SLIM SEMIMODULAR LATTICES. II. A DESCRIPTION BY PATCHWORK SYSTEMS

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Abstract. Rectangular lattices are special planar semimodular lattices introduced by G. Grätzer and E. Knapp in 2009. A *patch lattice* is a rectangular lattice whose weak corners are coatoms. As a variant of gluing, we introduce the concept of a *patchwork system*. We prove that every glued sum indecomposable, planar, semimodular lattice is a patchwork of its maximal patch lattice intervals. For a planar modular lattice, our patchwork system is the same as the S-glued system introduced by C. Herrmann in 1973. Among planar semimodular lattices, patch lattices are characterized as the patchwork-irreducible ones. They are also characterized as the indecomposable ones with respect to gluing over chains; this gives another structure theorem.

1. Introduction

Rectangular lattices were introduced by G. Grätzer and E. Knapp [14]. Intuitively, a *rectangular lattice* is a semimodular lattice that allows an esthetic planar diagram with rectangular contour. The smallest rectangular lattice is the four-element Boolean lattice $\mathbf{2}^2$. If L is a non-chain lattice such that each $x \in L - \{0, 1\}$ is incomparable with some element of L, then L is *glued sum indecomposable*.

Let L be a glued sum indecomposable, planar, distributive lattice. Then L can be decomposed into $\mathbf{2}^2$-intervals (that is, intervals isomorphic to $\mathbf{2}^2$), and for any two distinct $\mathbf{2}^2$-intervals I and J,

$$I \cap J \text{ is a chain, or } I \cap J = \varnothing. \tag{1.1}$$

The collection of these intervals is the simplest example of a patchwork system.

S-glued systems were introduced by C. Herrmann [16]. Let M be a glued sum indecomposable, planar, modular lattice. Then the maximal atomistic (equivalently, complemented) intervals of M are rectangular lattices of length two, and they form an S-glued system.

Motivated by these two examples, our goal is the develop a theory of patchwork systems for all planar *semimodular* lattices; see Figures 2 and 3 for a first impression. Rectangular lattices whose weak corners are coatoms will be called *patch lattices*. Patch lattices will also be characterized as semimodular lattices indecomposable with respect to gluing (also called the Hall-Dilworth gluing) over chains. Hence patch lattices give rise to two structure theorems for planar semimodular lattices: one of them, Theorem 3.6, is based on patchwork systems, while the other

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one, Corollary 3.5, is based on gluing over chains. Based on [6], we will give a constructive visual structure theorem, Theorem 3.4(vii), for patch lattices.

Outline. Section 2 surveys those known concepts and facts on planar semimodular lattices that we need in the paper. Section 3 gives the most important new concepts and the main results. Many of the concepts we deal with depend on the planar diagram chosen. This motivates the study of these diagrams and some related questions in Section 4. Section 5 is devoted to some properties of elements and rectangular intervals of L that do not depend on the diagram of L. Section 6 presents some properties of a planar semimodular lattice L that depend only on the slim semimodular lattice "canonically" derived from L. Section 7 proves that all properties of L that are really important from our perspective are independent from the diagram of L. Section 8 formulates and proves several properties of patchwork-indecomposable lattices; these properties are consequences of the main results stated in Section 3. Section 9 proves some properties of patchwork-indecomposable intervals of planar semimodular lattices. Section 10 completes the paper by proving the main results formulated in Section 3.

2. Preliminaries

We aim at *planar* semimodular lattices. Most questions of the planar case are easily deductible from the slim one; for example, see Proposition 2.1 and Lemma 6.1 later. This fact and the number "I" in [6] explains that the adjective "slim" rather than "planar" occurs in the title of the present paper. The importance of slim (and planar) semimodular lattices is surveyed in [6].

Basic concepts. Although the reader is assumed to be familiar with the concepts given in G. Grätzer [12] and [6], we recall many of them for emphasis, notation, or later reference. A lattice M is *slim* if it is finite and the order $\mathrm{Ji}\, M$ of (non-zero) join-irreducible elements of M contains no three-element antichain. (Orders are also called partially ordered sets or posets.) Equivalently, see R. P. Dilworth [10], G. Grätzer and E. Knapp [13] or [6], a finite lattice M is slim iff $\mathrm{Ji}\, M$ is the union of two chains. The systematic study of slim semimodular lattices started in G. Grätzer and E. Knapp [13]. By [5, Lemma 6], slim lattices are planar. All lattices occurring in this paper are assumed to be finite, even if this is not mentioned all the time. In particular, if a lattice is slim or planar, then it is finite by definition. In most of the cases, our lattices are assumed to have at least four elements.

A straightforward but extremely useful property of slim lattices, see [6, Lemma 2], is that

(2.1) each element of a slim lattice has at most two covers.

Another pleasant property is that

(2.2) every interval of a slim lattice is slim;

this follows from the fact that $\{a \vee x : x \in \mathrm{Ji}\, L,\ x \leq b\}$ join-generates $[a, b]$.

Let $\mathrm{Dgr}(L)$ stand for the set of all *planar* diagrams of L. The general convention throughout the paper is that a planar diagram $D \in \mathrm{Dgr}(L)$ is *fixed*, unless otherwise stated. Many concepts we are going to define depends on the choice of D, at least seemingly. However, in several cases we will prove that this dependence is only apparent without being real. The diagram D divides the plane into *cells*. In presence of semimodularity, they are 4-*cells* and *covering squares*. By [6, Prop. 1], for any planar semimodular lattice L and for an arbitrary $D \in \mathrm{Dgr}(L)$,

(2.3) L is slim iff all of its covering squares of are 4-cells.

This is the original definition of slimness in G. Grätzer and E. Knapp [13] for the semimodular case. Notice that the expression "4-cells" in (2.3) is a short form of the more precise "4-cells of the fixed diagram D" or "its 4-cells with respect to D"; similar terminology will frequently occur. A lattice L (in particular, a chain) is called *nontrivial* if it contains at least two elements. If L is a non-chain lattice and for each $x \in L - \{0, 1\}$, there is a $y \in L$ such that x and y are incomparable, then L is called *glued sum indecomposable*.

Given a fixed planar diagram D of a lattice L, it has a *left boundary chain* $\mathrm{C_l}(D)$, a *right boundary chain* $\mathrm{C_r}(D)$, and a *boundary* $\mathrm{Bnd}(D) = \mathrm{C_l}(D) \cup \mathrm{C_r}(D)$. We often write $\mathrm{C_l}(L, D)$ and, if there is no danger of confusion, $\mathrm{C_l}(L)$ instead of $\mathrm{C_l}(D)$, and we often do similarly for other subsets of D. The *interior* $L - \mathrm{Bnd}(L)$ of L is denoted by $\mathrm{int}(L) = \mathrm{int}(L, D)$. For a rigorous treatment of these concepts see D. Kelly and I. Rival [17]. Notice that, by [17, Prop. 2.2],

(2.4) $\mathrm{Ji}\, L \cap \mathrm{Mi}\, L \cap \mathrm{C_l}(L) \neq \varnothing \quad \text{and} \quad \mathrm{Ji}\, L \cap \mathrm{Mi}\, L \cap \mathrm{C_r}(L) \neq \varnothing,$

provided $|L| \geq 3$. We have to recall some further concepts and properties of planar lattices from [17]. Let L be a planar lattice with a fixed planar diagram D. If C is a maximal chain of L, then it has a *left side*, denoted by $\mathrm{LS}(C, D)$, and a *right side* $\mathrm{RS}(C, D)$. Notice that $\mathrm{LS}(C, D) \cup \mathrm{RS}(C, D) = L$ and $C = \mathrm{LS}(C, D) \cap \mathrm{RS}(C, D)$. The *strict sides* of C (with respect to D) are $\mathrm{LS}(C, D) - C$ and $\mathrm{RS}(C, D) - C$. If $a \leq b$ in L, $|[a, b]| \geq 3$ and C_1 and C_2 are maximal chains of $[a, b]$ such that $C_1 \cap C_2 = \{a, b\}$, then C_1 and C_2 determine a so-called *region* R of L. It is a convex sublattice with $\{\mathrm{C_l}(R), \mathrm{C_r}(R)\} = \{C_1, C_2\}$. Its interior, $\mathrm{int}(R)$, is $R - \mathrm{Bnd}(R)$. Assume that $u \in R$ and $v \in L - R$, or conversely. Let $a, b, c \in L$. Further, let C be a maximal chain of L, and let $x, y \in L$ such that x and y are on different sides of C. Then, by [17, Lemmas 1.2 and 1.5], by the definition of a region, and by (2.7),

(2.5) If $x \leq y$, then there is a $z \in C$ with $x \leq z \leq y$;

(2.6) every interval is a region;

(2.7) if $u \leq v$, then there is a $w \in \mathrm{Bnd}(R)$ with $u \leq w \leq v$;

(2.8) if $b \in \mathrm{int}(R)$ and $a \prec b \prec c$, then $a, c \in R$.

When referring to properties of regions, we often use (2.6) implicitly. By the exact definition of a region, given in [17], we also have that

(2.9) if R is a region, then $\mathrm{int}(R) \subseteq \mathrm{int}(L)$.

For slim lattices, we can assert even more. By [6, Lemma 6],

(2.10) $\mathrm{Ji}\, L \subseteq \mathrm{Bnd}(L, D)$, provided L is slim.

Also, if L is slim, then $\mathrm{Bnd}(L)$ is uniquely determined by [6, Lemma 7]. That is, $\mathrm{Bnd}(L, D) = \mathrm{Bnd}(L, F)$, for all $D, F \in \mathrm{Dgr}(L)$. By [6, Lemma 7], if L is slim and glued sum indecomposable, then even $\{\mathrm{C_l}(L), \mathrm{C_r}(L)\}$ is uniquely determined. That is, the left and the right boundary chains are determined up to symmetry. (For a stronger statement, see Lemma 4.7 later.)

