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SOME RESULTS ON SEMIMODULAR LATTICES

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Abstract. Some recent results on semimodular lattices are surveyed, and some related results are given. The topics include a strong form of the Jordan-Hölder theorem, the semimodularity of subdirect products, representations as cover-preserving join-homomorphic images of distributive lattices, cover-preserving embeddings into geometric lattices, and the congruence lattices of finite almost-geometric lattices.

1. Introduction

Semimodularity, because of its connection with combinatorics, is an important part of lattice theory. Our chief goal is to survey some recent results on semimodular lattices; however, some related new results are also given. A part of the results here are quite easy but, surprisingly, have been overlooked previously. Some proofs will be outlined only or omitted while some others will be simplified. For an extensive survey of the theory of semimodular lattices the reader can resort to M. Stern [20].

A lattice L is called (upper) *semimodular*, if, for all $a, b \in L$, the covering relation $a \wedge b \prec a$ implies $b \prec a \vee b$. Equivalently, if $a \preceq b$ implies $a \vee c \preceq b \vee c$ for all $a, b, c \in L$. The *length* of a lattice L is denoted by $\text{length}(L)$; it is defined to be the supremum of $\{n \in \mathbb{N} : \text{there is an } n+1\text{-element chain in } L\}$. A lattice L is said to be of *locally finite length* if all of its intervals are of finite length. The *height* $h(x)$ of an element is $\text{length}([0, x])$. Semimodularity is an interesting notion mainly for lattices of locally finite length, when the lattice order is the transitive closure of the covering relation.

2. A strong form of the Jordan-Hölder theorem

A classical theorem going back to R. Dedekind, C. Jordan, O. Hölder and H. Wielandt states that the factors of any composition series of a finite group are invariant. Formulations of this result for modular or semimodular lattices are well-known; they are called the *Jordan-Hölder theorem*.

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Recall some notions and notations. Two-element intervals are called *prime intervals*. For intervals $[a,b]$ and $[c,d]$ of a lattice, $[a,b]$ is up-perspective to $[c,d]$, in notation $[a,b] \nearrow [c,d]$, if $b \wedge c = a$ and $b \vee c = d$. Down-perspectivity is the reverse relation: $[a,b] \searrow [c,d]$ iff $[c,d] \nearrow [a,b]$. Perspectivity is the union of these two relations, and projectivity is the transitive closure of perspectivity. The relational product of $\nearrow$ and $\searrow$ is a recent notion; let us say that $[a,b]$ is *up-and-down projective* to $[c,d]$ if there is an interval $[x,y]$ such that $[a,b] \nearrow [x,y]$ and $[x,y] \searrow [c,d]$. For each $n \in \mathbb{N}$, there are a semimodular lattice and prime intervals I_0 and I_n such that I_0 is projective to I_n in n (perspectivity) steps but not in $n-1$ steps; for $n \leq 7$ see the snake lattice in Figure 1.

Theorem 1 (G. Grätzer and J. B. Nation [14])**.** *Let* $0 = c_0 \prec c_1 \prec \cdots \prec c_n = 1$ *and* $0 = d_0 \prec d_1 \prec \cdots \prec d_m = 1$ *be maximal chains in a semimodular lattice of finite length. Then* $n = m$*, and there is a permutation* π *of the set* $\{1,\dots,n\}$ *such that* $[c_{i-1}, c_i]$ *is up-and-down projective to* $[d_{\pi(i)-1}, d_{\pi(i)}]$*, for all* $i \in \{1,\dots,n\}$*.*

The subnormal subgroups (that is, normal subgroups of normal subgroups of ... of normal subgroups) of a finite group form a lower semimodular lattice, see J. B. Nation [19], so the dual of this theorem easily yields the original Jordan-Hölder theorem.

3. Subdirect products of semimodular lattices

While the operator of forming sublattices and that of forming homomorphic images do not preserve semimodularity, and therefore we need to modify them in subsequent sections, we have

Theorem 2 (cf. [7])**.** *Subdirect products of (arbitrary many) semimodular lattices are semimodular.*

Proof. Let L_i, $i \in I$, be semimodular lattices, and let $L \subseteq \prod_{i \in I} L_i$ be a subdirect product of these lattices. Assume that $f, g \in L$ and $h := f \wedge g \prec f$. (Covering is always understood in L or L_i, but never in the direct product.)

If we had $h(i) \not\preceq f(i)$ for some $i \in I$, then for each x with $h(i) < x < f(i)$ there would be a $p \in L$ with $p(i) = x$, and $h < (h \vee p) \wedge f < f$ would contradict $h \prec f$. Hence $h(i) \preceq f(i)$ for all $i \in I$, and the semimodularity of L_i gives $g(i) \preceq f(i) \vee g(i)$.

