The ring of an outer von Neumann frame in modular lattices

Gábor Czédli and Benedek Skublics

Dedicated to László Szabó on his sixtieth birthday

Abstract. We prove the following theorem. Let $(a_1,\dots,a_m,c_{12},\dots,c_{1m})$ be a spanning von Neumann m-frame of a modular lattice L, and let $(u_1,\dots,u_n,v_{12},\dots,v_{1n})$ be a spanning von Neumann n-frame of the interval $[0,a_1]$. Assume that either $m\geq 4$, or L is Arguesian and $m\geq 3$. Let R^* denote the coordinate ring of $(a_1,\dots,a_m,c_{12},\dots,c_{1m})$. If $n\geq 2$, then there is a ring S^* such that R^* is isomorphic to the ring of all $n\times n$ matrices over S^*. If $n\geq 4$ or L is Arguesian and $n\geq 3$, then we can choose S^* as the coordinate ring of $(u_1,\dots,u_n,v_{12},\dots,v_{1n})$.

1. The main result and historical background

Our goal is to prove the following theorem. Since it relies on classical notions only, the basic definitions are postponed to Section 2.

Theorem 1. (a) *Let L be a lattice with $0,1\in L$, and let $m,n\in\mathbb{N}$ with $n\geq 2$. Assume that*

$$L \text{ is modular and } m\geq 4. \tag{1.1}$$

Let $(a_1,\dots,a_m,c_{12},\dots,c_{1m})$ be a spanning von Neumann m-frame of L and $(u_1,\dots,u_n,v_{12},\dots,v_{1n})$ be a spanning von Neumann n-frame of the interval $[0,a_1]$. Let R^* denote the coordinate ring of $(a_1,\dots,a_m,c_{12},\dots,c_{1m})$. Then there is a ring S^* such that R^* is isomorphic to the ring of all $n\times n$ matrices over S^*. If

$$n\geq 4, \tag{1.2}$$

then we can choose S^* as the coordinate ring of $(u_1,\dots,u_n,v_{12},\dots,v_{1n})$.

(b) *The previous part of the theorem remains valid if* (1.1) *and* (1.2) *are replaced by*

$$L \text{ is Arguesian and } m\geq 3 \tag{1.3}$$

and

$$n\geq 3, \tag{1.4}$$

respectively.

2000 *Mathematics Subject Classification*: 06C05.

Key words and phrases: lattice, modularity, von Neumann n-frame.

This research was partially supported by the NFSR of Hungary (OTKA), grant nos. K 77432 and K 60148.

Notice that Arguesian lattices are necessarily modular. The statement of the Theorem will be illustrated by Example 7. The ring S^* in part (a) will be explicitely constructed, see (3.1) right before Lemma 8.

If $m = 2$, then $R^* = R\langle 1, 2\rangle = \{x \in L : xa_2 = 0 \text{ and } x + a_2 = a_1 + a_2\}$, see (2.5) and Section 4, is just a set, not a ring. If L is not Arguesian and $m = 3$, then R^* is not necessarily a ring. Hence the theorem does not make sense if $m = 2$, or $m = 3$ and L is not Arguesian. Nevertheless, the forthcoming proof still shows that

Remark 2. Lemma 9 holds even for $m = 2, 3$, provided L is modular.

Von Neumann normalized frames, *frames* shortly, are due to von Neumann [16]. In order to mimic the classical coordinatization of projective geometries in modular lattice environment, he associated a ring, the so-called *coordinate ring*, with each n-frame. Although von Neumann assumed that L is a *complemented* modular lattice and $n \geq 4$, his construction of the coordinate ring (without coordinatization) extends to arbitrary modular lattices without complementation, see Artmann [1] and Freese [6], and even to $n = 3$ if L is Arguesian, see Day and Pickering [4]. The equational theory of frame generated modular lattices is given by Herrmann [9]. Many of the deep results from [16], [4] and [9] are needed in the present paper.

To give a visual idea of the position of the two frames in Theorem 1, $(\vec{a}, \vec{c}) = (a_1, \ldots, a_m, c_{12}, \ldots, c_{1m})$ will be called the *outer frame*, while $(\vec{u}, \vec{v}) = (u_1, \ldots, u_n, v_{12}, \ldots, v_{1n})$ will be referred to as the *inner frame*. It has recently been proved in [3] that the inner and the outer frames lie in a uniquely determined *product frame*. Several statements on the product frame and the notation will be taken from [3].

A notion equivalent to frames is the notion of Huhn diamonds, see [11]. Frames and Huhn diamonds are used in the proof of several deep results showing how complicated modular lattices are, only to mention Freese [6], Huhn [12] and Hutchinson [14]. Frames or Huhn diamonds were also used in the theory of congruence varieties, see [15], [2], and Freese, Herrmann and Huhn [7]. Most of what we know on the equational properties of modular fractal lattices, see [3], depend on the following statement, which now becomes a corollary of Theorem 1.

Corollary 3 ([3])**.** *The coordinate ring of the outer frame and that of the inner frame have the same characteristic.*

2. Notation and basic notions

The lattice operations join and meet will be denoted by $+$ and $\cdot$ (mostly juxtaposition) such that meets take precedence over joins. The indices we use will be *positive* integers; so $i \leq n$ is understood as $1 \leq i \leq n$.

For definition, let $2 \leq m$, let L be a nontrivial modular lattice with 0 and 1, and let $\vec{a} = (a_1, \ldots, a_m) \in L^m$ and $\vec{c} = (c_{12}, \ldots, c_{1m}) \in L^{m-1}$. We say that $(\vec{a},\vec{c}) = (a_1,\ldots,a_m,\ c_{12},\ldots,c_{1m})$ is a *spanning m-frame* (or a frame of *order m*) of L, if $a_1 \neq a_2$ and the following equations hold for all $j \leq m$ and $2 \leq k \leq m$:

$$\sum_{i\leq m} a_i = 1, \qquad a_j \sum_{i\neq j} a_i = 0,$$
$$a_1 + c_{1k} = a_k + c_{1k} = a_1 + a_k, \qquad a_1 c_{1k} = a_k c_{1k} = 0. \tag{2.1}$$

Notice that if $(\vec{a},\vec{c})$ is a spanning m-frame, then

$$\text{the } a_i \text{ are the distinct atoms of a Boolean sublattice } \mathbf{2}^m, \tag{2.2}$$

and $\{a_1,\ c_{1k},\ a_k\}$ generates an M_3 with bottom $0 = 0_L$ for $k \in \{2,\ldots,m\}$. In particular, a frame of order two is simply a spanning M_3 without $0_{M_3} = 0_L$ and $1_{M_3} = 1_L$.

By the *order* of the frame we mean m. If $(\vec{a},\vec{c})$ is a spanning m-frame of a principal ideal of L, then we will call it a *frame in L*. Note that von Neumann [16], page 19, calls c_{1k} the axis of perspectivity between the intervals $[0,a_1]$ and $[0,a_k]$, and we will shortly call c_{1k} as the *axis of* $\langle a_1,a_k\rangle$-*perspectivity.*

Given an m-frame $(\vec{a},\vec{c})$, we define $c_{k1} = c_{1k}$ for $2 \leq k \leq n$, and, for 1, j, k distinct, let $c_{jk} = (c_{1j} + c_{1k})(a_j + a_k)$. From now on, a frame is always understood in this *extended sense*: $\vec{c}$ includes all the c_{ij}, $i \neq j$, $i,j \leq m$. Then, according to Lemma 5.3 in page 118 in von Neumann [16] (see also Freese [5]), for i,j,k distinct we have

$$\begin{gathered} c_{ik} = c_{ki} = (c_{ij} + c_{jk})(a_i + a_k) \\ a_i + c_{ij} = a_j + c_{ij} = a_i + a_j, \\ a_i c_{ij} = a_j c_{ij} = a_i a_j = 0. \end{gathered} \tag{2.3}$$

This means that the index 1 has no longer a special role.