Let L be a planar semimodular lattice with a fixed $D \in \mathrm{Dgr}(L)$. By a *weak corner* of D we mean a doubly irreducible element d on the boundary of L such that d is distinct from 0 and 1. (Sometimes we speak of weak corners of L even if they depend on D.) Following G. Grätzer and E. Knapp [14], by a *rectangular lattice* we mean a planar semimodular lattice L such that L has a planar diagram D such that $\mathrm{C_l}(L,D)$ has exactly one weak corner, denoted by $w_l(L)$, $w_l(D)$ or $w_l(L,D)$, $\mathrm{C_r}(L,D)$ has exactly one weak corner, denoted by $w_r(L)$, $w_r(D)$ or $w_r(L,D)$, and they are complementary, that is, $w_l(D)\wedge w_r(D)=0$ and $w_l(D)\vee w_r(D)=1$. (Although the weak corners depend on D, Lemma 5.5 will show later that all planar diagrams are equally appropriate to check whether L is rectangular.) Clearly, rectangular lattices have at least four elements and they cannot be chains. If L is slim, then, by the already mentioned [6, Lemma 7], $\{\mathrm{C_l}(L,D),\mathrm{C_r}(L,D)\}$ and $\{w_l(L,D),w_r(L,D)\}$ do not depend on the planar diagram chosen. (In fact, Lemma 4.7 will state even more.) It is easy to see (and we know it from [6, before Prop. 10]) that for a slim (not just planar) semimodular lattice L, L is rectangular iff $\mathrm{Ji}\,L$ is the union of two disjoint chains C and W such that every element of C is incomparable with all elements of W.

For a rectangular lattice L and $D\in\mathrm{Dgr}(L)$, we define the left and right *top boundary chains*

$$\mathrm{C_{ul}}(L,D)={\uparrow}w_l(D)\cap\mathrm{C_l}(D),\qquad \mathrm{C_{ur}}(L)={\uparrow}w_r(D)\cap\mathrm{C_r}(D),$$

the *bottom boundary chains*

$$\mathrm{C_{ll}}(L,D)={\downarrow}w_l(D)\cap\mathrm{C_l}(D),\qquad \mathrm{C_{lr}}(L,D)={\downarrow}w_r(D)\cap\mathrm{C_r}(D),$$

and the *upper and lower boundaries*

$$\mathrm{UBnd}(L,D)=\mathrm{C_{ul}}(D)\cup\mathrm{C_{ur}}(D),\qquad \mathrm{LBnd}(L,D)=\mathrm{C_{ll}}(D)\cup\mathrm{C_{lr}}(D).$$

We know from G. Grätzer and E. Knapp [14, Lemmas 3 and 4] and from the definition of a rectangular lattice that, for each rectangular lattice L,

(2.11) $\mathrm{C_{ul}}(L,D)$, $\mathrm{C_{ur}}(L,D)$, $\mathrm{C_{ll}}(L,D)$, and $\mathrm{C_{lr}}(L,D)$ are indeed chains,

$$\begin{aligned}&\mathrm{C_{ul}}(L,D)={\uparrow}w_l(D),\quad \mathrm{C_{ur}}(L,D)={\uparrow}w_r(D),\\ &\mathrm{C_{ll}}(L,D)={\downarrow}w_l(D),\quad \mathrm{C_{lr}}(L,D)={\downarrow}w_r(D),\end{aligned}\tag{2.12}$$

$$\mathrm{UBnd}(L,D)-\{1\}\subseteq\mathrm{Mi}\,L,\qquad \mathrm{LBnd}(L,D)-\{0\}\subseteq\mathrm{Ji}\,L,\tag{2.13}$$

(2.14) each element of $\mathrm{Bnd}(L,D)-\mathrm{UBnd}(L,D)$ has at least two covers, and

(2.15) each element of $\mathrm{Bnd}(L,D)-\mathrm{LBnd}(L,D)$ has at least two lower covers.

Let d be a doubly irreducible element of a slim semimodular lattice L. Then d belongs to (at least) one of the boundary chains $\mathrm{C_l}(I,D)$ and $\mathrm{C_r}(I,D)$ by (2.10). Since this boundary chain is a maximal chain, it contains the unique lower cover d_* and the unique upper cover d^* of d. If d_* has exactly two upper covers and d^* has exactly two lower covers, then d is called a *corner* of D. Note that corners are weak corners but (even for rectangular lattices) not conversely. A corner can be *removed* and a slim semimodular sublattice remains by [6, Prop. 10].

Some earlier structure theorems. Let S be a 4-cell of a planar diagram D of a planar lattice L. Replace this 4-cell by a copy of M_3, the five-element non-distributive modular lattice (with a fixed diagram). This means that we insert a new element, which is called an *eye*, into the interior of S, and this way we divide S into two new 4-cells. This way we obtain a new diagram that determines a new lattice. If $D^\bullet$ and $L^\bullet$ are obtained from D and L by inserting eyes one-by-one, then $D^\bullet$ and $L^\bullet$ are called an *anti-slimming* of D and L, respectively, and $D^\bullet\in\mathrm{Dgr}(L^\bullet)$.

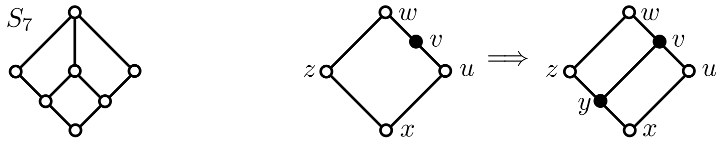

FIGURE 1. S_7 and the downward-going procedure

A 0-1-*sublattice* means a sublattice with the same 0 and 1. We recall the following statement.

Proposition 2.1 (G. Grätzer and E. Knapp [13]). *Each planar semimodular lattice L is an anti-slimming of one of its slim semimodular* 0-1-*sublattices, L'.*

Proposition 2.1 reduces most of the questions on planar semimodular lattices to the slim case. Let $D \in \mathrm{Dgr}(L)$ be fixed. Then the sublattice L' (with the corresponding diagram D') above is called a *full slimming sublattice* of L. More exactly, D' is obtained from D by omitting all elements from the interiors of intervals of length two. For a fixed L' (which depends only on D), the elements of $L - L'$ (or those of $D - D'$) are called *eyes.* Clearly, for each eye $e \in L - L'$, if e_* and e^* denote the unique lower and upper cover of e, respectively, then

$$e_* \text{ and } e^* \text{ belong to a unique 4-cell } \{e_*, a, b, e^*\} \text{ of } L'. \tag{2.16}$$

Let us emphasize the difference between *a full slimming sublattice* of L, which is a sublattice (a concrete subset of L) and depends on D, and *the full slimming* of L, which is an abstract lattice, not a concrete sublattice of L. While L can have many full slimming sublattices, as witnessed by $L = M_3$, the full slimming of L will turn out to be unique, see Remark 4.2.

The first structure theorem for slim semimodular lattices is due to G. Grätzer and E. Knapp [13], and it was soon generalized in [4]. (We have recently discovered that even the generalized version was already present but well-hidden in M. Stern [19]. However, it is [13] that initiated a rapid development leading to the present work.) Other structure theorems were given in [6] (two theorems), [2], and [7]; we will need and recall only one of them. Let S be a 4-cell of a slim semimodular lattice L, with respect to $D \in \mathrm{Dgr}(L)$. Then S is a covering square $\{a = b_1 \wedge b_2,\ b_1,\ b_2,\ c = b_1 \vee b_2\}$. We change L to a new lattice L^* as follows. Firstly, we replace S by a copy of S_7; see Figure 1 for its definition. This way we get three new 4-cells instead of S. Secondly, as long as there is a chain $u \prec v \prec w$ such that v is a new element and $T = \{x = u \wedge z,\ z,\ u,\ w = u \vee z\}$ is a 4-cell in the original lattice L but $x \prec z$ at the present stage, see Figure 1, we insert a new element y such that $x \prec y \prec z$ and $y \prec v$. (This way we get two 4-cells to replace the 4-cell T.) When this "downward-going" procedure terminates, we obtain L^*. The collection of all new elements, which is an order (also called poset), will be called a *fork.* We say that L^* is obtained from L by *adding a fork to L* (*at the* 4-*cell S*). For an illustration, see see Figure 2, where L_i is obtained from L_{i-1} by adding a single fork; the new elements of L_i, which form a fork, are the black-filled ones. Adding forks to L means adding several forks to L one by one. For example, L_3 in Figure 2 is obtained from $L_0 = \mathbf{2}^2$ by adding forks, in three steps. By a *grid* we mean the direct product of two finite, nontrivial chains. (The smallest grid is $\mathbf{2}^2$.) We are now ready to recall

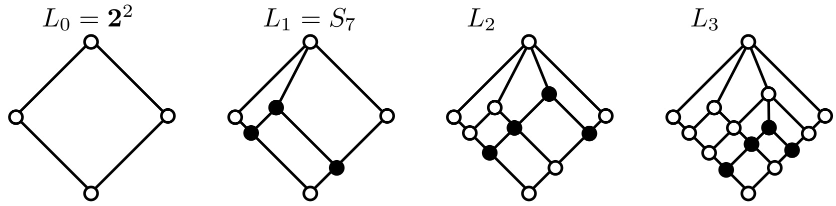

FIGURE 2. Forks and slim patch lattices

Proposition 2.2 ([6, Theorem 12]). *Let L be a slim semimodular lattice consisting of at least three elements. Then L can be obtained from a grid such that*

(i) *first we add finitely many (possibly zero) forks one by one,*
(ii) *and then we remove some (possibly zero) corners, one by one.*

For later reference, we formulate a trivial statement, see also [8, Figure 1].