Next, suppose that $b \in L$ and $g \leq b < f \vee g$. From the previous covering relation we obtain that $b(i) \in \{g(i), f(i) \vee g(i)\}$ for all $i \in I$. This together with $b < f \vee g$ yield a $j \in I$ such that $b(j) = g(j) < f(j) \vee g(j) = f(j) \vee b(j)$, implying $f(j) \not\leq b(j)$. Hence $f \neq f \wedge b$. But $h \leq f \wedge b \leq f$ and $h \prec f$, so $f \wedge b = h$.

Armed with $f \wedge b = h$, assume that $b(i) = f(i) \vee g(i)$ for a given $i \in I$. Then $h(i) = f(i) \wedge b(i) = f(i) \wedge \big(f(i) \vee g(i)\big) = f(i)$, which implies $f(i) \leq g(i)$ and $g(i) = f(i) \vee g(i)$, so $b(i) = g(i)$. Since $b(i) \in \{g(i), f(i) \vee g(i)\}$ for all

$i \in I$, we conclude that $b(i) = g(i)$ for all i, that is, $b = g$. This shows $g \prec f \vee g$ and the semimodularity of L. $\square$

4. Semimodular lattices of sectionally finite length as homomorphic images

Let K and L be lattices. We say that $\varphi\colon L \to K$ is a *cover-preserving join-homomorphism*, if $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ and $x \preceq y$ implies $\varphi(x) \preceq \varphi(y)$, for all $a, b, x, y \in L$. Like semimodularity, this notion is interesting only when our lattice has sufficiently many covering pairs. Hence we consider the class

$$\mathcal{C}^{sm}_{sf\ell} = \{\text{all semimodular lattices of sectionally finite length}\}.$$

(A lattice L is said to be of *sectionally finite length*, if $\operatorname{length}(\{x : x \le b\})$ is finite for all $b \in L$. Clearly, L is of locally finite length and $0 \in L$ in this case.) For a class $\mathcal{Y}$ of lattices, let

$$\mathbf{H}^{\vee}_{\prec}\,\mathcal{Y} = \{\text{all cover-preserving join-homomorphic images of members of } \mathcal{Y}\}.$$

The adequate morphism concept for the class $\mathcal{C}^{sm}_{sf\ell}$ is revealed by

Lemma 3. $\mathbf{H}^{\vee}_{\prec}\,\mathcal{C}^{sm}_{sf\ell} \subseteq \mathcal{C}^{sm}_{sf\ell}$.

This lemma is an easy generalization of Lemma 16 in G. Grätzer and E. Knapp [13] from the class of finite semimodular lattices to $\mathcal{C}^{sm}_{sf\ell}$. The following example shows that the class of *all* semimodular lattices is *not* closed with respect to forming cover-preserving join-homomorphic images.

Example 4. Let L be the direct square of the chain $(\mathbb{R}^+, \le)$ of non-negative real numbers. It is a semimodular lattice since it is distributive. The equivalence Θ whose non-singleton blocks are all the sublattices $S_c = \{(c, y) : y < 1\}$, $c \in \mathbb{R}^+$, and the principal filter $[(1, 1))$ is a join-congruence. The natural join-homomorphism $L \to L/\Theta$ is cover-preserving since $\prec_L$ is the empty relation. However, $[(0, 0)]\Theta \prec [(0, 1)]\Theta$ together with $[(0, 0)]\Theta \vee [(1, 0)]\Theta = [(1, 0)]\Theta < [(2, 0)]\Theta < [(1, 1)]\Theta = [(0, 1)]\Theta \vee [(1, 0)]\Theta$ show that L/Θ is not semimodular.

Proof of Lemma 3. Assume that $L \in \mathcal{C}^{sm}_{sf\ell}$ and $\varphi\colon L \to K$ is a cover-preserving join-homomorphism, $x, y, z \in K$ and $x \prec y$. Let $A = \{u \in L : \varphi(u) = x\}$, $B = \{u \in L : \varphi(u) = y\}$ and $C = \{u \in L : \varphi(u) = z\}$, and choose $a \in A$, $b \in B$ and $c \in C$. Then $\varphi(a \vee b) = \varphi(a) \vee \varphi(b) = x \vee y = y$ shows that $a \vee b \in B$. Consider a maximal chain $a = v_0 \prec v_1 \prec \cdots \prec v_n = a \vee b$ in the interval $[a, a \vee b]$, and let v_i be the first member of this chain outside A. Then $x < \varphi(v_i) \le \varphi(a \vee b) = y$ and $x \prec y$ yield that $v_i \in B$. Finally, since φ is cover-preserving, we conclude that $x \vee z = \varphi(v_{i-1}) \vee \varphi(c) = \varphi(v_{i-1} \vee c) \preceq \varphi(v_i \vee c) = \varphi(v_i) \vee \varphi(c) = y \vee z$. Hence K is semimodular, and it is trivially of sectionally finite length. $\square$