Example 4 (Canonical m-frame)**.** Let K be a ring with 1. Let v_i denote the vector $(0,\ldots,0,1,0,\ldots,0) \in K^m$ (1 at the ith position). Letting $a_i = Kv_i$ and $c_{ij} = K(v_i - v_j)$, we obtain a spanning m-frame of the submodule lattice $\mathrm{Sub}(K^m)$, where K^m is, say, a left module over K in the usual way. This frame is called the *canonical m-frame* of $\mathrm{Sub}(K^m)$.

This example shows that, sometimes, to unify some definitions or arguments, it is reasonable to allow the formal definition of a *trivial axis* $c_{ii} = 0$, $i \leq m$; this convention makes formula (2.3) valid also for $k \in \{i,j\}$. However, according to tradition, the trivial axes do not belong to the frame.

From now on, the general assumption throughout the paper is that $n \geq 2$, L is a modular lattice, and either $m \geq 4$, or $m = 3$ and L is Arguesian.

Next, we define the coordinate ring of $(\vec{a},\vec{c})$ in two, slightly different ways. For $p,q,r \in \{1,\ldots,m\}$ distinct, consider the following projectivities:

$$\begin{aligned} R\tbinom{p\ q}{r\ q}&\colon [0,a_p+a_q] \to [0,a_r+a_q], \quad x \mapsto (x + c_{pr})(a_r + a_q), \\ R\tbinom{p\ q}{p\ r}&\colon [0,a_p+a_q] \to [0,a_p+a_r], \quad x \mapsto (x + c_{qr})(a_p + a_r); \end{aligned} \tag{2.4}$$

these are almost the original notations, see von Neumann [16] and Freese [5], the only difference is that we write R rather than P. They are lattice isomorphisms between the indicated principal ideals. For $i, j, k \in \{1, \dots, m\}$ distinct, let

$$\begin{aligned} R\langle i,j\rangle &= R\langle a_i,a_j\rangle = \{x\in L : x+a_j = a_i+a_j,\ \ xa_j=0\},\\ x\oplus_{ijk} y &= (a_i+a_j)\big((x+a_k)(c_{ik}+a_j)+yR\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big)\big) \text{ and}\\ x\otimes_{ijk} y &= (a_i+a_j)\big(xR\big(\begin{smallmatrix} i & j\\ i & k\end{smallmatrix}\big)+yR\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big)\big) \text{ for } x,y\in R\langle i,j\rangle. \end{aligned} \tag{2.5}$$

Then the operations $\oplus_{ijk}$ and $\otimes_{ijk}$ do not depend on the choice of k, and this definition turns $R\langle i,j\rangle$ into a ring. Moreover, $R\langle i,j\rangle \cong R\langle i',j'\rangle$ for every $i' \neq j'$, see von Neumann [16] or Herrmann [9]. (Notice that von Neumann uses the opposite multiplication.) This $R\langle i,j\rangle$ is called the *coordinate ring* of the frame.

While the above definition seems to be the frequently used one, see Herrmann [9], our needs are better served by von Neumann's original definition, which is more complicated but carries much more information. Following Freese [5], for $i, j, k, h \in \{1, \dots, m\}$ pairwise distinct, let

$$R\big(\begin{smallmatrix} i & j\\ k & h\end{smallmatrix}\big) = R\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big)\circ R\big(\begin{smallmatrix} k & j\\ k & h\end{smallmatrix}\big).$$

We always compose mappings from left to right, that is, $x(R\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big)\circ R\big(\begin{smallmatrix} k & j\\ k & h\end{smallmatrix}\big)) = (xR\big(\begin{smallmatrix} i & j\\ k & j\end{smallmatrix}\big))R\big(\begin{smallmatrix} k & j\\ k & h\end{smallmatrix}\big)$. Now, the notation $R\big(\begin{smallmatrix} i & j\\ k & h\end{smallmatrix}\big)$ makes sense whenever $i \neq j$ and $k \neq h$; notice that $R\big(\begin{smallmatrix} i & j\\ i & j\end{smallmatrix}\big)$ is the identical mapping. To make a distinction from what will be associated with the inner frame, we will call the $R\big(\begin{smallmatrix} i & j\\ k & h\end{smallmatrix}\big)$ mappings as *outer projectivities*.

Next, we consider two small categories. The first one, $\mathcal{C}_1(\vec{a},\vec{c})$, consists of the pairs (i,j), $i \neq j$ and $i,j \le m$, as objects, and for any two (not necessarily distinct) objects (i,j) and (k,h), there is exactly one $(i,j) \to (k,h)$ morphism. The second category, $\mathcal{C}_2(\vec{a},\vec{c})$, consists of the coordinate rings $R\langle i,j\rangle$ of our frame, $i \neq j$, as objects, and all ring isomorphisms among them, as morphism. For a morphism $(i,j) \to (k,h)$ in the first category, let R send this morphism to the mapping $R\big(\begin{smallmatrix} i & j\\ k & h\end{smallmatrix}\big)$. Of course, for an object (i,j) in $\mathcal{C}_1(\vec{a},\vec{c})$, R sends (i,j) to $R\langle i,j\rangle$. The crucial point is captured in the following lemma.

Lemma 5 (von Neumann [16], Day and Pickering [4]). *R is a functor from the category $\mathcal{C}_1(\vec{a},\vec{c})$ to the category $\mathcal{C}_2(\vec{a},\vec{c})$.*

Proof. The notion of categories came to existence only after von Neumann's fundamental work in lattice theory, recorded later in [16]. Hence it is not useless to give some hints how to extract the above lemma from [16]. If $m \geq 4$, then it follows from pages 119–123 that R is functor, see also Freese [5]. Although von Neumann does not consider $R\langle i,j\rangle$ a ring in itself, it is implicit in [16] that the $R\big(\begin{smallmatrix} i & j\\ k & h\end{smallmatrix}\big)$ are ring isomorphisms. (This becomes a bit more explicit in Freese [5]. With slightly different notation, it is fully explicit in

Thm. 2.2 of Herrmann [9].) If $m = 3$, then the lemma follows from Lemma (4.1) of Day and Pickering [4]. $\square$

By an L-*number* (related to the frame $(\vec{a}, \vec{c})$) von Neumann means a system $(x^{ij} : i, j \le m,\ i \ne j)$ of elements such that $x^{ij} \in R\langle i, j\rangle$ and $x^{ij} R\left(\begin{smallmatrix} i & j \\ k & h \end{smallmatrix}\right) = x^{kh}$ for all $i \ne j$ and $k \ne h$. (Because there will be lattice entries later, here we use superscripts rather than von Neumann's subscripts.) Clearly, for every (i, j), $i \ne j$, each L-number x is determined by its (i, j)th component x^{ij}. Conversely,

Lemma 6 (page 130 of [16], see also Lemma 2.1 in [5])**.** *If $u \in R\langle i, j\rangle$, then there is a unique L-number x such that $x^{ij} = u$.*

Let R^* be the set of L-numbers related to $(\vec{a}, \vec{c})$. Von Neumann made R^* into a ring $(R^*, \oplus_{R^*}, \otimes_{R^*})$ such that $R^* \to R\langle i, j\rangle$, $x \mapsto x^{ij}$ is a ring isomorphism for every $i \ne j$. (Of course, von Neumann defined $(R^*, \oplus_{R^*}, \otimes_{R^*})$ first, and later others, including Herrmann [9], transferred the ring structure of R^* to $R\langle i, j\rangle$ by the bijection $R^* \to R\langle i, j\rangle$, $x \mapsto x^{ij}$.)

