geq \mathrm{sd}(N_5) > \mathrm{sd}(N_6) > \mathrm{sd}(N_7) > \dots$, whereby $r := \lim_{k\to\infty} \mathrm{sd}(N_k) \in A(\mathcal{W})$, and $1/8 \leq r < 1$. Thus (2.12) implies $|A(\mathcal{W})| = \aleph_0$, as required.

Assume now that $\mathcal{M}_\omega \subseteq \mathcal{W}$. By Lemma 10 or trivially, $M_k \in \mathcal{W}$ for all $3 \leq k \in \mathbb{N}$. Denoting the atoms of M_k by $a_1, \dots, a_k$, observe that for every subset X of $\{a_1, \dots, a_k\}$, $X \cup \{0, 1\} \in \mathrm{Sub}(M_k)$. Hence $\mathrm{sd}(M_k) \geq 2^k/2^{k+2} = 1/4$. Lemma 3 implies that $1 \geq \mathrm{sd}(M_3) > \mathrm{sd}(M_4) > \mathrm{sd}(M_5) > \dots$, allowing us to change 1/8 to 1/4 in the previous paragraph to obtain $|A(\mathcal{W})| = \aleph_0$.

To complete the proof, assume that $\mathcal{N}_5 \not\subseteq \mathcal{W}$ and $\mathcal{M}_\omega \not\subseteq \mathcal{W}$; we need to show that $A(\mathcal{W}) \cap (0,1)_{\mathbb{R}} = \emptyset$. Suppose the contrary. We have $N_5 \notin \mathcal{W}$ and, by Lemma 10, there is a $k_0 \in \mathbb{N} \setminus \{1, 2\}$ with $M_{k_0} \notin \mathcal{W}$. Pick a positive $r \in A(\mathcal{W})$, and let $(r_1, r_2, r_3, \dots)$ be a sequence in $\mathrm{SSD}(\mathcal{W})$ such that $r/2 \leq r_n \neq r$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = r$. We may assume that

$$\text{for every } n \in \mathbb{N},\ r_n \text{ occurs in the sequence only once.} \tag{2.13}$$

Indeed, otherwise, modify the sequence as follows. Keep r_1, and remove all later terms equal to r_1. Since $r_1 \neq r$ but $\lim_{n\to\infty} r_n = r$, infinitely many terms remain. Next, in the resulting sequence, keep the (new) r_2 and remove all later terms equal to r_2. Repeating this process, we obtain a subsequence of the original one that satisfies (2.13) and still converges to r.

For each $n \in \mathbb{N}$, pick a lattice $L_n \in \mathcal{W}$ of minimum size such that $\mathrm{sd}(L_n) = r_n$. Denote $\mathrm{Ip}(L_n) = L_n \setminus \mathrm{Rp}(L_n)$ by I_n. By Lemma 4, every L_n is a lattice with reducible bounds. Since $\mathrm{sd}(L_n) = r_n \geq r/2$, Lemma 13 rules out $|\mathrm{Rp}(L_n)| \geq k(r/2)$. Hence $|\mathrm{Rp}(L_n)| < k(r/2)$ for all $n \in \mathbb{N}$. Since there are only finitely many lattices with fewer than $k(r/2)$ elements, the pigeonhole principle yields a finite lattice S such that $\mathrm{Rp}(L_n) \cong S$ for infinitely many $n \in \mathbb{N}$. Thus, removing the rest of the lattices L_n from the sequence, we may assume that $\mathrm{Rp}(L_n) = S$ for all $n \in \mathbb{N}$. By Lemma 5, each L_n is determined by the associated function F_{L_n}.