It was proved in G. Grätzer and E. Knapp [13] that finite planar semimodular lattices are cover-preserving join-homomorphic images of finite distributive lattices. Soon afterwards, the assumption planarity was dropped in [3]. Now, in the representation theorem below, we replace finiteness by "belonging to $\mathcal{C}^{sm}_{sf\ell}$". Let

$$\mathcal{D} = \{\text{all distributive lattices}\}.$$

Theorem 5. $\mathcal{C}^{sm}_{sf\ell} = \mathbf{H}^{\vee}_{\prec}(\mathcal{D} \cap \mathcal{C}^{sm}_{sf\ell})$.

Proof. Lemma 3 yields that $\mathcal{C}^{sm}_{sf\ell} \supseteq \mathbf{H}^{\vee}_{\prec}(\mathcal{D} \cap \mathcal{C}^{sm}_{sf\ell})$. To show the converse direction, let $L \in \mathcal{C}^{sm}_{sf\ell}$, and let $H = \{C_i : i \in I\}$ be the set of all maximal chains of L. Let $D := \{f \in \prod_{i\in I} C_i : f(i) = 0$ for all but finitely many $i \in I\}$. Since the C_i are of sectionally finite length, so is D. Hence $D \in \mathcal{D}\cap\mathcal{C}^{sm}_{sf\ell}$.

Next, define a mapping $\varphi : D \to L$, $f \to \bigvee_{i\in I} f(i)$. This makes sense, since only finitely many $f(i)$ are distinct from 0. Clearly, φ is a join-homomorphism. It is surjective since $L \subseteq \bigcup_{i\in I} C_i$.

Next, assume that $f \preceq g$ in D. Then there is an $i \in I$ such that $f(i) \preceq g(i)$ in C_i and $f(j) = g(j)$ for all j distinct from i. Since C_i is a maximal chain, $f(i) \preceq g(i)$ is valid also in L. By semimodularity, this relation is preserved by the unary algebraic function $x \vee \bigvee_{j\neq i} f(j)$, so we obtain

$$\varphi(f) = f(i) \vee \bigvee_{j\neq i} f(j) \preceq g(i) \vee \bigvee_{j\neq i} f(j) = g(i) \vee \bigvee_{j\neq i} g(j) = \varphi(g).$$

This shows that φ is cover-preserving, whence $L \in \mathbf{H}^{\vee}_{\prec}(\mathcal{D} \cap \mathcal{C}^{sm}_{sf\ell})$. □

Instead of homomorphic images, semimodular lattices are better understood as quotient lattices. A join-congruence Θ of L will be called a *cover-preserving join-congruence* if the natural mapping $L \to L/\Theta$, $x \mapsto [x]\Theta$ is a cover-preserving join-homomorphism. By a Θ-*forbidden covering square* of L we mean a quadruple $(a, b, a \wedge b, a \vee b) \in L^4$ such that $a \wedge b \prec a$, $a \wedge b \prec b$, $a \prec a\vee b$, $b \prec a\vee b$, the Θ-classes $[a]\Theta$, $[b]\Theta$ and $[a\wedge b]\Theta$ are pairwise distinct but $[a]\Theta = [a \vee b]\Theta$. For finite L, the next lemma is in [3].

Lemma 6. *Let $L \in \mathcal{C}^{sm}_{sf\ell}$, and let Θ be a join-congruence of L. Then Θ is cover-preserving iff L has no Θ-forbidden covering square.*

Proof. If $(a, b, a\wedge b, a\vee b)$ is a Θ-forbidden covering square of L then $a\wedge b \prec a$ together with $[a\wedge b]\Theta < [b]\Theta < [a\vee b]\Theta = [a]\Theta$ show that neither $L \to L/\Theta$, nor Θ is cover-preserving.