According to Lemma 6 and the previous paragraph, we can perform computations with L-numbers componentwise, and it is sufficient to consider only one component. For $w \in R\langle i, j\rangle$, let $w^* \in R^*$ denote the unique L-number in R^* such that $(w^*)^{ij} = w$. However, we usually make no difference between w and w^*.

Next, we give an example to enlighten Theorem 1; for $n \ge 4$, the details can be checked based on Theorems II.4.2 and II.14.1 of von Neumann [16].

Example 7. Let R be the ring of all $n \times n$ matrices over a field S. Consider the canonical m-frame, with R instead of K, defined in Example 4. The coordinate ring R^* of this m-frame is isomorphic to R. Remember from Example 4 that $a_1 = R(E, 0, \dots, 0) \in \mathrm{Sub}(R^m)$, where E is the unit matrix in R. Hence the interval $[0, a_1]$ in $\mathrm{Sub}(R^m)$ is isomorphic to the lattice of all left ideals of R. The lattice of these left ideals is known to be isomorphic to the subspace lattice $\mathrm{Sub}(S^n)$ of the vector space S^n. Fix an appropriate isomorphism; it sends the canonical n-frame of $\mathrm{Sub}(S^n)$ to a spanning n-frame $(u_1, \dots, u_n, v_{12}, \dots, v_{1n})$ of $[0, a_1]$. Clearly, the coordinate ring S^* of this n-frame is isomorphic to S. Hence R^* is isomorphic to the ring of all $n \times n$ matrices over S^*.

While $\mathrm{Sub}(R^m)$ is coordinatizable by its construction in Example 7, it is worth pointing out that L in Theorem 1 is *not coordinatizable* in general. Although some ideas of the proof have been extracted from Example 7, Linear Algebra in itself seems to be inadequate to prove Theorem 1. (Even if it was an adequate tool, modular lattice theory would probably offer a more elegant treatment, see the last paragraph of Section 2 in [3].) Notice that Herrmann [9] reduces many problems of frame generated modular lattices to Linear Algebra, but our L is not frame-generated in general by evident cardinality reasons.

To help the reader to understand our calculations in modular lattices while we save a lot of space, the following notations will be in effect. We use

$$=^{i}, \quad =^{\mathrm{f}}, \quad \text{or} \quad =^{\mathrm{L11}}$$

to indicate that formula (i), some basic property of frames, or Lemma 11 is used, respectively. In many cases, $=^{\mathrm{f}}$ means the same as $=^{2.3}$. When an application of the modular law uses the relation $x \le z$ then, beside using $=^{\mathrm{m}}$, x resp. z will be underlined resp. doubly underlined. For example,

$$(\underline{x} + y)(\underline{\underline{x + z}}) =^{\mathrm{m}} x + y(x + z).$$

The use of the shearing identity (see Thm. IV.1.1 in Grätzer [8]) is indicated by $=^{\mathrm{s}}$ and underlining the subterm "sheared":

$$x(\underline{y} + z) =^{\mathrm{s}} x\big(y(x + z) + z\big).$$

Even in some other cases, subterms worth noticing are also underlined. If $x_1 \ge x_2 \dots x_k$ for some easy reason, then we write

$$\overline{x_1} x_2 \dots x_k$$

to indicate that this expression is considered as $x_2 \dots x_k$. In other words, overlined meetands will be omitted in the next step. Combining our notations like

$$=^{\mathrm{m},5,7,\mathrm{L8}},$$

we can simultaneously refer to properties like modularity, formulas and lemmas. Formulas, like (2.2), will also be used for the product frame, which comes in the next section.

3. The product frame

By Theorem 1 of [3], the outer frame $(\vec{a}, \vec{c})$ and the inner frame $(\vec{u}, \vec{v})$ determine a unique mn-frame $(\vec{b}, \vec{d})$, called the *product frame*. We will point out at the end of this section why the product frame is the relevant tool here even without quoting its complicated construction from [3]. First we formulate the most important property of the product frame in the next lemma. From now on,

$$\begin{gathered} S^* = (S^*, \oplus_{S^*}, \otimes_{S^*}) \text{ is the coordinate ring of the product frame } (\vec{b}, \vec{d}), \\ \text{and } M_n(S^*) = (M_n(S^*), \oplus_{M_n}, \otimes_{M_n}) \text{ is the } n \times n \text{ matrix ring over } S^*. \end{gathered} \tag{3.1}$$

This makes sense, since $mn \ge 4$. The following result, taken from [3], justifies that we can base the proof of Theorem 1 on this S^*.

Lemma 8 ([3])**.** *If $n \ge 4$, or $n \ge 3$ and L is Arguesian, then $(\vec{b}, \vec{d})$ and $(\vec{u}, \vec{v})$ have isomorphic coordinate rings.*

In the product frame, $\vec{b}$ is the system of its components

$$b_i^p, \qquad \text{where } i \le n \text{ and } p \le m,$$

and $\vec{d}$ is the system of its components

$$d_{i\,j}^{p\,q}, \qquad \text{where } i, j \le n,\ p, q \le m \text{ and } (p, i) \ne (q, j).$$

Further,

$$d_{i\,j}^{p\,q} \text{ is the axis of } \langle b_i^p, b_j^q \rangle \text{ perspectivity.}$$

(To comply with forthcoming notations, we suggest to read the indices of b_i^p downwards, "pi", and column-wise for $d_{i\,j}^{p\,q}$, "$pi\ qj$".)

Let us agree that, unless otherwise stated, the superscripts of b and d belong to $\{1, \dots, m\}$, while all their subscripts to $\{1, \dots, n\}$. For example, if $d_{j\,k}^{p\,r}$ occurs in a formula, then $p, r \le m$ and $j, k \le n$, and also $(p, j) \ne (r, k)$, are automatically stipulated. Similarly, the subscripts of a and c are automatically in $\{1, \dots, m\}$. This convention allows us, say, to write $\sum_i a_i$ instead of $\sum_{i=1}^m a_i$ without causing any ambiguity. Let us also agree that, unless otherwise stated, we understand our formulas with universally quantified indices, that is, for all meaningful values for the occurring indices. Define

$$B_k^p = \sum_{i \ne k} b_i^p.$$

Then, by (5) and (8) of [3], we have

$$a_p = \sum_{i=1}^n b_i^p = B_k^p + b_k^p, \qquad d_{i\,i}^{p\,q} \le c_{pq}. \tag{3.2}$$

Analogously to Lemma 5, the product frame gives rise to a functor and the $S\left\langle\begin{smallmatrix} p & q \\ i & j \end{smallmatrix}\right\rangle = S\langle b_i^p, b_j^q \rangle$ coordinate rings. The previous notations tailored to the product frame are as follows:

$$\begin{aligned} &S\left\langle\begin{smallmatrix} p & q \\ i & j \end{smallmatrix}\right\rangle = \{x \in L : x b_j^q = 0,\ x + b_j^q = b_i^p + b_j^q\}, \\ &S\left(\begin{smallmatrix} pi & qj \\ pi & rk \end{smallmatrix}\right) \colon S\left\langle\begin{smallmatrix} p & q \\ i & j \end{smallmatrix}\right\rangle \to S\left\langle\begin{smallmatrix} p & r \\ i & k \end{smallmatrix}\right\rangle, \quad x \mapsto (x + d_{j\,k}^{q\,r})(b_i^p + b_k^r), \\ &S\left(\begin{smallmatrix} pi & qj \\ rk & qj \end{smallmatrix}\right) \colon S\left\langle\begin{smallmatrix} p & q \\ i & j \end{smallmatrix}\right\rangle \to S\left\langle\begin{smallmatrix} r & q \\ k & j \end{smallmatrix}\right\rangle, \quad x \mapsto (x + d_{i\,k}^{p\,r})(b_k^r + b_j^q). \end{aligned} \tag{3.3}$$

(Since we have agreed in reading the indices of, say, $d_{j\,k}^{q\,r}$ column-wise, the space-saving entries qj and rk in $S\left(\begin{smallmatrix} pi & qj \\ pi & rk \end{smallmatrix}\right)$, rather than $\begin{smallmatrix} q \\ j \end{smallmatrix}$ and $\begin{smallmatrix} r \\ k \end{smallmatrix}$, should not be confusing.)

