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ON THE UNIQUENESS OF MAL'CEV POLYNOMIALS

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1. INTRODUCTION

In the classical paper [4] [5] A.I. MAL'CEV proved that congruence permutability of a variety T of algebras is equivalent to the existence of a polynomial $(x, y, z)P$ on T, a *Mal'cev (completion of a) parallelogram polynomial*, satisfying the identities $(x,x,z)P = z$ and $(x,z,z)P = x$. For example, if T is the variety of groups, $(x,y,z)P$ may be taken to be $xy^{-1}z$ or $zy^{-1}x$. Some of the theory of varieties with such a polynomial, eponymously called *Mal'cev varieties*, was presented in [9], to which reference is implicit for certain features mentioned in this paper. An important part of the study of a Mal'cev variety T involves the subvariety $Z(T)$ [9, 2.3] consisting of those algebras A in T for which the diagonal $\hat{A} = \{(a,a) \mid a \in A\}$ is a normal subalgebra of the direct square $A \times A$ (i.e. a V-class for

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a congruence V on $A \times A$). For example, if T is the variety of groups, $Z(T)$ is the variety of abelian groups. Now in $Z(T)$ the parallelogram polynomial is unique (as an element of the free algebra in $Z(T)$ on three generators); in the case of groups, commutativity of the multiplication implies for example that $xy^{-1}z = zy^{-1}x$. In [3] the second author raised the question of the converse: does uniqueness of the Mal'cev polynomial in a Mal'cev variety T imply that $T = Z(T)$? For example, in a variety of groups the equality $xy^{-1}z = zy^{-1}x$ becomes the commutative law when the identity is substituted for y. This paper sets out to answer the question. It transpires that a deeper and more detailed analysis of the uniqueness is required than might be thought at first sight. The answer given furnishes uniqueness conditions tailored to the individual variety T sufficient to prove that $T = Z(T)$, and provides an example demonstrating the necessity for such specific conditions.

2. THE MAIN THEOREMS

This section presents the main theorem stating that a Mal'cev variety T is equal to $Z(T)$ when certain uniqueness hypotheses appropriate to T are satisfied. Before introducing the hypotheses it is necessary to consider more closely the various kinds of Mal'cev polynomial. Mal'cev deduced the existence of a parallelogram polynomial $(x,y,z)P$ from the permutability of congruences in the Mal'cev variety T by considering the free algebra $F_X(T)$ in T on the three-element set $X = \{x, y,z\}$. If ζ and φ denote the congruences on $F_X(T)$ which are the respective kernels of the morphisms

$$F_X(T) \to F_{\{x,z\}} \;;\; x \mapsto x,\ y \mapsto x,\ z \mapsto z$$

and

$$F_X(T) \to F_{\{x,z\}} \;;\; x \mapsto x,\ y \mapsto z,\ z \mapsto z$$

then $x\ \zeta\ y\ \varphi\ z$. The permutability of ζ and φ then gives the existence of an element $(x,y,z)P$ such that $x\ \varphi\ (x,y,z)P\ \zeta\ z$. Such an element of $F_X(T)$ is what was defined above as a *parallelogram polynomial*, and T satisfies the identities $(x,x,z)P = z$ and $(x,z,z)P = x$. If $a\ \alpha\ b\ \beta\ c$ for elements a,b,c and congruences α,β of any algebra A in T, then $a\ \beta\ (a,b,c)P\ \alpha\ c$. Suppose now more generally that instead of considering x,y,z as elements of $F_X(T)$ one considers them as elements of $F_{X\cup W(\iota)}(T)$ for an ordinal $\iota \leq \omega$, where $W(\iota)$ is the set $\{w_i \mid i < \iota\}$ of distinct elements disjoint from X. Letting ζ and φ denote the congruences on $F_{X\cup W(\iota)}(T)$ which are the respective kernels of the morphisms

$$F_{X\cup W(\iota)}(T) \to F_{\{x,z\}\cup W(\iota)} \;;$$

$$x \mapsto x,\ y \mapsto x,\ z \mapsto z,\ w_i \mapsto w_i$$

and

$$F_{X\cup W(\iota)}(T) \to F_{\{x,z\}\cup W(\iota)} \;;$$

$$x \mapsto x,\ y \mapsto z,\ z \mapsto z,\ w_i \mapsto w_i$$

$(i < \iota)$, $x\ \zeta\ y\ \varphi\ z$. The permutability of these ζ and φ now gives the existence of an element $P(x,y,z;w_i \mid i < \iota)$ of $F_{X\cup W(\iota)}$ such that $x\ \varphi\ P(x,y,z;w_i \mid i < \iota)\ \zeta\ z$.