Conversely, by way of contradiction, assume that L has no Θ-forbidden covering square but $a \prec b$ and $[a]\Theta < [c]\Theta < [b]\Theta$ holds for some $a, b, c \in L$, see Figure 1. We may assume that $a \leq c$, for otherwise we could replace c by $a \vee c$. Take a maximal chain $x_0 = a \prec x_1 \prec \cdots \prec x_n = c$ in the interval $[a, c]$. Let $i \in \{1, 2, \ldots, n\}$ be the smallest subscript such that $x_i \notin [a]\Theta$. Then an easy consideration based on semimodularity yields that the black-filled elements in Figure 1 form a Θ-forbidden covering square, a contradiction. □

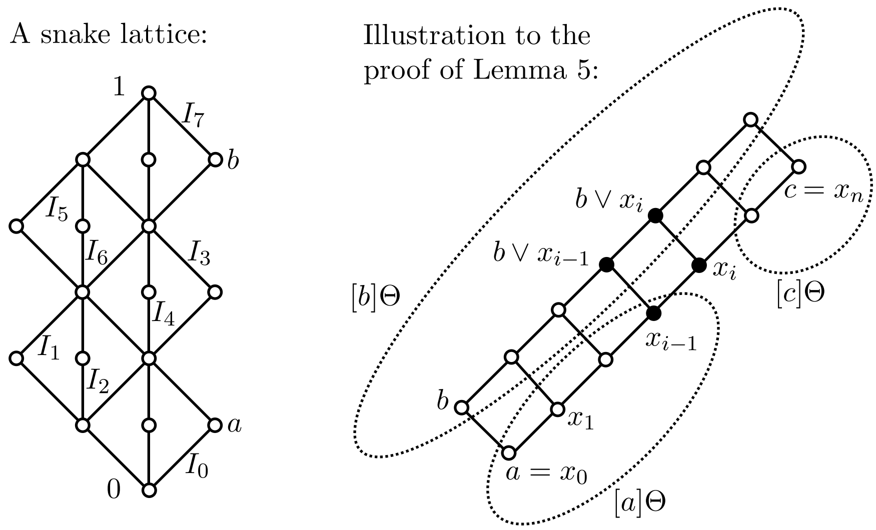

FIGURE 1

In the following three statements, we will consider *finite* lattices only. Recall that each finite distributive lattice is (up to isomorphism) uniquely determined by $(J(L), \leq)$, the partially ordered set of its nonzero join-irreducible elements.

Theorem 7 (cf. [3]). *For each finite semimodular lattice L, there is a unique finite distributive lattice D and a unique surjective cover-preserving join-homomorphism $\varphi\colon D \to L$ such that $J(L) = J(D)$ and φ acts identically on $J(D)$.*

A lattice is called *atomistic* if each of its element is a join of atoms. By a *geometric lattice* we mean a complete atomistic semimodular lattice in which each atom is a compact element.

Corollary 8 (cf. [3]). *Finite geometric lattices are characterized as cover-preserving join-homomorphic images of finite boolean lattices.*

Combining Theorem 5 and Lemmas 3 and 6 we can immediately see

Corollary 9 (cf. [3]). *Each finite semimodular lattice is (order-isomorphic to) D/Θ for a unique finite distributive lattice D and for some cover-preserving join-congruence Θ described in Lemma 6 such that $(J(D), \leq) \cong (J(L), \leq)$ and $\Theta\lceil_{J(L)\cup\{0\}}$ is the equality relation on $J(D) \cup \{0\}$. Conversely, if Θ is a cover-preserving join-congruence of a finite semimodular lattice L, then the join-semilattice L/Θ is a semimodular lattice.*

Restricting ourselves to lattice methods, it is not so easy to find enlightening small examples of semimodular lattices. A well-known method given right before Corollary IV.2.3 in G. Grätzer [10], now a consequence of Corollary 9 and Lemma 6, is the following: if L is a semimodular lattice of length

n and $k < n$, then $(\{x \in L : h(x) \le k \text{ or } x = 1\}, \le)$ is a semimodular lattice again. In order to extract a better knowledge of finite semimodular lattices from Corollary 9, which enables us to provide examples easily, the following three statements deal with join-congruences; finiteness will *not* be assumed.