Next, we strengthen (3.2) a bit:

$$\sum_i d_{i\,i}^{p\,q} = c_{pq}. \tag{3.4}$$

Indeed, (2.3), applied also to the product frame, yields

$$\begin{aligned} a_q + \sum_i d_{i\,i}^{p\,q} &= \sum_i b_i^q + \sum_i d_{i\,i}^{p\,q} = \sum_i (b_i^q + d_{i\,i}^{p\,q}) = \sum_i (b_i^q + b_i^p) \\ &= \sum_i b_i^q + \sum_i b_i^p = a_q + a_p = 1. \end{aligned}$$

This, together with $a_q c_{pq} = 0$ and (3.2) show that $\sum_i d_{i\,i}^{pq}$ and c_{pq} are comparable complements of a_q, whence modularity yields (3.4).

It is clear from Lemma 8 that it suffices to deal with the product frame $(\vec{b}, \vec{d})$ and the outer frame $(\vec{a}, \vec{c})$. Since the product frame determines the outer frame via (3.2) and (3.4), it is sufficient and convenient to work with the product frame without a closer look at its construction in [3].

4. A pair of reciprocal mappings

For $i, j \le n$, we define a mapping $\varphi_{ij} \colon R^* \to S^*$ as follows. We identify R^* with $R\langle 1,2\rangle = R\langle a_1, a_2\rangle$. So we define $x\varphi_{ij}$ for $x \in R\langle 1,2\rangle$, and, without over-complicating our formulas with writing x^*, we understand $x^*\varphi_{ij}$ as $x\varphi_{ij}$. Similarly, we define the value $x\varphi_{ij}$ in $S\left\langle\begin{smallmatrix} 1 & 1 \\ i & j \end{smallmatrix}\right\rangle$ but we understand it as $(x\varphi_{ij})^* \in S^*$ without making a notational distinction between $x\varphi_{ij}$ and $(x\varphi_{ij})^*$. Finally, we will put these φ_{ij} together in the natural way to obtain a mapping $\varphi \colon R^* \to M_n(S^*)$: the (i,j)th entry of the matrix $x\varphi$ is defined as $x\varphi_{ij}$. So, the definition of φ is completed by

$$\varphi_{ij} \colon R\langle 1,2\rangle \to S\left\langle\begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix}\right\rangle, \quad x \mapsto x_{ij} = (x + B_j^2)(b_i^1 + b_j^2). \tag{4.1}$$

(We will prove soon that φ_{ij} maps $R\langle 1,2\rangle$ into $S\left\langle\begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix}\right\rangle$.)

In the reverse direction, we will rely on the possibility offered by L-numbers even more: distinct entries of a matrix in $M_n(S^*)$ will be represented with their components of different positions. Let $(e_{ij} : i, j \le n)$ be a matrix over S^*, that is, an element of $M_n(S^*)$. The truth is that e_{ij} belongs to S^*. However, we identify e_{ij} with its component belonging to $S\left\langle\begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix}\right\rangle$, and, again, we do this without notational difference between e_{ij} and its corresponding component in $S\left\langle\begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix}\right\rangle$. Introduce the notation

$$E_{*k} = \sum_i e_{ik}.$$

With this convention, we define

$$\psi \colon M_n(S^*) \to R^*, \quad (e_{ij} : i, j \le n) \mapsto \prod_k (E_{*k} + B_k^2). \tag{4.2}$$

We will prove soon that $\prod_k (E_{*k} + B_k^2)$ belongs to $R\langle 1,2\rangle$, which is identified with R^*.

Next, we formulate an evident consequence of modularity:

$$R\langle i,j\rangle \text{ and } S\left\langle\begin{smallmatrix} p & q \\ i & j \end{smallmatrix}\right\rangle \text{ are antichains in } L. \tag{4.3}$$

Indeed, if, say, we had $x < y$ and $x, y \in R\langle i,j\rangle$, then x and y would be comparable complements of a_j, a contradiction. We will often have to prove that two elements of $R\langle i,j\rangle$ or $S\left\langle\begin{smallmatrix} p & q \\ i & j \end{smallmatrix}\right\rangle$ are equal; then (4.3) reduces this task to showing that the two elements are comparable.

The rest of this section is devoted to the following lemma.

Lemma 9. *φ and ψ are bijections, and they are inverse mappings of each other.*

Before proving this lemma, two preliminary statements are necessary.

Lemma 10. *Let $j \le m$, and suppose that, for all $i \in \{1, \dots, m\} \setminus \{j\}$, $w_i \in R\langle i, j\rangle$. Then $a_j \sum_{i \ne j} w_i = 0$.*

Proof. Let I_k denote the induction hypothesis "if $|\{i : w_i \ne a_i\}| \le k$, then $a_j \sum_{i \ne j} w_i = 0$". Then I_0 clearly holds by (2.2), and I_{m-1} is our target.

Assume I_{k-1} for an arbitrary $k < m$. We will refer to it with the notation $=^{\mathrm{ih}}$. We want to show I_k. By symmetry, we can assume that $j = m$ and $w_i \ne a_i$ holds only for $i \le k$. Then

$$\begin{aligned} a_j \sum_{i \ne j} w_i &= a_m(\underline{w_1 + \dots + w_{k-1} + a_{k+1} + \dots + a_{m-1}} + w_k) \\ &=^{\mathrm{s}} a_m\big((w_1 + \dots + w_{k-1} + a_{k+1} + \dots + a_{m-1})(w_k + a_m) + w_k\big) \\ &=^{2.5} a_m\big((w_1 + \dots + w_{k-1} + a_{k+1} + \dots + a_{m-1})(a_k + \underline{a_m}) + w_k\big) \\ &=^{\mathrm{s,ih}} a_m\big((w_1 + \dots + w_{k-1} + a_{k+1} + \dots + a_{m-1})a_k + w_k\big) \\ &\le^{2.5} a_m\big((a_1 + a_m + \dots + a_{k-1} + a_m + a_{k+1} + \dots + a_{m-1})a_k + w_k\big) \\ &=^{2.2} a_m w_k =^{2.5} 0. \end{aligned}$$

□

The following easy statement on elements of a *modular* lattice belongs to the folklore; it also occurs as (1) in Huhn [13].

