# ON THE SEMIDISTRIBUTIVITY OF ELEMENTS IN WEAK CONGRUENCE LATTICES OF ALGEBRAS AND GROUPS

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Abstract. Weak congruence lattices and semidistributive congruence lattices are both recent topics in Universal Algebra. This motivates the main result of the present paper, which asserts that a finite group $G$ is a Dedekind group if and only if the diagonal relation is a join-semidistributive element in the lattice of weak congruences of $G$. A variant in terms of subgroups rather than weak congruences is also given. It is pointed out that no similar result is valid for rings. An open problem and some results on the join-semidistributivity of weak congruence lattices are also included.

## 1. Introduction and the main result

A *weak congruence* on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$. By a subuniverse we mean a subset closed with respect to all operations, so a nonempty subset is a subuniverse iff it is a subalgebra. Weak congruences of $\mathcal{A}$ form an algebraic lattice $\mathrm{Cw}(\mathcal{A})$ with respect to inclusion, cf. [14] or [10]. The diagonal relation $\Delta = \{(x,x) : x \in \mathcal{A}\} \in \mathrm{Cw}(\mathcal{A})$ plays a special role. The filter $[\Delta)$ is just $\mathrm{Con}(\mathcal{A})$, the congruence lattice, while the ideal $(\Delta]$ is isomorphic to $\mathrm{Sub}(\mathcal{A})$, the subalgebra lattice. Hence $\mathrm{Cw}(\mathcal{A})$ carries a lot of information on the algebra $\mathcal{A}$. The diagonal relation $\Delta$ is always a codistributive element in $\mathrm{Cw}(\mathcal{A})$, i.e.,

$$\text{(D)} \qquad \Delta \wedge (\alpha \vee \beta) = (\Delta \wedge \alpha) \vee (\Delta \wedge \beta)$$

for all $\alpha, \beta \in \mathrm{Cw}(\mathcal{A})$. If the dual of condition (D) holds then $\mathcal{A}$ is said to satisfy the *congruence intersection property* (CIP for short), cf. [9] or [10]. Notice that the CIP simply means that $\Delta$ is a distributive element in $\mathrm{Cw}(\mathcal{A})$.

Distributivity is a rather strong assumption in lattice theory, so two weaker conditions satisfied by free lattices, the meet- and join-semidistributive laws, are also important. For definition, an element $d$ of a lattice $L$ will be called a *join-semidistributive element* if for every $x$ and $y$ in $L$ the implication

$$d \vee x = d \vee y \Rightarrow d \vee (x \wedge y) = d \vee x$$

holds. If all the elements of $L$ are join-semidistributive then $L$ is called a *join-semidistributive lattice. Meet-semidistributive* elements and lattices are defined dually.

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Obviously, codistributive elements are meet-semidistributive while distributive elements are join-semidistributive.

Many results in Universal Algebra show that the distributivity of congruence lattices is a very important condition, cf. e.g. Jónsson [3] for an overview. Somewhat later the join- and meet-semidistributivity of congruence lattices became also important. These two conditions are closely connected with types of finite algebras, cf. Hobby and McKenzie [2], and with "the shape of congruence lattices" of not necessarily finite algebras, cf. Kearnes and Kiss [4]. Sometimes old results with congruence distributivity are generalized for congruence semidistributivity, cf. e.g., Kearnes and Willard [5].

Motivated by these recent trends, the target of the present paper is to replace CIP with a weaker condition, the join-semidistributivity of $\Delta$ in $\mathrm{Cw}(\mathcal{A})$. It will be shown that this is practically impossible for general algebras and even for rings. However, the join-semidistributivity of $\Delta$ will be shown to be an important property for finite groups.

Notice that there are many known connections between group properties and subgroup lattices, ranging e.g. from Suzuki [12] to Lukács and Pálfy [6]. This together with the fact that the weak congruence lattice includes the subgroup lattice foretell future results on weak congruence lattices of groups.

Given a group $G$ and a subgroup $X$ of $G$, $(X)_G$ will denote the normal subgroup generated by $X$. The lattice of all subgroups resp. all normal subgroups of $G$ will be denoted by $\mathrm{Sub}(G)$ resp. $\mathcal{N}(G)$. With the usual notation, $\mathcal{N}(G) = \{X \in \mathrm{Sub}(G) : X \lhd G\}$. A group is called a *Dedekind group* if all of its subgroups are normal. Non-abelian Dedekind groups are called *Hamiltonian* ones. These groups are characterized by a nice structure theorem due to Dedekind and Baer, cf. e.g. Theorem 5.3.7 in Robinson [8]. Namely, a group is Hamiltonian if and only if it is the direct product of the eight element quaternion group, an elementary abelian 2-group and an abelian group with all its elements of odd order. Our main theorem offers quite a different characterization in the finite case.