Such an element of $F_{X \cup W(\iota)}(T)$ will be called a *Mal'cev polynomial*, for which T satisfies the identities $P(x, x, z; w_i \mid i < \iota) = z$ and $P(x, z, z; w_i \mid i < \iota) = x$. If $a \; \alpha \; b \; \beta \; c$ for elements a, b, c and congruences α, β of any algebra A in T, then $a \; \beta \; P(a, b, c; d_i \mid i < \iota) \; \alpha \; c$ for any set $\{d_i \mid i < \iota\}$ of elements of A. For example,

$$xy^{-1}z[[x, y, w_0], [y, z, w_1]]$$

(where $[a, b, c] = [[a, b], c]$) is a Mal'cev polynomial $P(x, y, z; w_0, w_1)$ in the variety of groups. It will often prove convenient to distinguish between a parallelogram polynomial $(x, y, z)P$ (almost always assumed unique) and the Mal'cev polynomials $P(x, y, z; w_i \mid i < \iota)$ *having* ι *irrelevant variables* w_i $(i < \iota)$. With this distinction made, the uniqueness hypotheses may now be introduced:

2.1. DEFINITION. Let T be a Mal'cev variety. For $\iota \leq \omega$, T is said to satisfy the *generalised uniqueness hypothesis* $U(\iota)$ if there is at most one Mal'cev polynomial $P(x, y, z; w_i \mid i < \iota)$ as an element of $F_{X \cup W(\iota)}(T)$.

Note that $U(0)$ merely says that the parallelogram polynomial is unique. In particular, $(x, y, z)P = (z, y, x)P$, since $F(x, y, z) = (z, y, x)P$ is a Mal'cev polynomial. If $U(\iota)$ is satisfied, then $U(\rho)$ is satisfied for all $\rho < \iota$. $U(\omega)$ is the conjunction of the $U(\iota)$ for all finite ι. If $U(\iota)$ is satisfied by T, then the equality of the parallelogram polynomial with each Mal'cev polynomial having ι irrelevant variables is an identity in T.

The other hypothesis that plays an important rôle in these considerations is that the Mal'cev variety T satisfy the so-called *Big Dipper identity*

$$(y,x,(x,y,z)P)P = z$$

for the parallelogram polynomial $(x,y,z)P$ (Cf. [8, P2*]). This identity says that the parallelogram polynomial is in a certain sense invertible.

Satisfaction of generalised uniqueness hypotheses and the Big Dipper identity are reasonable requirements because of the following

2.2. THEOREM. *If a Mal'cev variety* T *is equal to* $Z(T)$, *then* T *satisfies the generalised uniqueness hypothesis* $U(\omega)$ *and the Big Dipper identity.*

The main theorem of the paper provides the vonverse of Theorem 2.2. However, the strength of the hypothesis $U(\omega)$ means that such a direct converse is of little practical value. The theorem is therefore stated with weaker hypotheses $U(\iota)$ tailored to the peculiarities of the variety T under consideration. For the concept of *clone* of polynomials of a variety, cf. [2, III.3].

2.3. THEOREM. *Let* T *be a Mal'cev variety of type* τ *satisfying the Big Dipper.*

(a) *If* τ *consists of at most ternary operations and includes at least one nullary operation, let* T *satisfy* $U(0)$.

(b) *If all operations of* τ *have arity less than* $4+\iota$ *for some* $0 < \iota \leq \omega$, *let* T *satisfy* $U(\iota)$.

Then $T = Z(T)$, *and the clone of* T *is generated by the parallelogram and polynomials of arity at most* 2.