Lemma 2.3. *Each nontrivial finite lattice is uniquely decomposable as a glued sum of nontrivial chains and glued sum indecomposable lattices.*

Rectangular lattices are of separate interest, not only in the present paper but also in G. Grätzer and E. Knapp [14], [15], [3] and [18]. In connection with parts (iv) and (vii) of (the forthcoming) Theorem 3.4, we present the following structure theorem for them. Remember that grids are defined right before Proposition 2.2.

Proposition 2.4 (Mainly [6, Lemma 22], as detailed in Section 10). *Let L be an arbitrary* slim *rectangular lattice. Then*

(i) *there is a grid G such that L can be obtained from G by adding forks;*
(ii) *For all $D \in \mathrm{Dgr}(L)$, G is (isomorphic to) $\uparrow w_l(L,D) \times \uparrow w_r(L,D)$. Consequently, G is uniquely determined up to isomorphism.*
(iii) *Every lattice obtained from a grid by adding forks is a slim rectangular lattice.*
(iv) *Each rectangular lattice is an anti-slimming of a* slim *rectangular lattice, which is unique up to isomorphism.*

3. PATCHWORK SYSTEMS AND THE NEW RESULTS

An interval is called a *rectangular interval*, if it is a rectangular lattice. As usual, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the set of positive integers and $\mathbb{N} \cup \{0\}$, respectively. We will deal only with glued sum indecomposable lattices.

Definition 3.1. Let L be a *glued sum indecomposable*, planar, semimodular lattice, and let $\boldsymbol{\mathcal{H}}$ be a collection of rectangular intervals of L. For $I, J \in \boldsymbol{\mathcal{H}}$, I and J are *adjacent* if $I \cap J \neq \varnothing$. Let $\mathcal{E}(\boldsymbol{\mathcal{H}})$ denote the set of adjacent pairs of rectangular intervals, that is, $\mathcal{E}(\boldsymbol{\mathcal{H}}) = \{(I,J) \in \boldsymbol{\mathcal{H}}^2 : I \neq J \text{ and } I \cap J \neq \varnothing\}$. We say that $\boldsymbol{\mathcal{H}}$ is a *patchwork system* for L, if the following three conditions hold:

(i) For each covering square S of L, there exists an $I \in \boldsymbol{\mathcal{H}}$ such that $S \subseteq I$.
(ii) For all $(I,J) \in \mathcal{E}(\boldsymbol{\mathcal{H}})$, $I \cap J$ is a chain.
(iii) The lattice L has a planar diagram D such that, for all $(I,J) \in \mathcal{E}(\boldsymbol{\mathcal{H}})$, we have that $I \cap J \subseteq \mathrm{UBnd}(I,D) \cap \mathrm{LBnd}(J,D)$ or $I \cap J \subseteq \mathrm{LBnd}(I,D) \cap \mathrm{UBnd}(J,D)$.

If $\boldsymbol{\mathcal{H}}$ is a patchwork system of L such that $D \in \mathrm{Dgr}(L)$ witnesses (iii), then we also say that $\boldsymbol{\mathcal{H}}$ is a *patchwork system for the diagram D*. Hence, $\boldsymbol{\mathcal{H}}$ is a patchwork system for L iff it is a patchwork system for some $D \in \mathrm{Dgr}(L)$. A patchwork system $\boldsymbol{\mathcal{H}}$ is *nontrivial* if $|\boldsymbol{\mathcal{H}}| \geq 2$.

If there is a patchwork system for L, then we also say that L *allows a patchwork system*. An example of a patchwork system for L is provided by Figure 3; this system consists of eleven rectangular intervals: four light grey ones, five dark gray ones, and two striped ones. The following statement sheds more light on this concept, and offers several equivalent definitions.

Proposition 3.2. *Assume that $\boldsymbol{\mathcal{H}}$ is a set of rectangular intervals of a glued sum indecomposable, planar, semimodular lattice L such that $\boldsymbol{\mathcal{H}}$ satisfies* 3.1(i) *and* 3.1(ii). *Then the following four conditions are equivalent.*

- (i) *$\boldsymbol{\mathcal{H}}$ satisfies* 3.1(iii), *that is, $\boldsymbol{\mathcal{H}}$ is a patchwork system for L.*
- (ii) *For* all *planar diagrams D of L and for all $(I, J) \in \mathcal{E}(\boldsymbol{\mathcal{H}})$, we have that $I \cap J \subseteq \mathrm{UBnd}(I, D) \cap \mathrm{LBnd}(J, D)$ or $I \cap J \subseteq \mathrm{LBnd}(I, D) \cap \mathrm{UBnd}(J, D)$. That is, $\boldsymbol{\mathcal{H}}$ is a patchwork system for* all *$D \in \mathrm{Dgr}(L)$.*
- (iii) *There* exists *a planar diagram D of L such that for each $(I, J) \in \mathcal{E}(\boldsymbol{\mathcal{H}})$,*
  - (a) *$I \cap J \subseteq \mathrm{UBnd}(I, D) \cap \mathrm{LBnd}(J, D)$ or $I \cap J \subseteq \mathrm{LBnd}(I, D) \cap \mathrm{UBnd}(J, D)$, and*
  - (b) *$I \cap J \subseteq \mathrm{C_l}(I, D) \cap \mathrm{C_r}(J, D)$ or $I \cap J \subseteq \mathrm{C_r}(I, D) \cap \mathrm{C_l}(J, D)$.*
- (iv) *The previous two sub-conditions,* 3.2(iiia) *and* 3.2(iiib), *hold for* all *planar diagrams D of L and for each $(I, J) \in \mathcal{E}(\boldsymbol{\mathcal{H}})$.*

Remark 3.3. Let L and $\boldsymbol{\mathcal{H}}$ be as in Proposition (3.2).

- (i) If I and J are distinct members of a patchwork system $\boldsymbol{\mathcal{H}}$, then I and J are incomparable (in notation, $I \parallel J$), that is, $I \not\subseteq J$ and $J \not\subseteq I$. (This follows from 3.1(ii) since a rectangular interval is never a chain.)
- (ii) Since $\mathrm{C_l}(I)$ and $\mathrm{C_r}(I)$ are always chains, 3.2(iiib) implies 3.1(ii).
- (iii) It will follow from 3.1(i) and Lemma 4.3 that $\boldsymbol{\mathcal{H}}$ covers L in the sense that $L = \bigcup\{I : I \in \boldsymbol{\mathcal{H}}\}$.
- (iv) Let $\boldsymbol{\mathcal{G}}$ be a set of rectangular intervals of L. Then 3.1(i) holds for $\boldsymbol{\mathcal{G}}$ iff L has a planar diagram D such that each 4-cell of D is a subset of some member of $\boldsymbol{\mathcal{G}}$ iff for every planar diagram D of L, each 4-cell of D is a subset of some member of $\boldsymbol{\mathcal{G}}$.
- (v) The purpose of 3.1(i) is to ensure something like "$\boldsymbol{\mathcal{H}}$ is simply connected" (in other words, 1-connected) in the topological sense. For example, if $L = \mathbf{3}^2$ and $\boldsymbol{\mathcal{G}}$ is the collection of all covering squares, then $\boldsymbol{\mathcal{G}}$ is a patchwork system for L. However, if the middle square S is removed, then $\boldsymbol{\mathcal{G}} - \{S\}$ is not a patchwork system since 3.1(i) fails (while 3.1(ii) and 3.1(iii) hold).

We call a slim semimodular lattice L *patchwork-irreducible*, if it *allows* a patchwork system and, in addition, for every patchwork system $\boldsymbol{\mathcal{H}}$ for L, $|\boldsymbol{\mathcal{H}}| = 1$. In other words, if L is rectangular and it allows only the trivial patchwork system. For example, the lattice S_7 in Figure 1 is patchwork-irreducible. To define two related but more classical concepts, let L be a nontrivial lattice. If there are a proper ideal I and a proper filter F such that $I \cap F$ is nonempty and $L = I \cup F$, then L is *decomposable with respect to gluing* (in the general sense), *gluing decomposable* for short. If L is not a chain (equivalently, $|L| \geq 3$ or, still equivalently, $|L| \geq 4$) and L is not gluing decomposable, then we say that L is *indecomposable with respect to gluing*, *gluing indecomposable* for short. Notice that the two-element lattice is neither gluing decomposable, nor gluing indecomposable.

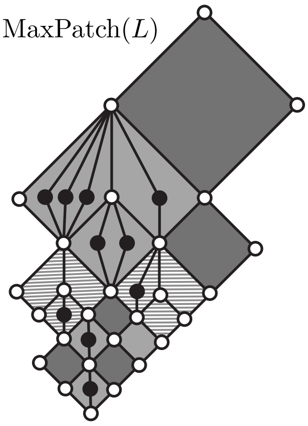

FIGURE 3. A patchwork system

Similarly, assume that L is not a chain (equivalently, $|L| \geq 3$ or, still equivalently, $|L| \geq 4$), and whenever I is an ideal and F is a filter of L such that $I \cap F$ is a chain and $L = I \cup F$, then $L \in \{I, F\}$. Then we say that L is *indecomposable with respect to gluing over chains*, *GC-indecomposable* for short.

Assume that the lattice L is patchwork-irreducible, or gluing indecomposable, or GC-indecomposable. Then, as a consequence of our definitions, L is not a chain, L is glued sum indecomposable, and L consists of at least four elements.