**Theorem 1.** *For any finite group $G$ the following five conditions are equivalent.*

*(i) $G$ is a Dedekind group;*

*(ii) $G$ has the CIP;*

*(iii) $\Delta$ is a join-semidistributive element in* $\mathrm{Cw}(G)$*;*

*(iv) for every normal subgroup $N$ of $G$,*

$$C_N := \{K \in \mathrm{Sub}(G) : \exists H \in \mathcal{N}(K) \text{ with } (H)_G = N\}$$

*is a sublattice of* $\mathrm{Sub}(G)$*;*

*(v) for every normal subgroup $N$ of $G$, $C_N$ is closed with respect to intersection.*

Notice that although (ii) and (iii) are easier to formulate than (v), they are harder to test.

*Proof.* Since group congruences are determined by normal subgroups, it is worth introducing the following lattice, cf. [10]:

$$\mathcal{N}w(G) = \{(H, K) \in \mathrm{Sub}(G) \times \mathrm{Sub}(G) : H \lhd K\}.$$

The lattice structure is defined so that $\mathcal{N}w(G)$ should be isomorphic with $\mathrm{Cw}(G)$, i.e., for $(H_1, K_1)$ and $(H_2, K_2)$ in $\mathcal{N}w(G)$ we have

$$(H_1, K_1) \le (H_2, K_2) \Longleftrightarrow (H_1 \subseteq H_2 \text{ and } K_1 \subseteq K_2),$$

$$(H_1, K_1) \wedge (H_2, K_2) = (H_1 \cap H_2, K_1 \cap K_2) \text{ and}$$
$$(H_1, K_1) \vee (H_2, K_2) = ((H_1 \vee H_2)_{K_1 \vee K_2}, K_1 \vee K_2)$$

where $H_1 \vee H_2$ and $K_1 \vee K_2$ are understood in $\mathrm{Sub}(G)$. Denoting the one-element subgroup by 1, the element of $\mathcal{N}w(G)$ corresponding to $\Delta$ is $(1, G)$. In what follows, we will work with $(1, G)$ and $\mathcal{N}w(G)$ rather than with $\Delta$ and $\mathrm{Cw}(G)$.

Suppose (i). Then $\mathcal{N}w\, G = \{(H, K) \in \mathcal{N}(G) : H \lhd K\}$, being a sublattice of the direct square of $\mathcal{N}(G)$, is modular. Hence either we can recall the fact that the modularity of the weak congruence lattice, of any algebra, implies CIP, cf. [10] or [14], or we can check by a trivial calculation that $(1, G)$ is a distributive element of $\mathcal{N}w(G)$. Therefore (i) implies (ii).

The implication (ii) $\Rightarrow$ (iii) is trivial, for distributive elements are always join-semidistributive in any lattice.

Now assume (iii), i.e., $(1, G)$ is a join-semidistributive element of $\mathcal{N}w(G)$. For $i = 1, 2$ let $K_i \in C_N$ witnessed by $H_i \in \mathcal{N}(K_i)$ with $(H_i)_G = N$. Define $H = N \cap (K_1 \vee K_2)$. Then $H_1 \subseteq H \in \mathcal{N}(K_1 \vee K_2)$ and we conclude from $N = (H_1)_G \subseteq (H)_G \subseteq (N)_G = N$ that $(H)_G = N$. Hence $K_1 \vee K_2$ belongs to $C_N$. (So far we have not used that $(1, G)$ is join-semidistributive.)

Now let us compute in $\mathcal{N}w(G)$: $(H_1, K_1) \vee (1, G) = ((H_1)_G, G) = (N, G) = ((H_2)_G, G) = (H_2, K_2) \vee (1, G)$. Using the join-semidistributivity of $(1, G)$ we obtain

$$(N, G) = ((H_1, K_1) \wedge (H_2, K_2)) \vee (1, G) = (H_1 \cap H_2, K_1 \cap K_2) \vee (1, G) = ((H_1 \cap H_2)_G, G),$$

implying $(H_1 \cap H_2)_G = N$. This together with $H_1 \cap H_2 \in \mathcal{N}(K_1 \cap K_2)$ yields $K_1 \cap K_2 \in C_N$, proving that (iii) implies (iv).

The implication (iv) $\Rightarrow$ (v) is trivial.