The latter result on generation of the clone appears naturally in deriving equality of T and $Z(T)$ from the hypotheses. Combining it with Theorem 2.2 gives the following generalisation of a result of OSTERMANN and SCHMIDT [8, Satz 2, (8), (9)] about affine spaces (regarded as algebras under the operations of taking weighted means):

2.4. COROLLARY. *If a Mal'cev variety* T *is equal to* $Z(T)$, *then its clone is generated by the parallelogram and polynomials of arity at most* 2.

The next two sections of the paper are devoted to proving Theorems 2.2 and 2.3, while the last section provides an example of a Mal'cev variety T satisfying the Big Dipper and $U(0)$ for which $Z(T)$ is a proper subvariety of T.

3. PROOF OF THEOREM 2.2

Polynomials $f(x_i \mid i < m)$ and $g(y_j \mid j < n)$ in a variety are said *to commute* [2, III.3] if $f(g(y_{ij} \mid j < n) \mid i < m) = g(f(y_{ij} \mid i < m) \mid j < n)$. This notion helps formulate a characterisation of varieties $T = Z(T)$ that will be useful in what follows:

3.1. PROPOSITION. *A Mal'cev variety* T *is equal to* $Z(T)$ *iff* T *satisfies* $U(0)$ *and the (unique) parallelogram commutes with all polynomials of* T.

PROOF. Firstly, suppose that T satisfies $U(0)$ and the unique parallelogram commutes with all polynomials of T. For an algebra A in T define a relation V on $A \times A$ by

$$(x,y)\ V\ (u,v) \quad \text{iff} \quad v = (y,x,u)P.$$

V is clearly reflexive, and if $(x_i,y_i)\ V\ (u_i,v_i)$ for $i < n$, then for an n-ary polynomial $f(x_i \mid i < n)$

$$f(v_i \mid i < n) = f((y_i,x_i,u_i)P \mid i < n) =$$
$$= (f(y_i \mid i < n), f(x_i \mid i < n), f(u_i \mid i < n))P$$

since P commutes with f, so that V is a subalgebra of $(A \times A)^2$. It follows by [9, 143] that V is a congruence on $A \times A$. Note that for all x,y in A, $y = (x,x,y)P$, so $\hat{A}$ is contained in a V-class. Conversely if $(x,x)\ V\ (u,v)$, then $v = (x,x,u)P = u$, so $\hat{A}$ is itself a V-class. It follows that $T = Z(T)$.

Conversely, suppose A is an algebra in a variety T for which $T = Z(T)$. Let V denote the congruence on $A \times A$ having $\hat{A}$ as a congruence class, and let $P(\ ,\ ,\ )$ be a ternary Mal'cev polynomial in T. Then

$$(0) \qquad (x,y)\ V\ (z,t) \quad \text{iff} \quad t = P(y,x,z).$$

For

$$(x,y) = P((x,y),(x,x),(x,x))\ V$$
$$V\ P((x,y),(x,x),(z,z)) =$$
$$= (P(x,x,z),P(y,x,z)) = (z,P(y,x,z)) = (z,t),$$

while on the other hand if $(x,y)\ V\ (z,t)$ then

$$\begin{aligned}(t,t) &= P((t,t),(t,z),(t,z)) = \\ &= P((t,t),P((t,t),(z,t),(z,z),(t,z))\ V \\ &\ V\ P((y,y),P((y,y),(x,y),(x,x)),(t,z)) = \\ &= P((y,y),(y,x),(t,z)) = (t,P(y,x,z))\end{aligned}$$

so that $t = P(y,x,z)$. In particular taking A to be the free T-algebra on $\{x,y,z\}$ shows the uniqueness of the Mal'cev polynomial $P(\ ,\ ,\ )$, so this will be written as a parallelogram polynomial $(\ ,\ ,\ )P$. Finally let $f(x_i \mid i < n)$ be an n-ary polynomial of T. By (0) $(x_i,y_i)\ V\ (z_i,(y_i,x_i,z_i)P)$ for $i < n$, so since V is a congruence $(f(x_i \mid i < n),f(y_i \mid i < n))\ V\ (f(z_i \mid i < n),f((y_i,x_i,z_i)P \mid i < n))$. (0) then shows that $f((y_i,x_i,z_i)P \mid i < n)) = (f(y_i \mid i < n),f(x_i \mid i < n), f(z_i \mid i < n))P$, as required for the commuting of P with f.