Lemma 11. *If $f_i \le g_j$ for all $i \ne j$, $i, j \le k$, then*

$$\prod_{i \le k} g_i + \sum_{i \le k} f_i = \prod_{i \le k} (g_i + f_i).$$

Proof of Lemma 9. Let $\prod_k (E_{*k} + B_k^2)$ from (4.2) be denoted by e, and remember that $e_{ij} \in S\left\langle \begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix} \right\rangle$. We have to show that $e \in R\langle 1, 2\rangle$. Let us compute:

$$\begin{aligned} a_2 e = \prod_k \big(\underline{\underline{a_2}}(E_{*k} + \underline{B_k^2})\big) &=^{\mathrm{m}} \prod_k (a_2 E_{*k} + B_k^2) =^{3.2} \prod_k \big((\underline{b_k^2} + B_k^2)E_{*k} + B_k^2\big) \\ =^{\mathrm{s}} \prod_k \big((\underline{b_k^2(E_{*k} + B_k^2)} + B_k^2)E_{*k} + B_k^2\big). \end{aligned}$$

Focusing on the last underlined subterm, observe that the summands e_{ik} of E_{*k} belong to $S\left\langle \begin{smallmatrix} 1 & 2 \\ i & k \end{smallmatrix} \right\rangle$, and the summands b_j^2, $j \ne k$, of B_k^2 belong to $S\left\langle \begin{smallmatrix} 2 & 2 \\ j & k \end{smallmatrix} \right\rangle$. Hence, applying Lemma 10 to the product frame, we conclude that $b_k^2(E_{*k} + B_k^2) = 0$. Therefore,

$$a_2 e = \prod_k (B_k^2 E_{*k} + B_k^2) = \prod_k B_k^2 =^{2.2} 0. \tag{4.4}$$

Next, we compute

$$\begin{aligned} a_2 + e &=^{3.2} \sum_k b_k^2 + \prod_k (E_{*k} + B_k^2) =^{\mathrm{L11}} \prod_k (E_{*k} + B_k^2 + b_k^2) \\ &=^{3.2} \prod_k \sum_j (e_{jk} + b_k^2 + a_2) =^{3.3} \prod_k \sum_j (b_j^1 + b_k^2 + a_k) \\ &=^{3.2} \prod_k (a_1 + a_2) = a_1 + a_2. \end{aligned}$$

This and (4.4) imply $e \in R\langle 1, 2\rangle$. Hence ψ maps into R^*, as desired.

Next, let $x \in R\langle 1, 2\rangle$. To show that φ maps into $M_n(S^*)$, we have to show that $x_{ij} = x\varphi_{ij} = (x + B_j^2)(b_i^1 + b_j^2)$ belongs to $S\left\langle \begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix} \right\rangle$. This follows easily, since

$$\begin{aligned} x_{ij} b_j^2 &= (\underline{x} + B_j^2)\overline{(b_i^1 + b_j^2)} b_j^2 =^{\mathrm{s}} \big(x(b_j^2 + B_j^2) + B_j^2\big) b_j^2 \\ &=^{3.2} \ (x a_2 + B_j^2) b_j^2 =^{2.5} B_j^2 b_j^2 =^{\mathrm{f}} 0, \quad \text{and} \\ x_{ij} + b_j^2 &= (x + B_j^2)(\underline{\underline{b_i^1 + b_j^2}}) + \underline{b_j^2} =^{\mathrm{m}} (b_i^1 + b_j^2)(x + B_j^2 + b_j^2) \\ &=^{3.2} \ (b_i^1 + b_j^2)(x + a_2) =^{2.5} (b_i^1 + b_j^2)(a_1 + a_2) =^{3.2} b_i^1 + b_j^2. \end{aligned}$$

Next, we show that $\varphi \circ \psi$ is the identical mapping. Let $x \in R\langle 1, 2\rangle$. Then

$$\begin{aligned} x(\varphi \circ \psi) &= (x\varphi)\psi = (x\varphi_{ij} : i, j \le n)\psi \\ &=^{4.1} \big((x + B_j^2)(b_i^1 + b_j^2) : i, j \le n\big)\psi \\ &=^{4.2} \prod_k y_k, \quad \text{where } y_k = B_k^2 + \sum_i (x + B_k^2)(b_i^1 + b_k^2). \end{aligned}$$

Observe that it suffices to show that $x \le y_k$ for all $k \le n$, since then (4.3) implies $x = y$. Let us compute:

$$\begin{aligned} y_k &= \sum_i \big(\underline{B_k^2} + (\underline{\underline{x + B_k^2}})(b_i^1 + b_k^2)\big) =^{\mathrm{m}} \sum_i (x + B_k^2)(b_i^1 + b_k^2 + B_k^2) \\ &\ge^{3.2} \sum_i x(b_i^1 + a_2) = \underline{\underline{x}}(b_1^1 + a_2) + \underline{\sum_{2\le i} x(b_i^1 + a_2)} \\ &=^{\mathrm{m}} x\Big(b_1^1 + a_2 + \sum_{2\le i} x(b_i^1 + a_2)\Big) = x\Big(b_1^1 + \sum_{2\le i}\big(\underline{a_2} + x(\underline{\underline{b_i^1 + a_2}})\big)\Big) \\ &=^{\mathrm{m}} x\Big(b_1^1 + \sum_{2\le i}(a_2 + x)(b_i^1 + a_2)\Big) =^{2.5} x\Big(b_1^1 + \sum_{2\le i}(a_2 + a_1)(b_i^1 + a_2)\Big) \\ &=^{3.2} x\Big(b_1^1 + \sum_{2\le i}(b_i^1 + a_2)\Big) = x\Big(a_2 + \sum_i b_i^1\Big) =^{3.2} x(a_2 + a_1) =^{2.5} x. \end{aligned}$$

Hence $x \le y_k$, as requested, and $\varphi \circ \psi$ is the identical mapping.

Next, to show that $\psi \circ \varphi$ is the identical mapping, let $e_{ij} \in S\left\langle \begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix} \right\rangle$ for $i, j \le n$, and denote $(e_{ij} : i, j \le n)\psi = \prod_k (E_{*k} + B_k^2)$ by e. We have already shown that $e \in R\langle 1, 2\rangle$, see (4.2), and $e\varphi_{ij} = (e + B_j^2)(b_i^1 + b_j^2) \in S\left\langle \begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix} \right\rangle$, see (4.1). Since $e_{ij} \le b_i^1 + b_j^2$ by (3.3), $e_{ij} \le e + B_j^2$ would imply $e_{ij} \le e\varphi_{ij}$, and we could derive $e_{ij} = e\varphi_{ij}$ by (4.3). So, it suffices to show that $e_{ij} \le e + B^2_j$. Let us compute:

$$\begin{aligned} e + B^2_j &= B^2_j + \prod_k (E_{*k} + B^2_k) = \underline{B^2_j} + \underline{\underline{(E_{*j} + B^2_j)}} \prod_{k\neq j}(E_{*k} + B^2_k) \\ &=^{\mathrm{m}} (E_{*j} + B^2_j)\Big(\sum_{k\neq j} b^2_k + \prod_{k\neq j}(E_{*k} + B^2_k)\Big) \\ &=^{\mathrm{L11}} (E_{*j} + B^2_j)\prod_{k\neq j}(E_{*k} + B^2_k + b^2_k) \\ &=^{3.2} (E_{*j} + B^2_j)\prod_{k\neq j}\sum_{\ell}(e_{\ell k} + b^2_k + a_2) \\ &=^{3.3} (E_{*j} + B^2_j)\prod_{k\neq j}\sum_{\ell}(b^1_\ell + b^2_k + a_2) =^{3.2} (E_{*j} + B^2_j)\prod_{k\neq j}(a_1 + a_2). \end{aligned}$$

Since $a_1 + a_2 \ge^{(3.2)} b^1_i + b^2_j \ge^{(3.3)} e_{ij}$ and $E_{*j} \ge e_{ij}$, the above calculation shows that $e_{ij} \le e + B^2_j$. This completes the proof of Lemma 9. □

5. Addition and further lemmas

Lemma 12. *φ and, therefore, ψ are additive.*

Proof. Let $x, y \in R\langle 1, 2\rangle$, $z = x \oplus_{123} y$, $x' = x\varphi_{ij} = (x + B^2_j)(b^1_i + b^2_j)$, $y' = y\varphi_{ij} = (y + B^2_j)(b^1_i + b^2_j)$ and $z' = z\varphi_{ij} = (z + B^2_j)(b^1_i + b^2_j)$. It suffices to show that, in $S\langle \begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix} \rangle$, we have $x' \oplus^{123}_{iji} y' = z'$. Let us compute:

$$\begin{aligned} x' \oplus^{123}_{iji} y' &= (b^1_i + b^2_j)\big((x' + b^3_i)(d^{\,13}_{\,ii} + b^2_j) + y' S\big(\begin{smallmatrix} 1i & 2j \\ 3i & 2j \end{smallmatrix}\big)\big) \\ &= (b^1_i + b^2_j)\big((x' + b^3_i)(d^{\,13}_{\,ii} + b^2_j) + (y' + d^{\,13}_{\,ii})(b^3_i + b^2_j)\big). \quad (5.1) \end{aligned}$$