Lemma 10 (Folklore)**.** *Let Θ be a congruence of a join-semilattice $(L, \vee)$. Then the blocks of Θ are convex subsemilattices.*

Lemma 11. *Let α_i, $i \in I$, be congruences of a join-semilattice $(L, \vee)$, and let β denote their join in the congruence lattice of $(L, \vee)$. Then, for each x, y in L, $(x, y) \in \beta$ iff there is a $k \in \mathbb{N}_0$ and there are elements $x = u_0 \le u_1 \le \cdots \le u_k = v_k \ge v_{k-1} \ge \cdots \ge v_0 = y$ in L such that $\{(u_{j-1}, u_j), (v_{j-1}, v_j)\} \subseteq \bigcup_{i\in I} \alpha_i$ for $j = 1, \ldots, k$.*

Proof. Let Φ denote the relation described in the lemma. Clearly, Φ is reflexive, symmetric, and $(x, y) \in \Phi$ easily implies $(x \vee c, y \vee c) \in \Phi$ for any $c \in L$. In order to prove the transitivity of Φ, assume $(x, y) \in \Phi$ as described in the lemma, and $(y, z) \in \Phi$ is witnessed by $y = w_0 \le w_1 \le \cdots \le w_k = s_k \ge s_{k-1} \ge \cdots \ge s_0 = z$ (the same k, by reflexivity). Then it is easy to see that

$$\begin{aligned} x = u_0 \le u_1 \le \cdots \le u_k = v_k \vee w_0 \le v_k \vee w_1 \le \cdots \le v_k \vee w_k \\ \ge w_k \vee v_{k-1} \ge \cdots \ge w_k \vee v_0 = w_k = s_k \ge s_{k-1} \ge \cdots s_0 = z \end{aligned}$$

witnesses $(x, z) \in \Phi$. Thus, Φ is a congruence. Evidently, $\beta \subseteq \Phi$. If $(x, y) \in \Phi$, then Lemma 10 gives $(x, x \vee y), (y, x \vee y) \in \beta$, so $(x, y) \in \beta$. Hence $\Phi \subseteq \beta$. □

For $a \le b$ in $(L, \vee, \wedge)$, the (principal) join-congruence resp. lattice congruence generated by $\{(a, b)\}$ will be denoted by $\mathrm{con}^{\vee}(a, b)$ resp. $\mathrm{con}^{\vee\wedge}(a, b)$. If b covers a, then $\mathrm{con}^{\vee}(a, b)$ will be called a *prime join-congruence*. Each join-congruence of $L \in \mathcal{C}^{sm}_{sf\ell}$ is the join of some prime join-congruences, so it is worth describing prime join-congruences at least for distributive lattices.

Theorem 12. *Let D be a distributive lattice, and let $a \prec b \in D$. Then, for any $x, y \in L$, (x, y) belongs to the prime join-congruence $\mathrm{con}^{\vee}(a, b)$ if and only if*

(1) *either $x = y$,*
(2) *or $a \le x \prec y$ and $y = b \vee x$,*
(3) *or $a \le y \prec x$ and $x = b \vee y$.*

In particular, every block of a prime join-congruence consists of at most two elements. If part (2) resp. (3) holds then $a = x \wedge b$ resp. $a = y \wedge b$.

Proof. Assume that $x \ne y$ and $(x, y) \in \mathrm{con}^{\vee}(a, b)$. Let $x' := x \wedge y$ and $y' := x \vee y$. Note that $x' < y'$. Since $\mathrm{con}^{\vee}(a, b) \subseteq \mathrm{con}^{\vee\wedge}(a, b)$, we obtain that $(x', y') \in \mathrm{con}^{\vee\wedge}(a, b)$. The classical description of principal congruences of distributive lattices in [16] (see also Thm. II.3.3 in Grätzer [10]) says that $a \wedge x' = a \wedge y'$ and $b \vee x' = b \vee y'$. Since $(a \vee x') \vee y' = a \vee y'$ and

$$x' \le (a \vee x') \wedge y' = (a \wedge y') \vee (x' \wedge y') = (a \wedge x') \vee (x' \wedge y') \le x',$$

we have $[x', y'] \nearrow [a \vee x', a \vee y']$. Since $b \wedge (a \vee x') = \big(b \wedge (a \vee y')\big) \wedge (a \vee x')$ and, using modularity at the second equation sign,

$$a \vee y' \geq (a \vee x') \vee \big(b \wedge (a \vee y')\big) = (a \vee x' \vee b) \wedge (a \vee y') = (y' \vee b) \wedge (a \vee y') = a \vee y',$$

we obtain that $[a \vee x', a \vee y'] \searrow [b \wedge (a \vee x'), b \wedge (a \vee y')]$. Hence $[x', y']$ is projective (and therefore isomorphic) to a subinterval of the prime interval $[a, b]$, and we conclude that $x' \prec y'$. This excludes $x \parallel y$, and we obtain that $x \prec y$ or $y \prec x$. Hence every block of $\mathrm{con}^{\vee}(a, b)$ has at most two elements.