Now the proof of Theorem 2.2 follows rapidly. Suppose that the Mal'cev variety T is equal to $Z(T)$. Let $(x,y,z)P$ be the parallelogram polynomial in T, unique by Proposition 3.1, and let $F(x,y,z;w)$ be a Mal'cev polynomial with a vector w of irrelevant variables. Then $(x,y,z)P = (F(x,y,y;w),F(y,y,y;w),F(y,y,z;w))P = F((x,y,y)P,(y,y,y)P,(y,y,z)P;(w,w,w)P) = F(x,y,z;w)$, the penultimate equality holding since by Proposition 3.1 P and F commute. Thus T satisfies $U(\omega)$. To show that T satisfies the Big Dipper identity, note that the commuting of P with itself yields $(y,x,(x,$

$y,z)P)P = ((y,y,y)P,(x,y,y)P,(x,y,z)P)P = ((y,x,x)P,$
$(y,y,y)P,(y,y,z)P)P = (y,y,z)P = z.$

4. PROOF OF THEOREM 2.3

The proof of the main Theorem 2.3 will be carried out in two stages corresponding to the two cases (a) and (b). Both cases require the following lemma which shows the use of the Big Dipper identity:

4.1. LEMMA. *Suppose that* T *is a Mal'cev variety satisfying the Big Dipper identity for the parallelogram polynomial* $(x,y,z)P$. *Let* $f(\mathrm{x})$, $g(\mathrm{x})$, $h(\mathrm{x})$ *be polynomials of* T *of the same arity for which* T *satisfies* $(f(\mathrm{x}),g(\mathrm{x}),h(\mathrm{x}))P = f(\mathrm{x})$. *Then* T *also satisfies* $g(\mathrm{x}) = h(\mathrm{x})$.

PROOF. $g(\mathrm{x}) = (g(\mathrm{x}),f(\mathrm{x}),f(\mathrm{x}))P = (g(\mathrm{x}),f(\mathrm{x}),(f(\mathrm{x}),g(\mathrm{x}),h(\mathrm{x}))P)P = h(\mathrm{x})$.

The crucial idea underlying the proof of Theorem 2.3 is that of designing new Mal'cev polynomials in which two adjacent components are the two sides of an identity one wishes to derive, and then using Lemma 4.1 to prove equality of these components.

Suppose then that T is a Mal'cev variety satisfying the hypotheses of case (a). By Proposition 3.1 it suffices to show that the parallelogram $(\, , \, , \,)P$ commutes with all polynomials of T. Now if a polynomial commutes with the polynomials in a certain set, it commutes with all the polynomials in the clone generated by that set [2, Prop. III.3.2]. For the proof of case (a) it thus suffices to show that all polynomials of arity at most 3 lie in the clone generated by P and polynomials of arity at most 2, and that P commutes with these.

Let e denote a nullary polynomial of T (such exist by the hypothesis of case (a)). Then $(e,e,e)P = e$, so P certainly commutes with e. Next, let $f(x)$ be a unary polynomial of T. Define the ternary polynomial

$$F(x,y,z) = ((x,y,z)P, (f(x),f(y),f(z))P, f((x,y,z)P))P.$$

Now $F(x,x,z) = (z,f(z),f(z))P = z$ and $F(x,z,z) = (x,f(x),f(x))P = x$, so $F(x,y,z)$ is a ternary Mal'cev polynomial. $U(0)$ then gives that $(x,y,z)P = F(x,y,z)$, whence Lemma 4.1 yields $(f(x),f(y),f(z))P = f((x,y,z)P)$, i.e. P commutes with the unary $f(x)$.