Now assume (v), and consider an arbitrary subgroup $K$ of $G$. Then $K \lhd K$, so $K \in C_N$ where $N = (K)_G$. Let $K = K_1, K_2, \ldots, K_n$ be the list of all conjugates of $K$. Each of them generates the same normal subgroup $N$, so $K_i \in C_N$ for all $i \in \{1, \ldots, n\}$. Let $M = K_1 \cap \cdots \cap K_n$. Then $M \in C_N \cap \mathcal{N}(G)$. So there is an $H \in \mathcal{N}(M)$ with $(H)_G = N$. Now $N = (H)_G \subseteq (M)_G = M$ together with $M \subseteq K \subseteq N$ imply $K = N \in \mathcal{N}(G)$. This proves that $G$ is a Dedekind group. $\square$

Much less is known if we do not assume that the group in question is finite. Obraztsov [7] has given a group that has CIP without being a Dedekind group. However, we still have the following

**Open problem.** *Are the join-semidistributivity of $\Delta$ in $\mathrm{Cw}(G)$ and the CIP of $G$ equivalent for arbitrary groups?*

When infinitary meets are allowed, infinite groups do not give any problem. Indeed, the previous proof gives that an arbitrary group is Dedekind iff $\Delta$ is an infinitely distributive element of $\mathrm{Cw}(G)$ iff $\Delta$ is an infinitely join-semidistributive element of $\mathrm{Cw}(G)$; the details are omitted.

When studying weak congruence lattices, rings are essentially different from groups. The four element field easily witnesses that even in the finite case CIP does not imply that all subrings are ideals. Moreover, we have the following

**Example 1.** *There exists a finite commutative ring $R$ with unit such that $\Delta$ is join-semidistributive in* $\mathrm{Cw}(R)$ *but the CIP fails.*

*Proof.* Analogously to $\mathcal{N}w(G)$, we define

$$\mathcal{I}w(R) = \{(H,K) \in \mathrm{Sub}(R)^2 : H \lhd K\}.$$

The operations are the same except that $(X)_R$ stands for the ideal of $R$ generated by $X$. Clearly, $\mathrm{Cw}(R)$ is isomorphic to $\mathcal{I}w(R)$ and $\Delta$ corresponds to $D = (\{0\}, R)$.

Now let $I = (\{x^2+1\})_{\mathbf{Z}_2[x]}$, the principal ideal generated by $x^2+1$ in $\mathbf{Z}_2[x]$. With the notation $s = x + I$ the required ring is $R = \mathbf{Z}_2/I = \{0, 1, s, s+1\}$, where the rule of computation is $s^2 = 1$. This gives $s(s+1) = s+1$ and $(s+1)^2 = 0$. An easy calculation, starting with determining the cyclic subrings, shows that $\mathrm{Sub}(R)$ consists of $R$, $\{0\}$, $\{0,1\}$ and $\{0, s+1\}$. All of them but $\{0,1\}$ are ideals. Let $A = (\{0,1\},\{0,1\})$ and $B = (\{0,s+1\},\{0,s+1\})$. In $\mathcal{I}w(R)$ we have

$$(A \vee D) \wedge (B \vee D) = ((\{0,1\})_R, R) \wedge ((\{0,s+1\})_R, R) =$$
$$(R,R) \wedge (\{0,s+1\}, R) = (\{0,s+1\}, R) \neq (\{0\}, R) =$$
$$(\{0\},\{0\}) \vee (\{0\}, R) = (A \wedge B) \vee D,$$

whence CIP fails.

Now let $A_1 = (H_1, K_1)$ and $A_2 = (H_2, K_2)$ in $\mathcal{I}w(R)$ be arbitrary such that $A_1 \vee D = A_2 \vee D$, i.e. $((H_1)_R, R) = ((H_2)_R, R)$, i.e. $(H_1)_R = (H_2)_R$. This excludes $\{H_1, H_2\} = \{\{0,1\},\{0,s+1\}\}$. Therefore $H_1$ and $H_2$ are comparable and $H_1 \cap H_2 = H_i$ for $i = 1$ or $i = 2$. Hence

$$(A_1 \wedge A_2) \vee D = (H_i, K_1 \cap K_2) \vee (\{0\}, R) = ((H_i)_R, R) = A_i \vee D,$$

showing that $D$ is join-semidistributive. ☐

## 2. More about the semidistributivity of $\mathrm{Cw}(A)$

An element $d$ of a lattice $L$ is called a *dually modular element*, cf. e.g. Stern [11], page 74, if for all $x, y \in L$, $d \leq x$ implies $(d \vee y) \wedge x = d \vee (y \wedge x)$. Surprisingly, we could not find any reference to the following easy lattice theoretic statement so we give a proof.