Now let Φ denote the relation described by the disjunction of (1), (2) and (3) of the lemma. Clearly, $\Phi \subseteq \mathrm{con}^{\vee}(a, b)$. Further, $(a, b) \in \Phi$, Φ is reflexive and symmetric, and $(x, y) \in \Phi$ implies $(x \vee c, y \vee c) \in \Phi$ for any $c \in D$.

Finally, assume that $\{(x, y), (y, z)\} \subseteq \Phi$. Since every block of $\mathrm{con}^{\vee}(a, b)$ has at most two elements and $\Phi \subseteq \mathrm{con}^{\vee}(a, b)$, we obtain $|\{x, y, z\}| \leq 2$. Hence $(x, z) \in \Phi$, showing the transitivity of Φ. $\square$

Example 13. The intersection of cover-preserving join-congruences need not be cover-preserving, and we cannot speak of "generated cover-preserving join-congruences". Indeed, there is no smallest cover-preserving join-congruence that extends $\alpha \cap \beta$ where α and β are cover-preserving join-congruences that correspond to the partitions $\{\{a, 1\}, \{0, b\}\}$ and $\{\{a, b, 1\}, \{0\}\}$ of the four-element boolean lattice.

Method 14 (to obtain examples of small semimodular lattices). Start from a known finite semimodular lattice L, usually from a distributive one. Based on Lemma 11 and Theorem 12, find a join-congruence Θ. (If $J(L)$ should not change, then $\Theta\lceil_{J(L)\cup\{0\}}$ should be the equality relation, see Corollary 9.) If there is a Θ-forbidden covering square (see Lemma 6), then choose a covering pair $u \prec v$ with $(u, v) \notin \Theta$ in this square and replace Θ by $\Theta \vee \mathrm{con}^{\vee}(u, v)$ (see Theorem 12). Iterate this step until no Θ-forbidden covering square remains. Finally, the join-semilattice L/Θ is a semimodular lattice.

In the rest of this section, we mention some applications of representing semimodular lattices as join-homomorphic images. We say that a finite lattice L with $|L| \geq 2$ satisfies *Frankl's conjecture* from 1979, see P. Frankl [9], if the principal filter $\uparrow f = \{x \in L : f \leq x\}$ has at most $|L|/2$ elements, that is, $2 \cdot |\uparrow f| - |L| \leq 0$, for at least one $f \in J(L)$. We say that L satisfies the *averaged Frankl's property*, if $\sum_{f \in J(L)} \big(2 \cdot |\uparrow f| - |L|\big) \leq 0$, see [1]. (This is a much stronger property, which fails in many finite lattices.) After dozens of partial positive results supporting Frankl's conjecture, see the bibliography of [2], V.l Božin announced a counterexample at the 3rd Novi Sad Algebraic Conference, August, 2009. (Notice however that his construction neither has been checked, nor has been submitted at the time of this writing.) The maximum size of a lattice L is 2^m where $m = |J(L)|$.

Theorem 15 (cf. [1], [2] and [4]). *Let L be a finite lattice with $n = |L|$ and $m = |J(L)|$.*

(1) *If* $m \geq 3$ *and* $n \geq 2^m - \sqrt{2^m}$*, then* L *satisfies the averaged Frankl's property.*
(2) *If* $m \geq 3$*,* $n > 2 \cdot 2^m/3$*, and Frankl's conjecture holds for all lattices* K *with* $|J(K)| \leq m$*, then* L *satisfies the averaged Frankl's property.*
(3) *If* $n > 5 \cdot 2^m/8$ *and* L *is semimodular, then* L *satisfies Frankl's conjecture.*
(4) *Every planar semimodular lattice satisfies Frankl's conjecture.*

Part (4) of this theorem is based on the particular case of Theorem 5 that is present already in G. Grätzer and E. Knapp [13]. Parts (1) and (3) also rely on join-homomorphisms.

5. A cover-preserving embedding of semimodular lattices into geometric lattices

If K is a sublattice of a lattice L such that, for all $a, b \in K$, $a \prec_K b$ iff $a \prec_L b$, then K is called a *cover-preserving sublattice* of L and L is said to be a *cover-preserving extension* of K. If length(L) is finite, then cover-preserving sublattices are exactly those sublattices that have the same length as L. While sublattices of a semimodular lattice L are not semimodular in general, cover-preserving sublattices of L inherit semimodularity.

Theorem 16 (cf. [5])**.** *Each semimodular lattice of finite length has a cover-preserving embedding into an appropriate geometric lattice.*

The particular case of *finite* semimodular lattices was settled by G. Grätzer and E. W. Kiss [12], see M. Wild [21] for a different approach.