On the other hand,

$$\begin{aligned} z' &= (z + B^2_j)(b^1_i + b^2_j) \\ &= (b^1_i + b^2_j)\Big((a_1 + a_2)\big((x + a_3)(c_{13} + a_2) + yR\big(\begin{smallmatrix} 1 & 2 \\ 3 & 2 \end{smallmatrix}\big)\big) + B^2_j\Big) \\ &= (b^1_i + b^2_j)\Big(\underline{\underline{(a_1 + a_2)}}\big((x + a_3)(c_{13} + a_2) + (y + c_{13})(a_3 + a_2)\big) + \underline{B^2_j}\Big) \\ &=^{\mathrm{m}} (b^1_i + b^2_j)\overline{(a_1 + a_2)}\big((x + a_3)(c_{13} + a_2) + (y + c_{13})(a_3 + a_2) + B^2_j\big) \\ &= (b^1_i + b^2_j)\big((x + a_3)\underline{\underline{(c_{13} + a_2)}} + \underline{B^2_j} + (y + c_{13})\underline{\underline{(a_3 + a_2)}} + \underline{B^2_j}\big) \\ &=^{\mathrm{m}} (b^1_i + b^2_j)\big((\underline{x + B^2_j} + a_3)(c_{13} + a_2) + (\underline{y + B^2_j} + c_{13})(a_3 + a_2)\big). \quad (5.2) \end{aligned}$$

Now, we can see that the subterms obtained in (5.1) are less than or equal to the corresponding subterms obtained in (5.2). Indeed, $x' \le x + B^2_j$ and $y' \le y + B^2_j$ by definitions, and $b^3_i \le a_3$, $b^2_j \le a_2$ and $d^{\,13}_{\,ii} \le c_{13}$ by (3.2). Hence (4.3) yields $x' \oplus^{123}_{iji} y' = z'$. □

Lemma 13. *$b^i_j + c_{ik} = b^k_j + c_{ik}$ and $B^i_j + c_{ik} = B^k_j + c_{ik}$.*

Proof. It suffices to deal only with the first equation: $b^i_j + c_{ik} =^{3.2} b^i_j + d^{\,i\,k}_{\,j\,j} + c_{ik} =^{\mathrm{f}} b^k_j + d^{\,i\,k}_{\,j\,j} + c_{ik} =^{3.2} b^i_k + c_{ik}$. ☐

Lemma 14. *Assume that* $x, y \in S\left\langle\begin{smallmatrix}1&2\\u&v\end{smallmatrix}\right\rangle$. *Then* $xR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) = xS\left(\begin{smallmatrix}1u&2v\\1u&3v\end{smallmatrix}\right)$ *and, similarly,* $yR\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right) = yS\left(\begin{smallmatrix}1u&2v\\3u&2v\end{smallmatrix}\right)$.

Proof. If $i, j, k \le m$ are pairwise distinct, then we have

$$\begin{aligned} c_{jk}(\underline{a_i} + a_k) &=^{\mathrm{s}} c_{jk}\bigl(a_i(c_{jk} + a_k) + a_k\bigr) \\ &=^{2.3} c_{jk}\bigl(a_i(a_j + a_k) + a_k\bigr) =^{2.2} c_{jk}a_k =^{2.3} 0. \end{aligned} \tag{5.3}$$

The outer projectivities $R\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right)$ and $R\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right)$ are lattice isomorphisms that send the interval $[0, a_1 + a_2]$ onto $[0, a_1 + a_3]$ and $[0, a_3 + a_2]$, respectively. Since $S\left\langle\begin{smallmatrix}1&2\\u&v\end{smallmatrix}\right\rangle \subseteq [0, a_1 + a_2]$ is defined in the terminology of lattices and

$$\begin{aligned} b^1_u R\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) &= (\underline{b^1_u} + c_{23})(\underline{\underline{a_1 + a_3}}) =^{\mathrm{m}} b^1_u + c_{23}(a_1 + a_3) =^{5.3} b^1_u, \\ b^2_v R\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right) &= (b^2_v + c_{23})(a_1 + a_3) =^{\mathrm{L13}} (\underline{b^3_v} + c_{23})(\underline{\underline{a_1 + a_3}}) =^{\mathrm{m},5.3} b^3_v, \\ b^1_u R\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right) &= (b^1_u + c_{13})(a_3 + a_2) =^{\mathrm{L13}} (\underline{b^3_u} + c_{13})(\underline{\underline{a_3 + a_2}}) =^{\mathrm{m},5.3} b^3_u, \\ b^2_v R\left(\begin{smallmatrix}1&2\\3&2\end{smallmatrix}\right) &= (\underline{b^2_v} + c_{13})(\underline{\underline{a_3 + a_2}}) =^{\mathrm{m},5.3} b^2_v, \end{aligned}$$

we conclude that these outer projectivities send (the support set of) $S\left\langle\begin{smallmatrix}1&2\\u&v\end{smallmatrix}\right\rangle$ onto $S\left\langle\begin{smallmatrix}1&3\\u&v\end{smallmatrix}\right\rangle$ and $S\left\langle\begin{smallmatrix}3&2\\u&v\end{smallmatrix}\right\rangle$, respectively. Lattice terms are monotone, so we obtain

$$xS\left(\begin{smallmatrix}1u&2v\\1u&3v\end{smallmatrix}\right) = (x + d^{\,2\,3}_{\,v\,v})(b^1_u + b^3_v) \le^{3.2} (x + c_{23})(a_1 + a_3) = xR\left(\begin{smallmatrix}1&2\\1&3\end{smallmatrix}\right). \tag{5.4}$$

We have seen that both sides of (5.4) belong to $S\left\langle\begin{smallmatrix}1&3\\u&v\end{smallmatrix}\right\rangle$, whence they are equal in virtue of (4.3). The other equation of the lemma follows the same way. ☐

6. Multiplication

By an *almost zero matrix* we mean a matrix in which all but possibly one entries are zero. We say that ψ, defined in (4.2), *preserves the multiplication of almost zero matrices*, if $(E \otimes_{M_n} F)\psi = (E\psi) \otimes_{R^*} (F\psi)$ holds for all almost zero matrices $E, F \in M_n(S^*)$.