The next stage of the proof of (a) involves showing that the parallelogram P commutes with itself. Since this stage is also necessary for the proof of (b), it is formulated as the general

4.2. LEMMA. *Let* T *be a Mal'cev variety satisfying the Big Dipper.*

(a) *If* T *contains a nullary operation, let* T *satisfy* $U(0)$.

(b) *Otherwise, let* T *satisfy* $U(1)$.

Then the parallelogram operation P *commutes with itself.*

PROOF. In case (a), let w denote a nullary operation of T, and let A be the free T-algebra on a 9-element set X. In case (b), let A be the free T-algebra on $X \dot{\cup} \{w\}$. Define the maps

$$+ : A \times A \to A \; ; \; (x,y) \mapsto x+y = (x,w,y)P$$

and

$$- : A \to A \; ; \; x \mapsto (-x) = (w,y,w)P.$$

It will be shown in a series of steps that $(A,+,-,w)$ is an abelian group, and that $(x,y,z)P = x-y+z$ on A. Since the statement that P commutes with itself in T is a 9-variable identity, the result follows.

Step 1. *Commutativity of* $+$. By the remarks following Definition 2.1, $(x,y,z)P = (z,y,x)P$ is a consequence of $U(0)$. Substituting w for y gives the commutativity of $+$.

Step 2. w *is a unit for* $+$. $x+w = (x,w,w)P = x$.

Step 3. $x+(-x) = w$. This step shows directly how the Big Dipper works as an invertibility condition: $x + {} + (-x) = (x,w,(w,x,w)P)P = w$.

Step 4. *Associativity of* $+$. For this step, the identity

$$(0) \qquad ((x,w,z)P,w,y)P = ((x,w,y)P,w,z)$$

will be proved in T. The required result then follows on recalling the commutativity of $+$. Define the ternary polynomial

$$F(x,y,z) = ((x,y,z)P,((x,y,z)P,y,x)P, ((x,y,x)P,y,z)P)P$$

in T. Then $F(x,x,z) = (z,(z,x,x)P,(x,x,z)P)P = {} = (z,z,z)P = z$ and $F(x,z,z) = (x,(x,z,x)P,(x,z,x)P)P =$

$= x$, so that $F(x,y,z)$, being a Mal'cev polynomial, is equal to $(x,y,z)P$ by $U(0)$. Lemma 4.1 then shows that

$$(1) \qquad ((x,y,z)P,y,x)P = ((x,y,x)P,y,z)P$$

holds in T. Now define the polynomial (ternary for (a), quaternary for (b))

$$G(x,y,z;w) = ((x,y,z)P,((x,w,z)P,w,y)P, ((x,w,y)P,w,z)P)P$$

in T. Then

$$G(x,x,z;w) = $$
$$= (z,((x,w,z)P,w,x)P,((x,w,x)P,w,z)P)P = z$$

by (1) and

$$G(x,z,z;w) = $$
$$= (x,((x,w,z)P,w,z)P,((x,w,z)P,w,z)P)P = x,$$

so that $G(x,y,z;w)$, being a Mal'cev polynomial, is equal to $(x,y,z)P$, by $U(0)$ in case (a) and $U(1)$ in case (b). Lemma 4.1 then yields (0).

Step 4 completes the proof that $(A,+,-,w)$ is an abelian group.

Step 5. $(x,y,z)P = x-y+z$. This equality is equivalent to the identity

$$(x,y,z)P = ((x,w,(w,y,w)P)P,w,z)P,$$

which follows (from $U(0)$ in case (a) or from $U(1)$ in case (b)) if the right hand side can be shown to be a Mal'cev polynomial. But by the Big Dipper

$$((x,w,(w,x,w)P)P,w,z)P = (w,w,z)P = z$$

and by (0) and the Big Dipper

$$((x,w,(w,z,w)P)P,w,z)P =$$

$$= (x,w,((w,z,w)P,w,z)P)P = (x,w,w)P = x.$$

The proof of Lemma 4.2 is complete.