Theorem 3.4. *Let L be a planar semimodular lattice. Assume that $|L| \geq 4$. Then the following seven conditions are equivalent.*

- (i) *L is a patchwork-irreducible lattice;*
- (ii) *L is indecomposable with respect to gluing;*
- (iii) *L is indecomposable with respect to gluing over chains;*
- (iv) *L is a rectangular lattice whose weak corners $w_l(D)$ and $w_r(D)$, with respect to some planar diagram D of L, are coatoms;*
- (v) *L has a planar diagram such that the intersection of the leftmost coatom and the rightmost coatom is 0;*
- (vi) *for each planar diagram of L, the intersection of the leftmost coatom and the rightmost coatom is 0;*
- (vii) *L is an anti-slimming of a lattice obtained from the four-element Boolean lattice by adding finitely many forks one by one.*

By a *patch lattice* we mean a rectangular lattice L whose weak corners, with respect to some $D \in \mathrm{Dgr}(L)$, are coatoms; that is, a lattice satisfying 3.4(iv) above. Theorem 3.4 offers six equivalent definitions. Some slim patch lattices are given in Figure 2. Some non-slim patch lattices occur among the members of $\mathrm{MaxPatch}(L)$ in Figure 3. Theorem 3.4 trivially leads to the following structure theorem.

Corollary 3.5 (A structure theorem)**.** *Each planar semimodular lattice can be constructed as the last member of a list $L_1, L_2, \dots, L_n$ such that each L_i $(i = 1, \dots, n)$ is either a patch lattice* (*constructed according to Theorem* 3.4(vii))*, or there are $j, k < i$ such that L_i is a Hall-Dilworth gluing of L_j and L_k over a chain. Conversely, every lattice constructed this way is a planar semimodular lattice.*

A *patch* of a lattice is an interval that is a patch lattice. Let $\mathrm{Patch}(L)$ denote the set of all patches of L, and let $\mathrm{MaxPatch}(L)$ be the set of maximal patches of L (with respect to set inclusion). Now we state our third structure theorem.

Theorem 3.6. *Let L be a glued sum indecomposable planar semimodular lattice. Then* $\mathrm{MaxPatch}(L)$ *is a patchwork system for L.*

For an example, take the lattice L of Figure 3, which is shown with its patchwork system. This L is not slim. After deleting all the black-filled elements, we would obtain a slim lattice L' and $\mathrm{MaxPatch}(L')$.

Since S_7 is not a modular lattice, in the modular case we cannot add forks. Similarly, in the distributive case we cannot add eyes. Hence, Theorems 3.4 and 3.6 together with Proposition 2.1 clearly imply the following two corollaries (except for the last sentence of the second one). The first of them is a folklore result (with another terminology), see also G. Grätzer and E. Knapp [13, Introduction].

Corollary 3.7. *If L is a glued sum indecomposable, planar,* distributive *lattice, then* $\mathrm{MaxPatch}(L)$ *is the set of all 4-cells, and it is a patchwork system for L.*

The definition of Herrmann's S-glued systems will not be needed here; the reader can see [16] for details. The main result of C. Herrmann [16] asserts that the maximal complemented (equivalently, maximal atomistic) intervals of a modular lattice M of finite length form an S-glued system, which we denote by $\mathrm{Herrm}(M)$.

Corollary 3.8. *If L is a glued sum indecomposable, planar,* modular *lattice, then* $\mathrm{MaxPatch}(L)$ *is the set of all non-chain intervals of length 2. Moreover, the patchwork system* $\mathrm{MaxPatch}(L)$ *coincides with the S-glued system* $\mathrm{Herrm}(L)$.

Hence, Theorem 3.6 extends the main result of C. Herrmann [16] to planar semimodular lattices. However, there is an essential difference. If M is a modular lattice, then $\varrho := \bigcup\{A^2 : A \in \mathrm{Herrm}(M)\}$ is a lattice tolerance, see A. Day and C. Herrmann [9], and the quotient lattice L/ϱ in the sense of [1] is what Herrmann calls the "skeleton" of his construction. However, if L is (the planar semimodular) lattice given in Figure 3, then $\varrho := \bigcup\{A^2 : A \in \mathrm{MaxPatch}(L)\}$ is not a lattice tolerance. Hence, we do not associate "skeleton lattices" with patchwork systems.

4. More about planar diagrams

Lemma 4.1. *Let L_i' be a full slimming sublattice of a planar semimodular lattice L_i, for $i \in \{1, 2\}$. If L_1 is isomorphic to L_2, then L_1' is isomorphic to L_2'.*

Remark 4.2. This lemma allows us to speak of *the full slimming* of a slim semimodular lattice L: it is any of the full slimming sublattices of L, and it is considered an abstract lattice. Lemma 4.1 implies that the full slimming L' of L is uniquely determined up to isomorphism. In other words, the isomorphism type of L' does not depend on the planar diagram of L.

Proof of Lemma 4.1. We apply induction by $|L_1|$. Let D_i be a planar diagram of L_i, for $i \in \{1, 2\}$. Let $\varphi\colon L_1 \to L_2$ be an isomorphism. If L_1 is slim, then the statement is trivial. Assume that L_1 is not slim. Then there are $u < v \in L_1$ such that $[u, v]$ is an interval of length two, and $[u, v]$ contains a doubly irreducible element s_1 that belongs to $\mathrm{int}([u, v], D_1)$ (the interior of $[u, v]$ with respect to the diagram D_1). Let s_2 be a doubly irreducible element of L_2 that belongs to $\mathrm{int}([\varphi(u), \varphi(v)], D_2)$. Then $t := \varphi^{-1}(s_2)$ is a doubly irreducible element in L_1, and it belongs to $[u, v]$.

Obviously, there is an automorphism of L_1 that sends s_1 to t and t to s_1, and keeps any other element fixed. Let ψ denote the composite of this automorphism and φ. Then $\psi\colon L_1 \to L_2$ is an isomorphism and $\psi(s_1) = s_2$.

Let $L_i^- := L_i - \{s_i\}$, for $i \in \{1, 2\}$, and let D_i^- denote the diagram obtained from D_i by removing s_i. The restriction ψ^- of ψ to L_1^- is an isomorphism $\psi^-\colon L_1^- \to L_2^-$. Clearly, L_i' is the full slimming sublattice of L_i^- with respect to D_i^-, for $i = 1, 2$. Since $|L_1^-| < |L_1|$, the induction hypothesis implies that $L_1' \cong L_2'$. □

Two-element intervals are called *prime intervals*. That is, $[a, b]$ is a prime interval iff $a \prec b$. A covering square B is formed by four *edges*, which are the prime intervals of B. Covering squares need not be 4-cells.

Lemma 4.3. *Let $[a, b]$ be a prime interval of a glued sum indecomposable, planar, semimodular lattice. Then a is meet-reducible or b is join-reducible. Furthermore, $[a, b]$ is an edge of a covering square. Moreover, for any fixed planar diagram of L, $[a, b]$ is an edge of a 4-cell of D.*

Proof. By way of contradiction, assume that $a \prec b$ such that $a \in \operatorname{Mi} L$ and $b \in \operatorname{Ji} L$. Since L is glued sum indecomposable, we can select a minimal $y \in L$ such that $y \parallel b$. Then $y \neq 0$, so it has a lower cover x. By the minimality of y, we have that $x < b$, which gives that $x \leq a$. Semimodularity yields that $a = a \vee x \preceq a \vee y$. This means that $a \vee y$ is a or b since b is the only cover of a. However, both possibilities lead to $y \leq b$, a contradiction. Thus $a \notin \operatorname{Mi} L$ or $b \notin \operatorname{Ji} L$, proving the first part of the lemma.

If a is meet-reducible, then it has a cover c distinct from b, and $S = \{a, b, c, b \vee c\}$ is a covering square by semimodularity. The prime interval $[a, b]$ is an edge of S. If we chose c such that b and c are neighboring covers of a (in the fixed diagram D), then S is a 4-cell. Next, assume that b is join-reducible. Then, with respect to D, there is a $c \in L$ such that a and c are neighboring lower covers of b. Then $[a \wedge c, b]$ is a 4-cell by [6, Lemma 13], and $[a, b]$ is one of its edges. □

On the set $\operatorname{PrInt}(L)$ of all prime intervals of L, we define a relation μ as follows: for $\mathfrak{p}, \mathfrak{q} \in \operatorname{PrInt}(L)$, let $\mathfrak{p}\ \mu\ \mathfrak{q}$ mean that there is a covering square B such that both $\mathfrak{p}$ and $\mathfrak{q}$ are edges of B. We will also need a similar relation defined on $\operatorname{PrInt}(D) = \operatorname{PrInt}(L)$, where $D \in \operatorname{Dgr}(L)$. For $\mathfrak{p}, \mathfrak{q} \in \operatorname{PrInt}(L)$, let $\mathfrak{p}\ \varrho_D\ \mathfrak{q}$ mean that there is a 4-cell B in the diagram D such that both $\mathfrak{p}$ and $\mathfrak{q}$ are edges of B. Both μ and ϱ_D are reflexive and symmetric relations, provided L belongs to the scope of Lemma 4.3. Their transitive closures will be denoted by μ^* and ϱ_D^*.

Lemma 4.4. *Let L be a glued sum indecomposable, planar, semimodular lattice, and let $D \in \operatorname{Dgr}(L)$. Then μ^* and ϱ_D^* are the "full relation" $\operatorname{PrInt}(L) \times \operatorname{PrInt}(L)$ on the set of prime intervals of L.*

Proof. Clearly, $\mu \subseteq \varrho_D^*$. Therefore, it suffices to deal with μ. Let $[u, v] \in \operatorname{PrInt}(L)$. By induction on the height $h(v)$ of v, we are going to show that

$$\text{there is an atom } r \in L \text{ such that } [0, r]\ \mu^*\ [u, v]. \tag{4.1}$$

By reflexivity, this is trivial if $h(v) = 1$. So let $h(v) \geq 2$. Let a and b be the leftmost lower cover and the rightmost lower cover of u, respectively. (They are not distinct in general, and they are never distinct if $h(v) = 2$.) Let H be a maximal chain in $\uparrow v$. Then $W := \mathrm{C_l}(\downarrow u, D) \cup H$ and $E := \mathrm{C_r}(\downarrow u, D) \cup H$ are maximal chains of L, and $a \in W$ and $b \in E$. These two maximal chains divide L into the strict left side

$L_W := \mathrm{LS}(W, D) - W$ of W, the strict right side $R_E := \mathrm{RS}(E, D) - E$ of E, and $H \cup \downarrow u = \mathrm{RS}(W, D) \cap \mathrm{LS}(E, D)$. Since L is glued sum indecomposable, there is an $x \in L$ such that $x \parallel u$. We assume that x is minimal with respect to this property. By left-right symmetry, we can also assume that $x \in R_E$. There are two cases.