Lemma 15. *If ψ is additive and preserves the multiplication of almost zero matrices, then it is a ring homomorphism.*

Proof. Since each matrix in $M_n(S^*)$ is a sum of almost zero matrices, the lemma follows trivially by ring distributivity. ☐

Next, we introduce some notations, which will be permanent in the rest of the paper. Let $E = (e_{ij} : i, j < n) \in M_n(S^*)$ and $F = (f_{ij} : i, j < n) \in M_n(S^*)$ be two almost zero matrices. According to the earlier convention and keeping in mind that b_i^1 is the zero of the ring $S\big\langle {1\ 2 \atop i\ j} \big\rangle$, this means that there are indices p, q, r, s, fixed from now on, such that

$$\begin{aligned} &x := e_{pq} \in S\big\langle {1\ 2 \atop p\ q} \big\rangle, \qquad e_{ij} = b_i^1 \in S\big\langle {1\ 2 \atop i\ j} \big\rangle \ \text{ for } (i,j) \neq (p,q), \\ &y := f_{rs} \in S\big\langle {1\ 2 \atop r\ s} \big\rangle, \qquad f_{k\ell} = b_k^1 \in S\big\langle {1\ 2 \atop k\ \ell} \big\rangle \ \text{ for } (k,\ell) \neq (r,s). \end{aligned} \tag{6.1}$$

Let $G = (g_{ij} : i, j < n) = E \otimes_{M_n} F$. By definitions, including the everyday's definition of a product matrix, we have

$$g_{ij} = b_i^1, \text{ the zero of } S\big\langle {1\ 2 \atop i\ j} \big\rangle, \qquad \text{if } q \neq r \text{ or } (i,j) \neq (p,s); \tag{6.2}$$

$$g_{ps} = xS\big({1p\ 2q \atop 1p\ 2s} \big) \otimes^{1\,2\,\alpha}_{p\,s\,\beta} \ yS\big({1r\ 2s \atop 1p\ 2s} \big), \qquad \text{if } q = r; \tag{6.3}$$

where α and β are arbitrary, provided $(1,p) \neq (\alpha,\beta) \neq (2,s)$. We also define

$$e := E\psi, \quad f := F\psi, \quad \text{and, differently,} \quad g := e \otimes_{123} f.$$

The plan is to show that $g\varphi = G$, that is, $g\varphi_{ij} = g_{ij}$ for all $i, j \leq n$, since this is equivalent to $G\psi = g$. To prepare a formula for the $g\varphi_{ij}$, we need the following technical lemma.

Lemma 16. *For all $j \leq n$, we have*

$$B_j^2 + (y + B_r^1 + B_s^2) \prod_{k \neq s} (a_1 + B_k^2) = y + B_r^1 + B_j^2, \tag{6.4}$$

$$B_r^2 + (x + B_p^1 + B_q^2) \prod_{k \neq q} (a_1 + B_k^2) = x + B_p^1 + B_r^2. \tag{6.5}$$

Proof. It suffices to show (6.4), since it implies (6.5) by replacing (y, r, s, j) with (x, p, q, r). Let u denote the left hand side of (6.4). If $j = s$, then

$$\begin{aligned} u &= \underline{B_j^2} + (\underline{\underline{y + B_r^1 + B_j^2}}) \prod_{k \neq j} (a_1 + B_k^2) \\ &=^{\mathrm{m}} (y + B_r^1 + B_j^2)\Big(B_j^2 + \prod_{k \neq j} (a_1 + B_k^2)\Big) \\ &=^{\mathrm{L11}} (y + B_r^1 + B_j^2) \prod_{k \neq j} (a_1 + B_k^2 + b_k^2) \\ &=^{2.3} (y + B_r^1 + B_j^2) \prod_{k \neq j} (a_1 + a_2) =^{3.2,2.5} y + B_r^1 + B_j^2. \end{aligned}$$

If $j \neq s$, then

$$\begin{aligned}
u &= \underline{B_j^2} + (y + B_r^1 + B_s^2)(\underline{\underline{a_1 + B_j^2}}) \prod_{k\neq j,s} (a_1 + B_k^2) \\
&=^{\mathrm{m}} (a_1 + B_j^2)\Big(B_j^2 + (y + B_r^1 + B_s^2) \prod_{k\neq j,s} (a_1 + B_k^2)\Big) \\
&= (a_1 + B_j^2)\Big(b_s^2 + \sum_{k\neq j,s} b_k^2 + (y + B_r^1 + B_s^2) \prod_{k\neq j,s} (a_1 + B_k^2)\Big) \\
&=^{\mathrm{L11}} (a_1 + B_j^2)(y + B_r^1 + B_s^2 + b_s^2) \prod_{k\neq j,s} (a_1 + B_k^2 + b_k^2) \\
&=^{3.2} (a_1 + B_j^2)(\underline{y} + B_r^1 + a_2 + \underline{b_s^2}) \overline{\prod_{k\neq j,s} (a_1 + a_2)} \\
&=^{3.3,3.2} (a_1 + B_j^2)(\overline{a_1 + a_2}) =^{3.2} B_r^1 + b_r^1 + b_s^2 + B_j^2 \\
&=^{3.3} B_r^1 + y + b_s^2 + B_j^2 =^{3.2} y + B_r^1 + B_j^2
\end{aligned}$$

□

Lemma 17. *For every $i, j \leq n$, we have*

$$g\varphi_{ij} = (b_i^1 + b_j^2)\Big(B_p^1 + B_j^2 + B_r^3 + xS\big(\begin{smallmatrix} 1p & 2q \\ 1p & 3q \end{smallmatrix}\big) + yS\big(\begin{smallmatrix} 1r & 2s \\ 3r & 2s \end{smallmatrix}\big)\Big).$$

Proof. Firstly, we express e and, to obtain f, we replace (x, p, q) with (y, r, s):

$$\begin{aligned}
e = E\psi &=^{4.2} \prod_k (E_{*k} + B_k^2) = (E_{*q} + B_q^2) \prod_{k\neq q} (E_{*k} + B_k^2) \\
&=^{6.1,3.2} (x + B_p^1 + B_q^2) \prod_{k\neq q} (a_1 + B_k^2); \qquad (6.6) \\
f &= (y + B_r^1 + B_s^2) \prod_{k\neq s} (a_1 + B_k^2). \qquad (6.7)
\end{aligned}$$

We need some auxiliary equations:

$$\begin{aligned}
B_j^2 + fR\big(\begin{smallmatrix} 1 & 2 \\ 3 & 2 \end{smallmatrix}\big) &=^{2.4} \underline{B_j^2} + (f + c_{13})(\underline{\underline{a_3 + a_2}}) =^{\mathrm{m}} (a_3 + a_2)(B_j^2 + f + c_{13}) \\
&=^{6.7} (a_3 + a_2)\Big(c_{13} + B_j^2 + (y + B_r^1 + B_s^2) \prod_{k\neq s} (a_1 + B_k^2)\Big) \\
&=^{6.4} (a_3 + a_2)(c_{13} + y + B_r^1 + B_j^2) \\
&=^{\mathrm{L13}} (\underline{\underline{a_3 + a_2}})(c_{13} + y + \underline{B_r^3} + B_j^2) \\
&=^{\mathrm{m}} B_r^3 + (a_3 + a_2)(c_{13} + y + B_j^2), \quad \text{and}
\end{aligned}$$

(6.8)

$$\begin{aligned} B_r^3 + eR\binom{1\,2}{1\,3} &=^{2.4} \underline{B_r^3} + (e + c_{23})\underline{\underline{(a_1 + a_3)}} =^{\mathrm{m}} (a_1 + a_3)(B_r^3 + e + c_{23}) \\ &=^{\mathrm{L13}} (a_1 + a_3)(c_{23} + B_r^2 + e) \\ &=^{6.6} (a_1 + a_3)\Big(c_{23} + B_r^2 + (x + B_p^1 + B_q^2) \prod_{k\neq q}(a_1 + B_k^2)\Big) \\ &=^{6.5} \underline{\underline{(a_1 + a_3)}}(c_{23} + x + \underline{B_p^1} + B_r^2) \\ &=^{\mathrm{m}} B_p^1 + (a_1 + a_3)(c_{23} + x + B_r^2). \end{aligned} \tag{6.9}$$