Armed with the commutativity of P with itself the proof of case (a) of Theorem 2.3 continues much as before. For a typical binary polynomial $f(x,y)$ one defines

$$F(x,y,z) = ((x,y,z)P,f(x,z),$$
$$(f(x,y),f(y,y),f(y,z))P)P.$$

Then $F(x,x,z) = z$ and $F(x,z,z) = x$, so $(x,y,z)P = F(x,y,z)$, from which Lemma 4.1 gives the identity

(2) $$f(x,z) = (f(x,y),f(y,y),f(y,z))P.$$

In particular for the nullary operation e

$$f(x,z) = (f(x,e),f(e,e),f(e,z))P$$

is an identity holding in T. It shows that the binary polynomial $f(x,z)$ is in the clone generated by P and the set of polynomials of arity at most one in T. But by the earlier part of the proof P commutes with unary polynomials, and by Lemma 4.2 P commutes with itself. Thus [2, Prop. III.3.2] P commutes with the binary polynomial $f(x,z)$.

Finally, for a typical ternary polynomial $f(x,y,z)$, define

$$F(x,y,z) = ((x,y,z)P, f(x,y,z),$$
$$(f(x,y,y), f(y,y,y), f(y,y,z))P)P.$$

Once again $F(x,x,z) = z$ and $F(x,z,z) = x$, so $(x,y,z)P = F(x,y,z)$, from which Lemma 4.1 gives the identity

$$f(x,y,z) = (f(x,y,y), f(y,y,y), f(y,y,z))P$$

showing that the ternary f lies in the clone generated by P and the polynomials of arity at most 2. Since P commutes with all of these, it also commutes with f, and the proof of case (a) is complete.

For the proof of case (b), commuting of P with nullary and unary polynomials proceeds just as for case (a), as does the derivation of identity (2). The proof that P commutes with binary polynomials is somewhat different: define

$$F(x,y,z;w) = ((x,y,z)P, (f(x,w), f(y,w),$$
$$f(z,w))P, f((x,y,z)P,w))P$$

for an irrelevant variable w. Since $F(x,y,z;w)$ is a Mal'cev polynomial, $U(1)$ and Lemma 4.1 give the identity

$$(3) \qquad (f(x,w),f(y,w),f(z,w))P = f((x,y,z)P,w).$$

Then by (2), by Lemma 4.2, by (3) (applied to $f(x,y)$ and its opposite $g(x,y) = f(y,x)$) and again by (2), respectively, we get

$$\begin{aligned}
&(f(x,r),f(y,s),f(z,t))P = \\
&\quad = ((f(x,w),f(w,w),f(w,v))P, \\
&\quad (f(y,w),f(w,w),f(w,s))P, \\
&\quad (f(z,w),f(w,w),f(w,t))P)P = \\
&\quad = ((f(x,w),f(y,w),f(z,w))P, \\
&\quad (f(w,w),f(w,w),f(w,w))P, \\
&\quad (f(w,r),f(w,s),f(w,t))P)P = \\
&\quad = (f((x,y,z)P,w),f(w,w),f(w,(r,s,t)P))P = \\
&\quad = f((x,y,z)P,(r,s,t)P)
\end{aligned}$$

so that P commutes with the binary polynomial f.

Now for $0 \leq i < 2+\iota$ consider the induction hypothesis

$I(i)$: Polynomials of arity at most $2+i$ commute with P and lie in the clone generated by P and polynomials of arity at most 2.

$I(0)$ has been proved, and by [2, Prop. III.3.2] case (b) of Theorem 2.3 follows once $I(i)$ is proved for all $0 \leq i < 2+\iota$. Suppose then by induction that $I(i)$ has been proved for $0 \leq i < 1+\iota$. Let $f(x,y,z,w_j \mid j < i)$ be a $(3+i)$-ary polynomial of T. Define

$$F(x,y,z;w_j \mid j < i) = ((x,y,z)P, f(x,y,z,w_j \mid j < i),$$
$$(f(x,y,y,w_j \mid j < i), f(y,y,y,w_j \mid j < i),$$
$$f(y,y,z,w_j \mid j < i))P)P$$

for the irrelevant variables w_j $(j < i)$. Since $F(x,y,z;w_j \mid j < i)$ is a Mal'cev polynomial, $U(i)$ and Lemma 4.1 give the identity

$$f(x,y,z,w_j \mid j < i) = (f(x,y,y,w_j \mid j < i),$$
$$f(y,y,y,w_j \mid j < i), f(y,y,z,w_j \mid j < i))P,$$

showing that f lies in the clone generated by P and polynomials of arity at most $2+i$. $I(1+i)$ follows by $I(i)$ and the commuting of P with itself.