Let $(a, b) \in \mathrm{Sr}(S)$. We cannot have $F_{L_n}(a, b) = (x_1, x_2, \dots, x_t)$ with $t \geq k_0$, since otherwise M_{k_0} would be a sublattice of $[a, b]_{L_n}$ and L_n, and hence $M_{k_0} \in \mathcal{W}$, a contradiction. Thus for every $(a, b) \in \mathrm{Sr}(S)$,

$$\mathrm{len}(F_{L_n}(a, b)) < k_0. \tag{2.14}$$

Observe that

$$\text{if } \operatorname{len}(F_{L_n}(a,b)) > 1, \quad \text{then } \; F_{L_n}(a,b) = (1,1,\dots,1), \tag{2.15}$$

since otherwise N_5 would be a sublattice of $[a,b]_{L_n}$, and thereby N_5 would belong to $\mathcal{W}$. We now show that

$$\text{if } \operatorname{len}(F_{L_n}(a,b)) = 1, \text{ then } F_{L_n}(a,b) = (1). \tag{2.16}$$

Suppose the contrary. We claim that for every $d \in L_n \setminus (a,b)_{I_n}$,

$$\text{if } x < d \text{ for some } x \in (a,b)_{I_n}, \text{ then } y < d \text{ for all } y \in (a,b)_{I_n}. \tag{2.17}$$

To verify this, assume that $x, y \in (a,b)_{I_n}$, $d \in L_n \setminus (a,b)_{I_n}$, and $x < d$; we need to show that $y < d$. This is obvious if $y \le x$. Thus, since $\operatorname{len}(F_{L_n}(a,b)) = 1$ implies that $(a,b)_{I_n}$ is a chain, we may assume that $x < y < b$. As $x < d$, there is an $h \in \mathbb{N}$ and $s_0, \dots, s_h \in L_n$ such that $x = s_0 \prec s_1 \prec s_2 \prec \dots \prec s_h = d$. Every element of $(a,b)_{I_n}$ has only one upper cover and $(a,b)_{I_n}$ is a chain, whereby $s_i = y$ for some $i \in \{1,\dots,h-1\}$. Hence $y = s_i \prec s_{i+1} \prec \dots \prec d$ implies $y < d$, proving (2.17).

As we have assumed that $\operatorname{len}(F_{L_n}(a,b)) = 1$ but $F_{L_n}(a,b) \ne (1)$, there exist $u, v \in (a,b)_{I_n}$ with $u < v$. Suppose there is a $w \in L$ with $v \parallel w$. Since $(a,b)_{I_n}$ is a chain, we have $w \notin (a,b)_{I_n}$. Clearly, $w \not\le u$. We have $u \not\le w$, since otherwise (2.17) would yield $v \le w$. Hence, in addition to $v \parallel w$, we have $u \parallel w$. Let $q := u \vee w$. Since $q \in \operatorname{Jr}(L) \subseteq \operatorname{Rp}(L_n) = S$, we have $q \notin (a,b)_{I_n}$. By $u \parallel w$, we have $w < q$ and $u < q$. Applying (2.17) with (u,v,q) playing the role of (x,y,d), we obtain $v < q$. Hence the inequalities $q = u \vee w \le v \vee w \le q$ yield $v \vee w = q$. Dually, letting $p := v \wedge w$, we obtain $u \wedge w = p$. Hence $\{p,u,v,w,q\}$ is a sublattice isomorphic to N_5, contradicting $N_5 \notin \mathcal{W}$. This shows that there is no $w \in L_n$ with $v \parallel w$. This fact, $v \in I_n = \operatorname{Ip}(L_n)$, and Lemma 3 imply that $L'_n := L_n \setminus \{v\}$ is a sublattice with $\operatorname{sd}(L'_n) = \operatorname{sd}(L_n)$ and $|L'_n| = |L_n| - 1$. This contradicts the minimality of $|L_n|$ when $L_n \in \mathcal{W}$ with $\operatorname{sd}(L_n) = r_n$, and proves (2.16).