Proof of Theorem 16 (only the construction). Given a semimodular lattice L of finite length, we define a cover-preserving extension $\mathcal{G}(L)$ which is a geometric lattice. Denote the set of atoms by $A(L)$, and let $H(L) := J(L) \setminus A(L)$. In Figure 2, $H(L)$ is the set of black-filled elements. For each

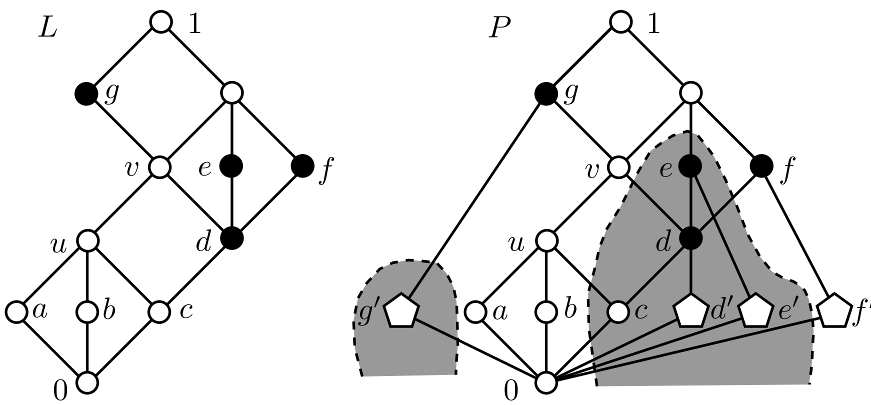

FIGURE 2. An example of L and P, and an ideal J

$x \in H(L)$, we insert a new element x' such that $0 \prec x' \prec x$. This way we obtain $P = (P; \leq)$, see Figure 2, where the new elements are the pentagon-shaped ones. Although P is an atomistic lattice, it is not semimodular in general, see Figure 2. Hence we will consider P as a *partial* join-semilattice $P = (P; \vee_P)$. Briefly speaking, $\vee_P$ will be the largest extension of $\vee_L$ to P such that $P = (P; \vee_P)$ is a "semimodular partial join-semilattice". The exact definition of $\vee_P$ is the following.

- If $x, y \in P$ are comparable or $\{x, y\} \subseteq L$, then $x \vee_P y$ is defined and it has the usual meaning.
- If $x, y \in P \setminus L$ and $x \neq y$, then $x \vee_P y$ is undefined.
- Suppose that $x \in L$, $y \in P \setminus L$, and $x \parallel y$. Then $y = z'$ for a unique $z \in H(L)$ and $x \vee_P y$ is defined iff $x \vee_L z$ covers x in L; if $x \vee_P y$ is defined, then it equals $x \vee_L z$, so it is the supremum of $\{x, y\}$.
- Suppose that $x \in P \setminus L$, $y \in L$, and $x \parallel y$. Then $x \vee_P y$ is defined iff $y \vee_P x$ is defined according to the previous case; if $x \vee_P y$ is defined then $x \vee_P y = y \vee_P x$.

For example, in Figure 2, $u \vee_P d' = v$, $c \vee_P d' = d$, and $g \vee_P f' = 1$, while $b \vee_P d'$ and $d' \vee_P e'$ are undefined.

Non-empty order-ideals closed with respect to $\vee_P$ are called *ideals* of P. Since the intersection of ideals is an ideal again, the ideals of P form a complete lattice $\mathcal{I}(P) = (\mathcal{I}(P), \subseteq)$. For $I \in \mathcal{I}(P)$, the largest element of $I \cap L$, that is $\bigvee(I \cap L)$, is called the *trunk* of I, while the set $\{x \in I : x \not\leq \mathrm{trunk}(I)\}$ is the *branch* of I. For example, the elements in the gray area with dotted boundary in Figure 2 constitute an ideal $J = \{0, c, d', e', d, e, g'\}$ with trunk e and branch $\{g'\}$. For brevity, an ideal with trunk t and branch $\{b_1, \ldots, b_k\}$ will be denoted by $t;b_1...b_k$; for example, $J = e;g'$. This makes sense since the trunk and the branch together determine the ideal.