Armed with the previous equations, we obtain

$$\begin{aligned} g\varphi_{ij} &=^{4.1} (b_i^1 + b_j^2)(g + B_j^2) \\ &=^{2.5} (b_i^1 + b_j^2)\Big(\underline{B_j^2} + \underline{\underline{(a_1 + a_2)}}\big(eR\tbinom{1\,2}{1\,3} + fR\tbinom{1\,2}{3\,2}\big)\Big) \\ &=^{\mathrm{m}} (b_i^1 + b_j^2)\overline{(a_1 + a_2)}\big(B_j^2 + eR\tbinom{1\,2}{1\,3} + fR\tbinom{1\,2}{3\,2}\big) \\ &=^{6.8} (b_i^1 + b_j^2)\big(eR\tbinom{1\,2}{1\,3} + B_r^3 + (a_3 + a_2)(c_{13} + y + B_j^2)\big) \\ &=^{6.9} (b_i^1 + b_j^2)\big(B_p^1 + (a_1 + a_3)(c_{23} + x + \underline{B_r^2}) + (a_3 + a_2)(c_{13} + y + B_j^2)\big) \\ &=^{\mathrm{L13}} (b_i^1 + b_j^2)\big(B_p^1 + \underline{\underline{(a_1 + a_3)}}(c_{23} + x + \underline{B_r^3}) + \underline{\underline{(a_3 + a_2)}}(c_{13} + y + \underline{B_j^2})\big) \\ &=^{\mathrm{m}} (b_i^1 + b_j^2)\big(B_p^1 + B_r^3 + (a_1 + a_3)(c_{23} + x) + B_j^2 + (a_3 + a_2)(c_{13} + y)\big) \\ &=^{2.4} (b_i^1 + b_j^2)\Big(B_p^1 + B_j^2 + B_r^3 + xR\tbinom{1\,2}{1\,3} + yR\tbinom{1\,2}{3\,2}\Big), \end{aligned}$$

whence Lemma 17 follows by Lemma 14. ☐

Lemma 18. *ψ preserves the multiplication of almost zero matrices.*

Proof. Keep the previous notations, and let

$$x' := xS\binom{1p\ 2q}{1p\ 3q} \in S\Big\langle{1\ 3 \atop p\ q}\Big\rangle, \qquad y' := yS\binom{1r\ 2s}{3r\ 2s} \in S\Big\langle{3\ 2 \atop r\ s}\Big\rangle. \tag{6.10}$$

We know from Lemma 17 that

$$g\varphi_{ij} = (b_i^1 + b_j^2)(B_p^1 + B_j^2 + B_r^3 + x' + y'). \tag{6.11}$$

According to (6.2), our first goal is to show that $g\varphi_{ij} = b_i^1$ whenever $q \neq r$ or $(i, j) \neq (p, s)$. Notice that if

$$b_i^1 \leq B_p^1 + B_j^2 + B_r^3 + x' + y', \tag{6.12}$$

then $g\varphi_{ij} = b_i^1$ follows from (4.3), so we can aim at (6.12). Since $x' \in S\langle{1\ 3 \atop p\ q}\rangle$ and $y' \in S\langle{3\ 2 \atop r\ s}\rangle$, (2.3) and (3.3) provide us with the following computation rules:

$$\alpha \neq q \Longrightarrow\ B_\alpha^3 + x' \geq b_p^1, \tag{6.13}$$

$$\beta \neq s \Longrightarrow\ B_\beta^2 + y' \geq b_r^3. \tag{6.14}$$

We can assume that $i = p$, since otherwise $b_i^1 \leq B_p^1$ gives (6.12). If $r \neq q$, then $B_r^3 + x' \geq^{6.13} b_p^1$ yields (6.12) again. Hence we assume that $q = r$. If $j \neq s$, then $B_j^2 + y' \geq^{6.14} b_r^3$ together with $b_r^3 + x' = b_q^3 + x' \geq^{3.3} b_p^1$ yields (6.12) once more. Therefore, we can assume that $j = s$.

Now, our task is restricted to the case $i = p$, $q = r$, $j = s$. Substituting these indices into (6.11) and computing:

$$\begin{aligned} g\varphi_{ps} &= (b_p^1 + b_s^2)(B_p^1 + B_s^2 + B_r^3 + x' + y') \\ &\geq (b_p^1 + b_s^2)(x' + y') =^{6.10} (b_p^1 + b_s^2)(xS\binom{1p\ 2r}{1p\ 3r} + yS\binom{1r\ 2s}{3r\ 2s}) \\ &=^{\mathrm{L5}} (b_p^1 + b_s^2)(xS\binom{1p\ 2r}{1p\ 2s}S\binom{1p\ 2s}{1p\ 3r} + yS\binom{1r\ 2s}{1p\ 2s}S\binom{1p\ 2s}{3r\ 2s}) \\ &=^{2.5} xS\binom{1p\ 2r}{1p\ 2s} \otimes_{p\,s\,r}^{1\,2\,3} yS\binom{1r\ 2s}{1p\ 2s} =^{6.3} g_{ps}. \end{aligned}$$

Hence (4.3) yields that $g\varphi_{ps} = g_{ps}$, indeed. □

Proof of Theorem 1. Lemmas 9, 12, 15 and 18. □

Added in the editorial process. Several comments of an anonymous referee and the historical remarks of Luca Giudici are acknowledged.

For the most recent developments in coordinatization theory the reader can see Herrmann [10], Wehrung [17] and [18], and their bibliography.

References

[1] Artmann, B.: On coordinates in modular lattices with a homogeneous basis. Illinois J. Math. **12**, , 626–648 (1968)
[2] Czédli, G.: How are diamond identities implied in congruence varieties. Algebra Universalis **30**, 291–293(1993)
[3] Czédli, G.: The product of von Neumann n-frames, its characteristic, and modular fractal lattices. Algebra Universalis **60**, 217–230 (2009)
[4] Day, A., Pickering, D.: The coordinatization of Arguesian lattices. Trans. Amer. Math. Soc. **278**, 507-522 (1983)
[5] Freese, R.: Projective geometries as projective modular lattices. Trans. Amer. Math. Soc. **251**, 329–342 (1979)
[6] Freese, R: The variety of modular lattices is not generated by its finite members. Trans. Amer. Math. Soc. **255**, 277–300 (1979)
[7] Freese, R., Herrmann, C., Huhn, A.P.: On some identities valid in modular congruence varieties. Algebra Universalis **12**, 322–334 (1981)
[8] Grätzer, G.: General Lattice Theory, 2nd edn. Birkhäuser Verlag, Basel (1998)
[9] Herrmann, C.: On the arithmetic of projective coordinate systems. Trans. Amer. Math. Soc. **284**, 759–785 (1984)
[10] Herrmann, C: Generators for complemented modular lattices and the von-Neumann-Jónsson coordinatization theorems. Algebra Universalis (in press)
[11] Huhn, A.P.: Schwach distributive Verbände. I. Acta Sci Math. (Szeged) **33**, 297–305 (1972)
[12] Huhn, A.P.: On G. Grätzer's problem concerning automorphisms of a finitely presented lattice. Algebra Universalis **5**, 65–71 (1975)
[13] Huhn, A.P.: On n-distributive systems of elements of a modular lattice. Publ. Math. Debrecen **27**, 107–115 (1980)
[14] Hutchinson, G: Embedding and unsolvability theorems for modular lattices. Algebra Universalis **7**, 47–84 (1977)
[15] Hutchinson, G., Czédli, G.: A test for identities satisfied in lattices of submodules. Algebra Universalis **8**, 269–309 (1978)
[16] von Neumann, J: Continuous Geometry. (Foreword by Israel Halperin.) Princeton University Press, Princeton (1960, 1998)
[17] Wehrung, F.: Von Neumann Coordinatization is not first-order. J. Math. Logic. **6**, 1–24 (2006)

[18] Wehrung, F.: Coordinatization of lattices by regular rings without unit and Banashewski functions. Algebra Universalis (in press)

Gábor Czédli

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720
e-mail: czedli@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/∼czedli/

Benedek Skublics

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720
e-mail: bskublics@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/∼bskublics/