**Lemma 1.** *Let $L$ be a lattice and $d \in L$. If $d$ is join-semidistributive and dually modular then $d$ is distributive.*

*Proof.* Let $x, y \in L$. The dual modularity of $d$ gives $(x\vee d)\wedge(y\vee d) = ((x\vee d)\wedge y)\vee d$ and $(x\vee d)\wedge(y\vee d) = ((y\vee d)\wedge x)\vee d$. Therefore from $((x\vee d)\wedge y)\vee d = ((y\vee d)\wedge x)\vee d$ and the join-semidistributivity of $d$ we obtain that $(x \vee d) \wedge (y \vee d) = ((y \vee d) \wedge x \wedge (x \vee d) \wedge y) \vee d = (x \wedge y) \vee d$. ☐

An algebra $\mathcal{A}$ is said to satisfy the weak CIP if $\Delta$ is a dually modular element of $\mathrm{Cw}(\mathcal{A})$. The congruence extension property will be abbreviated by CEP.

**Theorem 2.** *Let $A$ be an algebra.*

*(a) If $\mathcal{A}$ has the weak CIP and $\Delta$ is join-semidistributive in* $\mathrm{Cw}(\mathcal{A})$ *then $\mathcal{A}$ has the CIP.*

*(b) Suppose $\mathcal{A}$ satisfies the CIP and the CEP. Then* $\mathrm{Cw}(\mathcal{A})$ *is join-semidistributive if and only if both* $\mathrm{Sub}(\mathcal{A})$ *and* $\mathrm{Con}(\mathcal{A})$ *are join-semidistributive.*

*(c) If all subalgebras of $\mathcal{A}$ have the CIP and, in addition,* $\mathrm{Sub}(\mathcal{A})$ *and* $\mathrm{Con}(\mathcal{B})$ *for all $\mathcal{B} \in \mathrm{Sub}(\mathcal{A})$ are join-semidistributive then* $\mathrm{Cw}(\mathcal{A})$ *is join-semidistributive as well.*

In connection with part (b) of Theorem 2 it is worth mentioning that whenever $\mathrm{Cw}(\mathcal{A})$ is modular then $\mathcal{A}$ has the CIP and the CEP, cf. [14].

*Proof.* Part (a) is an evident consequence of Lemma 1.

It is shown in [13], cf. also [10] or [1], that if an algebra $\mathcal{A}$ satisfies the CIP and the CEP then $\mathrm{Cw}(\mathcal{A})$ is a subdirect product of its sublattices $\mathrm{Con}(\mathcal{A})$ and $\mathrm{Sub}(\mathcal{A})$. This clearly implies part (b).

Now, to prove part (c), let $\rho \in \mathrm{Con}(\mathcal{B})$, $\theta \in \mathrm{Con}(\mathcal{C})$ and $\sigma \in \mathrm{Con}(\mathcal{D})$ such that $\rho \vee \theta = \rho \vee \sigma$. Let $\mathcal{E}$ denote the subalgebra $\mathcal{B} \vee \mathcal{C}$, which equals $\mathcal{B} \vee \mathcal{D}$. By the join-semidistributivity of $\mathrm{Sub}(\mathcal{A})$, $\mathcal{E} = \mathcal{B} \vee (\mathcal{C} \wedge \mathcal{D})$. From

$$\Delta_{\mathcal{E}} = \Delta_{\mathcal{B}} \vee (\Delta_{\mathcal{C}} \wedge \Delta_{\mathcal{D}}) \leq \rho \vee (\theta \wedge \sigma) \leq \mathcal{B}^2 \vee \mathcal{C}^2 \leq \mathcal{E}^2$$

we conclude that $\rho \vee (\theta \wedge \sigma) \in \mathrm{Con}(\mathcal{E})$. Now, first using the join-semidistributivity of $\mathrm{Con}(\mathcal{E})$ and, later at $=^*$, the CIP for $\mathrm{Con}(\mathcal{E})$ we obtain from $(\rho \vee \Delta_E) \vee (\theta \vee \Delta_E) = (\rho \vee \Delta_E) \vee (\sigma \vee \Delta_E)$ that

$$\rho \vee \theta = (\rho \vee \Delta_E) \vee (\theta \vee \Delta_E) = (\rho \vee \Delta_E) \vee ((\theta \vee \Delta_E) \wedge (\sigma \vee \Delta_E)) =^* \\ \rho \vee (\theta \wedge \sigma) \vee \Delta_E = \rho \vee (\theta \wedge \sigma).$$

□

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