Assume first that $b \leq x$. Then $b < x$ since $x \parallel u$ and $b < u$. Take an atom x' in the interval $[b, x]$. Then $u \not\leq x$ gives that $x' \neq u$. Hence, as two covers of b, x' and u are incomparable. Since x was minimal with respect to this property, we obtain that $x' = x$. That is, we have the situation

$$\text{there is an } x \in R_E \text{ such that } x \neq u \text{ and } b \prec x. \tag{4.2}$$

Let $t = u \vee x$. Then $\{b, u, x, t\}$ is a covering square by semimodularity. If $t \neq v$, then $\{u, v, t, v \vee t\}$ is another covering square. Since $h(u) < h(v)$, the induction hypothesis yields an atom $r \in L$ such that $[0, r]\ \mu^*\ [b, u]$. The covering square $\{b, u, x, t\}$ gives that $[b, u]\ \mu^*\ [u, t]$. Hence, $[u, t]\ \mu^*\ [u, v]$ follows either from $v = t$ and reflexivity, see Lemma 4.3, or from the covering square $\{u, v, t, v \vee t\}$. By transitivity, $[0, r]\ \mu^*\ [u, v]$, as desired.

Secondly, we assume that $b \not\leq x$, that is, $b < b \vee x$. Since $x \parallel u$, x has a lower cover x_0. The minimality of x gives that $x_0 < u$. Hence, x_0 is on the left side of E while $x \in R_E$ is on the strict right side of E. We conclude from (2.5) and $x_0 \prec x$ that $x_0 \in E$. Hence, $x_0 \in \mathrm{C_r}(\downarrow u)$. Since $\mathrm{C_r}(\downarrow u)$ is a chain and $x \neq u$, we obtain that $x_0 \leq b$. Hence, $b = b \vee x_0 \prec b \vee x$ by semimodularity. Clearly, $b \vee x \neq u$ since $u \parallel x$. Moreover, $b \vee x \parallel u$ since $h(b \vee x) = h(b) + 1 = h(u)$. Furthermore, $E \cup R_E$ is a region (surrounded by E and $\mathrm{C_r}(L)$) that contains b and x. Hence, $b \vee x \in E \cup R_E$ since regions are (convex) sublattices. Since $b \vee x \parallel u \in E$, we obtain that $b \vee x \in R_E$. Therefore $b \vee x$ (instead of x) witnesses that (4.2) holds, which does the job. We have seen that (4.1) holds for each prime interval $[u, v]$.

Finally, for any two atoms, r_1 and r_2, $\{0, r_1, r_2, r_1 \vee r_2\}$ is a covering square and $[0, r_1]\ \mu^*\ [0, r_2]$. Hence, the lemma follows from (4.1) by transitivity. □

The following lemma it not surprising.

Lemma 4.5. *Assume that L is a planar lattice, that $D \in \mathrm{Dgr}(L)$, and that S and T are 4-cells of D. Assume also that S and T have a* common edge on the same side, *that is,* $\mathrm{PrInt}(\mathrm{C_l}(S, D)) \cap \mathrm{PrInt}(\mathrm{C_l}(T, D)) \neq \varnothing$ *or* $\mathrm{PrInt}(\mathrm{C_r}(S, D)) \cap \mathrm{PrInt}(\mathrm{C_r}(T, D)) \neq \varnothing$. *Then $S = T$.*

Proof. Suppose, for a contradiction, that $S \neq T$ and $a, b \in L$ such that $a \prec b$ and $a, b \in \mathrm{C_l}(S, D) \cap \mathrm{C_l}(T, D)$.

Firstly, assume that $a, b \in \mathrm{C_{ll}}(S, D) \cap \mathrm{C_{ll}}(T, D)$. Let $c = w_r(S) = w_r(S, D)$ and $d = w_r(T)$. Since $S \neq T$, we have that $c \neq d$. Let, say, d be strictly on the right of c, and extend $\{a, c, b \vee c\} = \mathrm{C_r}(S)$ to a maximal chain C of L. If $b \vee d = 1_T$ was strictly on the right of C, then $b \prec b \vee d$ would contradict (2.5). If 1_T was strictly on the left of C, then $d \prec b \vee d$ would induce the same contradiction. Hence, $1_T \in C$ together with $h(1_T) = h(1_S)$ gives that $1_T = 1_S$. However, then $c \in \mathrm{int}(T)$ is a contradiction since T is a 4-cell. By duality, $a, b \in \mathrm{C_{ul}}(S) \cap \mathrm{C_{ul}}(T)$ also leads to a contradiction.

Secondly, assume that $a, b \in \mathrm{C_{ll}}(S) \cap \mathrm{C_{ul}}(T)$. Let $c = w_r(S)$ again. Since $w_r(T)$ is strictly on the right of the previous C and b is strictly on the left of C, $w_r(T) \prec b$ contradicts (2.5). □

Definition 4.6. For $i = 1, 2$, let $D_i \in \mathrm{Dgr}(L_i)$, and let $\varphi : L_1 \to L_2$ be a lattice isomorphism. Then φ is a *directed diagram isomorphism* $(L_1, D_1) \to (L_2, D_2)$, if

(i) $\varphi(\mathrm{C_l}([a, b], D_1)) = \mathrm{C_l}([\varphi(a), \varphi(b)], D_2)$ and, similarly, $\varphi(\mathrm{C_r}([a, b], D_1)) = \mathrm{C_r}([\varphi(a), \varphi(b)], D_2)$ hold, for all $a < b \in L_1$, and
(ii) for each maximal chain C of L_1, we have that $\varphi(\mathrm{LS}(C, D_1)) = \mathrm{LS}(\varphi(C), D_2)$ and $\varphi(\mathrm{RS}(C, D_1)) = \mathrm{RS}(\varphi(C), D_2)$.

By reflecting the diagram D trough a vertical axis we obtain its *mirror image* D^{mir}. Let $\mathrm{id}_L\colon L \to L$ denote the identical $x \mapsto x$ map. We say that L *is uniquely oriented* if for any two planar diagrams D and F of L, $\mathrm{id}_L\colon (L, D) \to (L, F)$ or $\mathrm{id}_L\colon (L, D) \to (L, F^{\mathrm{mir}})$ is a directed diagram isomorphism.

For example, S_7 in Figure 2 is uniquely oriented but M_3 is not. We are interested in planar diagrams only up to directed diagram isomorphisms.

Lemma 4.7.

(i) *Let L_1 and L_2 be glued sum indecomposable,* slim*, semimodular lattices, and let $\varphi\colon L_1 \to L_2$ be a lattice isomorphism. Assume that $D_1 \in \mathrm{Dgr}(L_1)$ and $D_2 \in \mathrm{Dgr}(L_2)$. Then $\varphi\colon (L_1, D_1) \to (L_2, D_2)$ or $\varphi : (L_1, D_1) \to (L_2, D_2^{\mathrm{mir}})$ is a directed diagram isomorphism.*
(ii) *Each glued sum indecomposable, slim, semimodular lattice is uniquely oriented.*

Proof. Observe that part (i), applied to the identical mapping, implies part (ii). Hence, it suffices to prove part (i). It follows from [6, Lemma 7] that the set $\{\varphi(\mathrm{C_l}(L_1, D_1)), \varphi(\mathrm{C_r}(L_1, D_1))\}$ is equal to $\{\mathrm{C_l}(L_2, D_2), \mathrm{C_r}(L_2, D_2)\}$. Hence, after replacing D_2 by D_2^{mir} if necessary, we can assume that $\varphi(\mathrm{C_l}(L_1, D_1)) = \mathrm{C_l}(L_2, D_2)$ and $\varphi(\mathrm{C_r}(L_1, D_1)) = \mathrm{C_r}(L_2, D_2)$. For a prime interval $\mathfrak{p}$ of L_1, the distance of $\mathfrak{p}$ from $\mathrm{C_l}(L_1, D_1)$ will be measured by

$$(4.3)\qquad d(\mathfrak{p}) := \min\{n \in \mathbb{N}_0 : \text{there is a } \mathfrak{q} \in \mathrm{PrInt}(\mathrm{C_l}(L_1, D_1)) \text{ such that } \mathfrak{q}\ \mu^n\ \mathfrak{p}\},$$

where μ is defined right before Lemma 4.4. Notice that, in virtue of (2.3), the covering squares of L_1 and the 4-cells of D_1 are the same. Hence, μ in (4.3) can be, and sometimes will be, replaced by ϱ_{D_1}. For a 4-cell S of L_1 (with respect to D_1), we let $d(S) := \min\{d(\mathfrak{p}) : \mathfrak{p} \in \mathrm{PrInt}(S)\}$. By Lemma 4.4, $d(\mathfrak{p})$ and $d(S)$ are well-defined. We will show by induction on $d(S)$ that, for each 4-cell S of L_1,

$$(4.4)\qquad \varphi(\mathrm{C_l}(S, D_1)) = \mathrm{C_l}(\varphi(S), D_2) \quad\text{and}\quad \varphi(\mathrm{C_r}(S, D_1)) = \mathrm{C_r}(\varphi(S), D_2).$$