The *rank* $r(I)$ of an ideal I is defined to be $h(\mathrm{trunk}(I)) + |\mathrm{branch}(I)|$. For example, $\mathrm{rank}(J) = 3 + 1 = 4$, see Figure 2. We say that $I \in \mathcal{I}(P)$ is a *rank-jumper ideal*, if for all $J \in \mathcal{I}(P)$, $I \subset J$ implies $r(I) < r(J)$. For the lattice L in Figure 3 (the circle-shaped elements of P), all ideals but $0; b'c'd'$ are rank-jumper. It is shown in [5] that the rank-jumper ideals form a complete meet-subsemilattice $\mathcal{R}(P)$ of $\mathcal{I}(P)$, and $\mathcal{G}(L) := \mathcal{R}(P)$ is a geometric lattice with $\mathrm{length}(\mathcal{G}(L)) = \mathrm{length}(L)$. (In Figure 3, the square-shaped elements show how L is embedded in $\mathcal{G}(L)$.) $\square$

When applied to a *finite* lattice L, our construction preserves distributivity while G. Grätzer and E. W. Kiss [12] embed the three-element chain into M_3. However, even our $\mathcal{G}(L)$ is not the smallest cover-preserving geometric lattice extension of L. Indeed, $\mathcal{G}(L)$ in Figure 3 consists of twelve elements while L is a cover-preserving sublattice of the ten-element direct product of M_3 and the two-element chain.

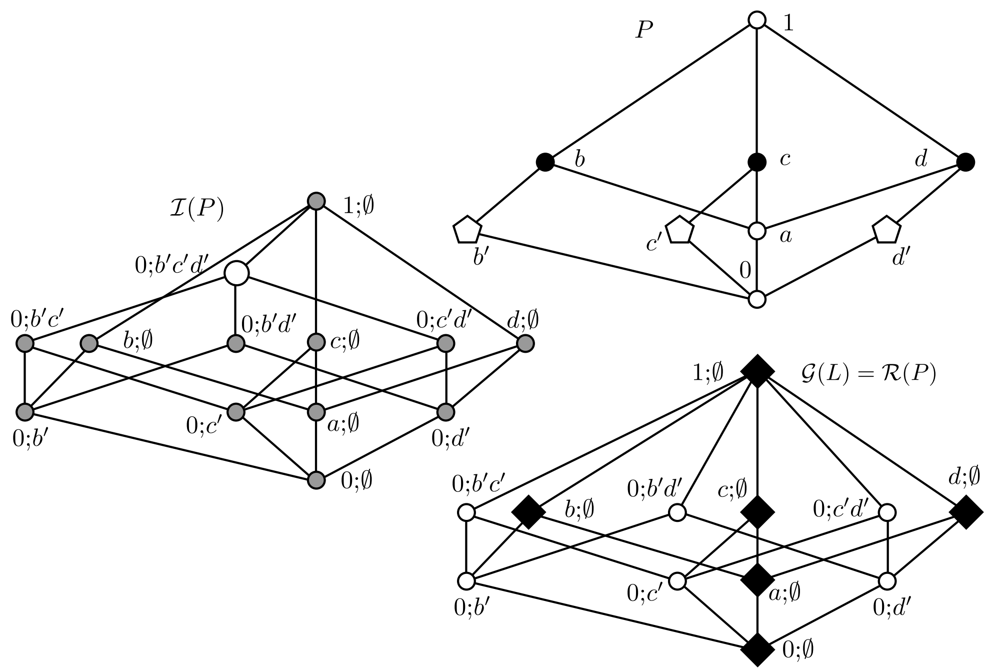

FIGURE 3. Rank-jumper ideals and $\mathcal{G}(L) = \mathcal{R}(P)$

6. Representing distributive lattices by almost-geometric lattices

A classical theorem of R. P. Dilworth states that each finite distributive lattice D is isomorphic to the congruence lattice of an appropriate finite lattice L. The first proof appeared in [15]. This result was followed by a great number of stronger results in which further properties of L are stipulated, see [18] or G. Grätzer [11] for an extensive overview. We conclude this paper with a result of this kind on semimodular lattices. The omitted proof is based on the chopped lattice technique developed in [17], see also [18] or G. Grätzer [11].

We say that a partially ordered set $P = (P, \leq)$ is a cardinal sum of at most two-element chains, if for every $a \in P$, both $\{x \in P : x \leq a\}$ and $\{x \in P : x \geq a\}$ are at most two-element. By an *almost-geometric lattice* we mean a semimodular lattice L of finite length such that $(J(L), \leq)$ is a cardinal sum of at most two-element chains. Geometric lattices of finite length are almost-geometric and simple. Hence we cannot drop "almost" from the following theorem.

Theorem 17 (cf. [6]). *Each finite distributive lattice is isomorphic to the congruence lattice of a finite almost-geometric lattice.*

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