If $\operatorname{len}(F_{L_n}(a,b)) = 0$, then $F_{L_n}(a,b) = ()$. Thus, by (2.15) and (2.16), $\operatorname{len}(F_{L_n}(a,b))$ determines $F_{L_n}(a,b)$. Hence (2.14) imply that $F_{L_n}(a,b)$ can take at most k_0 values. Thus the family $\{F_{L_n} : n \in \mathbb{N}\}$ contains at most $k_0^{|\operatorname{Sr}(S)|}$ distinct functions. In particular, it is finite, so there exist $n < m$ in $\mathbb{N}$ with $F_{L_n} = F_{L_m}$. By Lemma 5, $L_n \cong L_m$. Thus $r_n = \operatorname{sd}(L_n) = \operatorname{sd}(L_m) = r_m$, contradicting (2.13) and proving $A(\mathcal{W}) \cap (0,1)_{\mathbb{R}} = \emptyset$. □

Proof of Corollary 1. Let $B := \operatorname{SSD}(\mathcal{W}) \cup \operatorname{Acc}(\operatorname{SSD}(\mathcal{W}))$. Then B is the topological closure in question, and Theorem 2 implies $|B| = \aleph_0$. The rest of Corollary 1 follows from Theorem 1 and Lemma 14. □

Proof of Corollary 2. Since $\mathcal{M}_3 \subseteq \mathcal{M}_4 \subseteq \mathcal{M}_5 \subseteq \dots$, we have that

$$\operatorname{SSD}(\mathcal{M}_3) \subseteq \operatorname{SSD}(\mathcal{M}_4) \subseteq \operatorname{SSD}(\mathcal{M}_5) \subseteq \dots \tag{2.18}$$

Suppose, for contradiction, that only finitely many objects $\mathrm{SSD}(\mathcal{M}_k)$ occur in (2.18). Then there is an $n \in \mathbb{N} \setminus \{1, 2\}$ such that $\mathrm{SSD}(\mathcal{M}_k) \subseteq \mathrm{SSD}(\mathcal{M}_n)$ for all $k \in \mathbb{N} \setminus \{1, 2\}$. Thus, since Lemma 11 implies that $\mathrm{SSD}(\mathcal{M}_\omega)$ is the union of the sets occurring in (2.18), we obtain $\mathrm{SSD}(\mathcal{M}_\omega) = \mathrm{SSD}(\mathcal{M}_n)$. By this equality and Theorem 2, $|\mathrm{Acc}(\mathrm{SSD}(\mathcal{M}_n))| = \aleph_0$. Hence, by Theorem 2 again, $\mathcal{N}_5 \subseteq \mathcal{M}_n$ or $\mathcal{M}_\omega \in \mathcal{M}_n$. But $\mathcal{N}_5 \subseteq \mathcal{M}_n$ is impossible, since $\mathcal{M}_n$ contains only modular lattices. Thus $\mathcal{M}_\omega \subseteq \mathcal{M}_n$, implying $M_{n+1} \in \mathcal{M}_n$. As M_{n+1} is subdirectly irreducible (in fact, simple), Jónsson [11, Corollary 3.4] in this special case asserts that M_{n+1} is a homomorphic image of a sublattice of M_n. This contradicts $|M_{n+1}| > |M_n|$, completing the proof. □

3. Conclusion

For a finite (universal) algebra A in a variety $\mathcal{W}$, the *subalgebra density* $\mathrm{sd}(A)$ of A is the fraction $|\mathrm{Sub}(A)|/\max\{|B| : B \in \mathcal{W} \text{ and } |B| = |A|\}$. We proved results for the sets $\mathrm{SSD}(\mathcal{W}) := \{\mathrm{sd}(A) : A \text{ is finite and } A \in \mathcal{W}\}$ of varieties $\mathcal{W}$ of *lattices* and their accumulation points. Similar results are expected for some varieties of algebras other than lattices, including semilattices, as Czédli [4] and Ahmed, Salih, and Hale [2] suggest.

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