If $d(S) = 0$, which means that S has an edge on the left boundary chain, then (4.4) is evident. Assume that $n := d(S) > 0$, and $\mathfrak{p} \in \mathrm{PrInt}(S)$ such that $d(\mathfrak{p}) = n$. Then there are a $\mathfrak{q} \in \mathrm{PrInt}(\mathrm{C_l}(L_1, D_1))$ and an $\mathfrak{r} \in \mathrm{PrInt}(L_1)$ such that $\mathfrak{q}\ \varrho_{D_1}^{n-1}\ \mathfrak{r}$ and $\mathfrak{r}\ \varrho_{D_1}\ \mathfrak{p}$. Clearly, $d(\mathfrak{r}) \le n - 1$. (Actually, we have equality but we do not need it.) By the definition of ϱ_{D_1}, there is a 4-cell T of L_1 such that $\mathfrak{r}, \mathfrak{p} \in \mathrm{PrInt}(T)$. Since $d(T) \le d(\mathfrak{r}) \le n - 1$, we have that $T \ne S$. The induction hypothesis says that $\varphi(\mathrm{C_l}(T, D_1)) = \mathrm{C_l}(\varphi(T), D_2)$ and $\varphi(\mathrm{C_r}(T, D_1)) = \mathrm{C_r}(\varphi(T), D_2)$. By Lemma 4.5, the common edge $\mathfrak{r}$ of S and T determines how the left-right orientation of S depends on that of T, and this "determination" is preserved by φ. This fact together with the induction hypothesis implies that $\varphi(\mathrm{C_l}(S, D_1)) = \mathrm{C_l}(\varphi(S), D_2)$ and $\varphi(\mathrm{C_r}(S, D_1)) = \mathrm{C_r}(\varphi(S), D_2)$. This completes the proof of (4.4).

In the rest of the proof, we will focus mainly on the left sides; if the corresponding right sides are not mentioned then their treatment would be analogous. Let $a < b \in L_1$. By Lemma 2.3, we can assume that $[a, b]$ is a glued sum indecomposable lattice since otherwise we could deal with its glued summands. Hence, a is meet-reducible in $[a, b]$. Moreover, $[a, b]$ is a slim lattice by (2.2). Therefore, we conclude from (2.1) that a has exactly two covers, c and d, in $[a, b]$. By semimodularity and (2.3), $S := \{a, c, d, c \vee d\}$ is a 4-cell. The idea is that $[a, b]$ is slim, its boundary is determined, its bottom 4-cell S intersects the boundary of $[a, b]$ in $\{a, c, d\} = \mathrm{Bnd}(S, D_1) - \{1_S\}$, so S determines which one of the boundary chains is the left one and which one is the right one. More exactly, [6, Lemma 7] yields that $\{\varphi(\mathrm{C_l}([a, b], D_1)), \varphi(\mathrm{C_r}([a, b], D_1))\} = \{\mathrm{C_l}([\varphi(a), \varphi(b)], D_2), \mathrm{C_r}([\varphi(a), \varphi(b)], D_2)\}$. Hence, knowing that (4.4) holds for S and keeping $\{c, d\} \subseteq \mathrm{Bnd}([a, b], D_1)$ in mind, we conclude that φ satisfies 4.6(i).

Next, we prove the validity of 4.6(ii) for L_1 by induction on $\mathrm{length}\, L_1$. The case: $\mathrm{length}\, L_1 = 2$, where L_1 necessarily equals $\mathbf{2}^2$, is trivial. Let $\mathrm{length}\, L_1 \geq 3$, and let C be a maximal chain of L_1. Since L_1 is glued sum indecomposable, it has exactly two atoms, u and v, by (2.1). Let, say, $u \in C$. We have previously assumed that $\varphi(\mathrm{C_l}(L_1, D_1)) = \mathrm{C_l}(L_2, D_2)$. By the left-right symmetry, we can assume that $u \in \mathrm{C_l}(L_1, D_1)$; notice that we will not be allowed to use the left-right symmetry for the *right* side of C later. By the induction hypotheses, $\varphi(\mathrm{LS}(C - \{0\}, D_1 \cap {\uparrow} u))$ is equal to $\mathrm{LS}(\varphi(C) - \{0\}, D_2 \cap {\uparrow}\varphi(u))$. Hence

$$\begin{aligned}\varphi\bigl(\mathrm{LS}(C, D_1)\bigr) &= \varphi\bigl(\{0\} \cup \mathrm{LS}(C - \{0\}, D_1 \cap {\uparrow} u)\bigr)\\ &= \{0\} \cup \mathrm{LS}(\varphi(C) - \{0\}, D_2 \cap {\uparrow}\varphi(u)) = \mathrm{LS}(\varphi(C), D_2).\end{aligned}$$

Since φ is a bijection, the equation just obtained implies that $\varphi\bigl(\mathrm{RS}(C, D_1)\bigr) = \varphi\bigl(C \cup (L_1 - \mathrm{LS}(C, D_1))\bigr) = \varphi(C) \cup \bigl(L_2 - \mathrm{LS}(\varphi(C), D_2)\bigr) = \mathrm{RS}(\varphi(C), D_2)$. This shows that φ satisfies 4.6(ii), completing the induction. □

Let L be a planar semimodular lattice, and $x \in L$. We say that x is a *possible weak corner* of L, if x is a weak corner of L with respect to some planar diagram of L. The set of possible weak corners of L will be denoted by $\mathrm{Corn_{pw}}(L)$. Clearly, $\mathrm{Corn_{pw}}(L) \subseteq \mathrm{Ji}\, L \cap \mathrm{Mi}\, L$ but the converse inclusion does not hold in general. (For example, if L is obtained from a grid by inserting an eye e into a "middle 4-cell", then $e \notin \mathrm{Corn_{pw}}(L)$.) The set of non-chain intervals of length 2 will be denoted by $\mathrm{Ivl_2}(L)$. By the *trunk* of an $I \in \mathrm{Ivl_2}(L)$, denoted by $\mathrm{Trnk}(I)$, we mean the nontrivial antichain $I - \{0_I, 1_I\}$. As usual, the unique lower cover and upper cover of a doubly irreducible element x is denoted by x_* and x^*, respectively.

Lemma 4.8. *Let L be a glued sum indecomposable planar semimodular lattice with a fixed planar diagram D. Then* $\mathrm{Corn_{pw}}(L) = \{x \in L : x$ *is a double irreducible element,* $[x_*, x^*] \in \mathrm{Ivl_2}(L)$*, and* $\mathrm{Trnk}([x_*, x^*])$ *contains a weak corner of L with respect to D*$\}$.

Proof. Let U denote the set on the left of the equality sign in the lemma. Firstly, to prove the "$\supseteq$" inclusion, assume that $x \in U$. Let w be a weak corner of D witnessing that $x \in U$. Clearly, $[w_*, w^*] = [x_*, x^*]$. Since both x and w are doubly irreducible elements of $[w_*, w^*] \in \mathrm{Ivl_2}(L)$, there is an automorphism of L that interchanges w and x but keeps the rest of elements fixed. Therefore, if we interchange the labels w and x in the diagram D, we obtain a new diagram in which x is a weak corner. Hence, $x \in \mathrm{Corn_{pw}}(L)$, which proves that $\mathrm{Corn_{pw}}(L) \supseteq U$.

To prove the converse inclusion, assume that $v \in \mathrm{Corn}_{\mathrm{pw}}(L)$. Then there is an $F \in \mathrm{Dgr}(L)$ such that v is a weak corner with respect to F. This implies that $v \in \mathrm{Ji}\, L \cap \mathrm{Mi}\, L$. We obtain from Lemma 4.3 that v_* has a cover y that is distinct from v. Since $v = v \vee v_* \preceq v \vee y \neq v$ by semimodularity and v^* is the only cover of v, we obtain that $v \vee y = v^*$ and $I := [v_*, v^*] \in \mathrm{Ivl}_2(L)$. This yields that if v happens to be a weak corner of D, then $v \in U$. So we can assume that v is not a weak corner of D. Let L'_D be the full slimming sublattice of L with respect to D. To get a contradiction, suppose that $v \in L'_D$. Then v does not belong to $\mathrm{int}(L'_D, D)$ since otherwise it would be join-reducible by (2.10). Hence, $v \in \mathrm{Bnd}(L'_D, D) = \mathrm{Bnd}(L, D)$, implying that v is a weak corner of D, a contradiction again.

Therefore, $v \in L - L'_D$. Let a resp. b denote the left weak corner resp. the right weak corner of I with respect to D. In other words, a is the leftmost element of $\mathrm{Trnk}(I)$, and b is the rightmost one. Clearly, $a, b \in L'_D$. Hence, $|\{v, a, b\}| = 3$. It follows from $v \in \mathrm{Bnd}(L, F)$, (2.6) and (2.9) that v belongs to $\mathrm{Bnd}(I, F)$. This together with $|\mathrm{Bnd}(I, F)| \le 2$ and $|\{v, a, b\}| = 3$ yields that $\{a, b\} \not\subseteq \mathrm{Bnd}(I, F)$. Let, say, $b \notin \mathrm{Bnd}(I, F)$. We conclude from (2.7) that v_* is the only lower cover of b and v^* is the only upper cover of b. Hence, $b \in \mathrm{Ji}\, L \cap \mathrm{Mi}\, L$. In particular, $b \in \mathrm{Ji}\, L'_D$. Hence, (2.10) implies that $b \in \mathrm{Bnd}(L'_D, D) = \mathrm{Bnd}(L, D)$. Therefore, b is a weak corner of D. Thus $v \in U$. □

Lemma 4.9. *Let L be a rectangular lattice, and let F be a planar diagram of L. (Not necessarily the same that witnesses the rectangularity of L.) Then*

- (i) *F has exactly one left weak corner $w_l(F)$ and exactly one right weak corner $w_r(F)$, and they are complementary.*
- (ii) *Consequently, all planar diagrams are "equally appropriate" when we want to verify the rectangularity of a planar semimodular lattice.*
- (iii) $\mathrm{Bnd}(L, F) - \mathrm{LBnd}(L, F)$ *and* $\mathrm{Bnd}(L, F) - \mathrm{UBnd}(L, F)$ *do not depend on* $F \in \mathrm{Dgr}(L)$.