5. AN EXAMPLE

This section mentions an example of a Mal'cev variety T satisfying the uniqueness hypothesis $U(0)$ and the Big Dipper identity for which $Z(T)$ is a proper subvariety of T. Derivation of the properties of T used wanders too far from the subject of this paper; instead the reader is referred to [6] or [7] and [9].

A *totally symmetric quasigroup* or TS-*quasigroup* $(Q,\cdot)$ is a quasigroup for which the ternary relation $\{(x,y,z) \mid x \cdot y = z\}$ on Q is invariant under all permutations of x,y, and z. A TS-quasigroup $(Q,\cdot)$ is said to be *Abelian* iff for each element e in Q, $(Q,+,e)$ is an abelian group with identity e under the operation $x+y = e\cdot(x\cdot y)$. A *cubic hypersurface quasigroup* or CH-*quasigroup* is a TS-quasigroup in which each set of three elements generates an Abelian subquasigroup. A *commutative Moufang loop* or CML is a quasigroup $(L,+)$ having a nullary operation selecting the so-called *identity element* 1 and a unary operation assigning to each element x its so-called *inverse* $-x$, such that each set of two elements generates an abelian group and the *Moufang law*

$$x+(y+(x+z)) = (x+y+x)+z$$

is satisfied. TS-quasigroups, CH-quasigroups, and CML's each form Mal'cev varieties.

Let T denote the variety of CH-quasigroups and CML the variety of CML's. Note that $Z(CML)$ is the variety of abelian groups [9, p.43]. For arbitrary e in a CH-quasigroup $(Q,\cdot)$, let $F_e(Q,\cdot)$ denote the CML $(Q,+,e)$, where $x+y = e\cdot(x\cdot y)$ [6, I.5.1], [7, 5.1], [9, 431]; for another f in Q, the loops $F_e(Q,\cdot)$ and $F_f(Q,\cdot)$ are isomorphic, so the particular choice of e is irrelevant. By the definition of "Abelian", a CH-quasigroup $(Q,\cdot)$ is an Abelian quasigroup iff the CML $F_e(Q,\cdot)$ is an abelian group. For an algebra A in a Mal'cev variety, the *centre congruence* ζA [9, p.42] is the largest congruence on A having the diagonal $\hat{A}$ as a normal subalgebra; in particular $\hat{A}$ is normal in $A \times \times A$ iff $\zeta A = A \times A$. By [9, 437], $\zeta(Q,\cdot) = \zeta F_e(Q,\cdot)$. Thus $(Q,\cdot)$ is in $Z(T)$ iff $F_e(Q,\cdot)$ is an abelian group: $Z(T)$ is the variety of Abelian quasigroups.

Let $(L,+,1)$ be a CML. For $c\ \zeta(L,+,1)\ 1$, define $\cdot$ on L by $x \cdot y = c-x-y$ (such c associate with all pairs of elements of L). Then [6, I.5.2], [7, 5.2], [9, 432] $G_c(L,+,1)$, defined to be $(L,\cdot)$, is a CH-quasigroup, and [9, 437] $\zeta(L,+,1) = \zeta G_c(L,+,1)$. In particular, if $(L,+,1)$ is a CML not satisfying the associative law, $G_1(L,+,1)$ is a CH-quasigroup that is not Abelian. But there are CML's which are not associative [1, VIII.1], and so $Z(T)$ is a proper subvariety of T.

Finally, note that the hypothesis $U(0)$ and the Big Dipper identity are three-variable identities. T satisfies any three-variable identities that are satisfied in $Z(T)$ since any three elements of a T-algebra generate a $Z(T)$-algebra. But by Theorem 2.2 $Z(T)$ satisfies $U(0)$ and the Big Dipper, so the variety T does also.

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