Notice that 4.9(ii) will often be used implicitly.

Proof. We can assume that $\mathrm{length}\, L \ge 3$ since otherwise the statement is evident. Let D be a fixed planar diagram that witnesses the rectangularity of L. In particular, we know that $w_l(D)$ and $w_r(D)$ are complementary elements. Let $I_\ell^D := [w_l(D)_*, w_l(D)^*]$ and $I_r^D := [w_r(D)_*, w_r(D)^*]$ be the intervals of length 2 whose trunk contains $w_l(D)$ and $w_r(D)$, respectively.

Next, let $F \in \mathrm{Dgr}(L)$ be arbitrary. We know from (2.4) that there is a double irreducible element x in $\mathrm{C_l}(L, F) - \{0, 1\}$. Lemma 4.8 implies that $x \in \mathrm{Trnk}(I_\ell^D)$ or $x \in \mathrm{Trnk}(I_r^D)$. Let, say $x \in \mathrm{Trnk}(I_\ell^D)$; the other case would be left-right symmetric and needs no separate treatment.

To get a contradiction, suppose that x' is another weak corner of F such that x' is comparable with x. Let, say, $x' < x$. Since $x'^* \neq x$ by Lemma 4.3, we have that $x'^* \le x_*$. We obtain $x' \in \mathrm{Trnk}(I_r^D)$ from Lemma 4.8, and $w_r(D)^* = x'^* \le x_* = w_l(D)_*$. Hence, $w_r(D) < w_l(D)$ contradicts the fact that $w_l(D)$ and $w_r(D)$ are complementary elements and they are distinct from 0 and 1. Consequently,

$$\text{no two distinct weak corners of } F \text{ are comparable.} \tag{4.5}$$

In particular, since $\mathrm{C_l}(L,F)$ is a chain, x is the *only* weak left corner of F; so we denote it by $x = w_l(F)$. Clearly,

$$w_l(F)_* = w_l(D)_* \quad \text{and} \quad w_l(F)^* = w_l(D)^*. \tag{4.6}$$

Similarly, $\mathrm{C_r}(L,F)$ has a unique doubly irreducible element y, and $y \in \mathrm{Trnk}(I_\ell^D)$ or $y \in \mathrm{Trnk}(I_r^D)$. Suppose, for a contradiction, that $y \in \mathrm{Trnk}(I_\ell^D)$. Then $y_* = (w_l(D))_* = x_*$ and $y^* = (w_l(D))^* = x^*$. Since $x_* \in \mathrm{C_l}(L,F)$, $x_* = y_* \in \mathrm{C_r}(L,F)$, and L is glued sum indecomposable, it follows that $x_* = 0_L$. Dually, $x^* = y^* \in \mathrm{C_l}(L,F) \cap \mathrm{C_r}(L,F)$ yields that $x^* = 1_L$. This contradicts $\mathrm{length}\, L \geq 3$. Therefore $y = w_r(F) \in \mathrm{Trnk}(I_r^D)$ is the unique right weak corner of F and we have that

$$w_r(F)_* = w_r(D)_* \quad \text{and} \quad w_r(F)^* = w_r(D)^*. \tag{4.7}$$

We know from (4.5) that $w_l(F) \parallel w_r(F)$. Hence, (4.6) and (4.7) yield that $w_l(F) \wedge w_r(F) = w_l(F)_* \wedge w_r(F)_* = w_l(D)_* \wedge w_r(D)_* = 0$ and $w_l(F) \vee w_r(F) = w_l(F)^* \vee w_r(F)^* = w_l(D)^* \vee w_r(D)^* = 1$. That is, $w_l(F)$ and $w_r(F)$ are complementary elements. This proves 4.9(i).

Next, (2.12) yields that

$$\begin{aligned} \mathrm{Bnd}(L,F) - \mathrm{LBnd}(L,F) &= \uparrow w_l(F)^* \cup \uparrow w_r(F)^* \text{ and} \\ \mathrm{Bnd}(L,F) - \mathrm{UBnd}(L,F) &= \downarrow w_l(F)_* \cup \downarrow w_r(F)_*, \end{aligned} \tag{4.8}$$

proving 4.9(iii). Finally, (4.8), (4.6) and (4.7) imply 4.9(ii). □

5. Some properties that do not depend on the diagram chosen

Lemma 5.1. *Let I be a rectangular interval of a slim semimodular lattice L. Assume that $a \in I$ and $b \in L - I$ such that $a < b$. Then $[a,b] \cap \mathrm{UBnd}(L,D) \neq \varnothing$, for all $D \in \mathrm{Dgr}(L)$.*

Proof. Let $D \in \mathrm{Dgr}(L)$. We know from Lemma 4.9(ii) that the rectangularity of I is witnessed by D. It follows from (2.6) and (2.7) that there is a maximal $x \in \mathrm{Bnd}(I) = \mathrm{Bnd}(I,D)$ such that $a \leq x < b$. Let x^* be an atom in $[x,b]$; it is not in I by the choice of x. If $x \notin \mathrm{UBnd}(I)$, then x has at least two additional covers in I by (2.14), which contradicts (2.1). Hence, $x \in [a,b] \cap \mathrm{UBnd}(I)$ proves the statement. □

Lemma 5.2. *Assume that I is a rectangular interval of a slim semimodular lattice L and $D \in \mathrm{Dgr}(L)$. Then the intervals $[0_I, w_l(I,D)]_L$ and $[0_I, w_r(I,D)]_L$ are chains.*

Proof. To get a contradiction, suppose that, say, $[0_I, w_l(I)]_L$ is not a chain. Then there is an $x \in [0_I, w_l(I)]_L - \{w_l(I)\}$ with (at least) two distinct covers, y_1 and y_2, in $[0_I, w_l(I)]_L$. Let, say, $y_1 \in [0_I, w_l(I)]_I$ (the interval within I). Then $y_2 \notin I$, since $[0_I, w_l(I)]_I$ is a chain by (2.11) and (2.12). By (2.14), there is a $y_3 \in I - \{y_1\}$ such that $x \prec y_3$. Now we have three distinct covers of x, which contradicts (2.1). □

Lemma 5.3. *If L is a planar semimodular lattice and $u \in L - \mathrm{Mi}\, L$, then any two covers of u have the same join.*

Proof. Fix a $D \in \mathrm{Dgr}(L)$, and let L' be the full slimming sublattice of L with respect to D. Then $u \in L'$. We obtain from (2.1) that u has exactly two covers, a and b, within L'. Let $v = a \vee b \in L'$. All further covers of u in L are eyes belonging to $[u,v]$. Hence, the join of arbitrary two distinct covers of u equals v. □

For $x \in L$, the *height* of x is denoted by $h(x)$. Let $D \in \mathrm{Dgr}(L)$ be fixed. Let $x, y \in L$ with $h(x) = h(y)$. We say that x is *on the left* of y, with respect to D, if for every (equivalently, some) maximal chain C of L that contains y, x is on the left of C. (Equivalently, if $y \in \mathrm{RS}(C, D)$ for all maximal chains C that contain x.) Let us emphasize that "x is on the left of y" implies that $h(x) = h(y)$. If x is on the left of y, $x \neq y$, and there is no $z \in L - \{x, y\}$ such that x is on the left of z and z is on the left of y, then y is the *right neighbor* of x (with respect to D). Clearly,

$$\text{if } x \text{ belongs a maximal chain } C,\ h(x) = h(y), \text{ and } y \text{ is (strictly) on the left of } x, \text{ then } y \text{ is (strictly) on the left of } C; \tag{5.1}$$

$$\text{if } x \text{ is on the left of } y,\ y \text{ is on the left of } x, \text{ and } h(x) = h(y), \text{ then } x = y; \tag{5.2}$$

$$\text{if } x \in \mathrm{C_l}(L) \text{ and } h(x) = h(y), \text{ then } x \text{ is on the left of } y; \tag{5.3}$$

$$\text{each } x \in L - \mathrm{C_r}(L, D) \text{ has a unique right neighbor (with respect to } D). \tag{5.4}$$

Notice that these assertions imply, for a planar semimodular L, that

$$\text{if } a \in \mathrm{C_l}(L, D), \text{ then } \mathrm{C_l}(\downarrow a, D) = \mathrm{C_l}(L, D) \cap \downarrow a \text{ and } \mathrm{C_l}(\uparrow a, D) = \mathrm{C_l}(L, D) \cap \uparrow a. \tag{5.5}$$

Indeed, for $x \in \mathrm{C_l}(L, D) \cap \downarrow a$, let y denote the unique element of $\mathrm{C_l}(\downarrow a, D)$ such that $h(x) = h(y)$. Applying (5.3) to L and also to $\downarrow a$, we obtain that x and y are mutually on the left of each other. Hence, they are equal by (5.2), and the first equality of (5.5) follows. The second one holds by duality.

The following lemma, which does not assume semimodularity, is the counterpart of Lemma 5.3. Although it looks evident by our geometric intuition, its rigorous proof needs a result borrowed from D. Kelly and I. Rival [17].

Lemma 5.4. *Assume that a and b are the leftmost lower cover and the rightmost lower cover of an element v in some planar diagram of a planar lattice L, respectively. Then $a \wedge b$ is the meet of* all *lower covers of v.*

Proof. We can clearly assume that $a \neq b$. Let $u = a \wedge b$. By (2.6), $I := [u, v]$ is a region. Let C_0 and C_1 be maximal chains in $\downarrow u$ and in $\uparrow v$, respectively. Then $W := C_0 \cup \mathrm{C_l}(I) \cup C_1$ and $E := C_0 \cup \mathrm{C_r}(I) \cup C_1$ are maximal chains in L. It follows from D. Kelly and I. Rival [17] that

$$I = \{x \in L : x \text{ is on the right of } W,\ x \text{ is and on the left of } E,\ x \not\leq u \text{ and } x \not\geq v\}. \tag{5.6}$$

Let x be a lower cover of v. Then x cannot be strictly on the left of W since then x would be strictly on the left of the leftmost lower cover, a. Hence, x is on the right of W and, similarly, on the left of E. This together with (5.6) shows that $x \in I$. Hence, $u = 0_I \leq x$ for all lower covers x of v, proving the lemma. □

Although the boundary of a planar semimodular lattice L is not unique in general, see M_3, the following assertion holds.

Lemma 5.5. *Let L be a glued sum indecomposable, planar, semimodular lattice, and let D be a planar diagram of L. Let I and J be rectangular intervals such that $I \cap J$ is a chain. Then $I \cap J \subseteq \mathrm{Bnd}(I, D) \cap \mathrm{Bnd}(J, D)$.*

Proof. Let x be the least element of the chain $I \cap J$. Assume first that $x \in \mathrm{int}(I, D) \cap \mathrm{int}(J, D)$. Then $x \neq 0_L$, so x has a lower cover y. By (2.6) and (2.8), $y \in I \cap J$, contradicting the choice of x. This excludes that $x \in \mathrm{int}(I, D) \cap \mathrm{int}(J, D)$.

Secondly, we assume that $x \in \mathrm{Bnd}(I, D) \cap \mathrm{int}(J, D)$. By (2.6) and (2.8), all lower covers of x belong to J. Hence, by the choice of x, no lower cover of x belongs to I. This means that $x = 0_I$. However, then x has at least two covers in I, and these covers belong to J (and therefore to $I \cap J$) by (2.6) and (2.8). This is a contradiction since $I \cap J$ is a chain. Thus, taking the I-J symmetry into account, we conclude that $x \notin \mathrm{Bnd}(I, D) \cap \mathrm{int}(J, D)$ and $x \notin \mathrm{int}(I, D) \cap \mathrm{Bnd}(J, D)$.

Therefore, $x \in \mathrm{Bnd}(I, D) \cap \mathrm{Bnd}(J, D)$. To get a contradiction, suppose that x has more than one covers both in I and J. Let $a_1, a_2 \in I$ and $b_1, b_2 \in J$ be covers of x such that $a_1 \neq a_2$ and $b_1 \neq b_2$. Let $v := a_1 \vee a_2 \in I$. Since $a_1 \vee a_2 = b_1 \vee b_2$ by Lemma 5.3, $v \in J$. By the convexity of $I \cap J$, we have that $\{a_1, a_2\} \subseteq [u, v] \subseteq I \cap J$, which is a contradiction since $I \cap J$ is a chain. This proves that, say, x has at most one cover in I. This fact together with (2.14) implies that $x \in \mathrm{UBnd}(I, D)$.

We are now in the position to show that each $y \in I \cap J$ belongs to $\mathrm{Bnd}(I, D) \cap \mathrm{Bnd}(J, D)$. We already know this if $y = x$. Hence, we can assume that $y > x$. We obtain from $x \in \mathrm{UBnd}(I, D)$ and (2.12) that $y \in \mathrm{UBnd}(I, D) - \mathrm{LBnd}(I, D)$. Hence, (2.15) yields that y has at least two lower covers, z_1 and z_2, in I. If we had $y \in \mathrm{int}(J, D)$, then (2.6) and (2.8) would imply $\{z_1, z_2\} \subseteq J$, and the antichain $\{z_1, z_2\} \subseteq I \cap J$ would be a contradiction. Consequently, $y \in \mathrm{Bnd}(J, D)$, whence $y \in \mathrm{Bnd}(I, D) \cap \mathrm{Bnd}(J, D)$ proves the statement. □

6. Some properties that depend only on the full slimming of L

Given a rectangular interval or, in particular, a patch (interval) $I = [u, v]$ of L, its bottom and top will sometimes be denoted by $0_I = u$ and $1_I = v$, while $w_l(I) = w_l(I, D)$ and $w_r(I) = w_r(I, D)$ stand for its weak corners, with respect to $D \in \mathrm{Dgr}(L)$.

Lemma 6.1. *Let L be a planar semimodular lattice, and let L' be the full slimming of L. Then L' is a slim semimodular lattice, and the following five assertions hold.*

- (i) *L is a patchwork-irreducible lattice iff so is L';*
- (ii) *L is a rectangular lattice iff so is L';*
- (iii) *L is a patch lattice (that is, a rectangular lattice whose weak corners are coatoms, see also Lemma* 4.9(ii)) *iff so is L';*
- (iv) *L is glued sum indecomposable iff so is L'.*
- (v) *L is indecomposable with respect to the Hall-Dilworth gluing over chains iff so is L'.*

Moreover, if D is a fixed planar diagram of L, L' denotes the full slimming sublattice *of L with respect to D, and D' is the restriction of D to L', then the following three assertions also hold.*

- (vi) $\mathrm{Bnd}(L, D) = \mathrm{Bnd}(L', D')$.
- (vii) *Let $a < b \in L$. Then $[a, b]_L$ is a rectangular interval of L iff $a, b \in L'$ and $[a, b]_{L'}$ is a rectangular interval of L'. In particular, $[a, b]_L \in \mathrm{Patch}(L)$ iff $a, b \in L'$ and $[a, b]_{L'} \in \mathrm{Patch}(L')$. Hence, $[a, b]_L \in \mathrm{MaxPatch}(L)$ iff $a, b \in L'$ and $[a, b]_{L'} \in \mathrm{MaxPatch}(L')$.*
- (viii) *Let $\boldsymbol{\mathcal{H}} = \{[a_i, b_i]_L : 1 \leq i \leq n\}$ be a system of rectangular intervals of L, and let $\boldsymbol{\mathcal{H}}' = \{[a_i, b_i]_{L'} : 1 \leq i \leq n\}$ be the corresponding system of rectangular intervals of L', see part* (vii)*. Then $\boldsymbol{\mathcal{H}}$ is a patchwork system for D iff $\boldsymbol{\mathcal{H}}'$ is a patchwork system for D'.*

In connection with parts (vi)–(viii), notice that we often write D for D' when an interval or a sublattice of L is considered.

Proof. By Lemma 4.1 (see also Remark 4.2), we can assume that L' is the full slimming *sublattice* of L with respect to a fixed planar diagram D even in parts (ii)–(v) of the lemma. We will have to be more careful with statement (i) since the full slimming sublattice and the patchwork system may depend on different diagrams. Similarly, L could have intervals I whose rectangularity comes from diagrams distinct from the restriction of D to I, and this phenomenon would cause a lot of difficulty while proving (vii).

We know from Proposition 2.1 that L' is a slim semimodular lattice. We can assume that $L \neq L'$. Let $e \in L - L'$ denote an *arbitrary* eye. The 4-cell $\{e_*, a, b, e^*\}$ of L', see (2.16), will be denoted by S. The notation $a = a_e$ and $b = b_e$ are also used in the proof.

Since $e \in \mathrm{int}(S, D') \subseteq \mathrm{int}(L', D')$ by (2.9), we insert the eyes into the interior of L'. Hence, $\mathrm{Bnd}(L, D) = \mathrm{Bnd}(L', D')$. This gives (vi), which implies (ii) and (iii). We also obtain (iv) since glued sum indecomposability in the planar case means that the lattice in question has at least four elements and $\{0, 1\}$ is the intersection of the left and the right boundary chains.

Assume that an eye e belongs to a rectangular interval I. Since $e \in \mathrm{Mi}\, L \cap \mathrm{Ji}\, L$, we obtain that $e \notin \{0_I, 1_I\}$. Hence, $[e_*, e^*] \subseteq I$. Using (2.9), we conclude that $e \in \mathrm{int}([e_*, e^*], D) \subseteq \mathrm{int}(I, D)$. That is,

(6.1) e cannot be on the boundary of a rectangular interval of L.

In particular, (6.1) implies that an eye cannot be the bottom or the top of a rectangular interval. Therefore, if we consider an interval I as the pair $(0_I, 1_I)$, then we can say that L' and L has "exactly the same" rectangular intervals. This implies the first half (vii). The rest of (vii) is then evident since L' is a cover-preserving sublattice of L.

While proving (viii), we use the following notation: for $I \in \boldsymbol{\mathcal{H}}$, we let $I' := I \cap L' = [0_I, 1_I]_{L'}$; and for $J' \in \boldsymbol{\mathcal{H}}'$, we let $J := [0_{J'}, 1_{J'}]_L$. By the definition of $\boldsymbol{\mathcal{H}}'$, we have that

(6.2) $K \in \boldsymbol{\mathcal{H}}$ iff $K' \in \boldsymbol{\mathcal{H}}'$, and $(I, J) \in \mathcal{E}\big(\boldsymbol{\mathcal{H}}\big)$ iff $(I', J') \in \mathcal{E}(\boldsymbol{\mathcal{H}}')$.

Since the tops and the bottoms of covering squares are the same in L as in L', (6.2) yields that $\boldsymbol{\mathcal{H}}$ satisfies 3.1(i) iff so does $\boldsymbol{\mathcal{H}}'$.

If $\boldsymbol{\mathcal{H}}$ satisfies 3.1(ii), then so does $\boldsymbol{\mathcal{H}}'$, evidently. Before proving the converse implication, we claim that, for all $I, J \in \boldsymbol{\mathcal{H}}$,

(6.3) if $I' \cap J'$ is a chain, then $I \cap J = I' \cap J'$, whence $I \cap J$ is also a chain.

By way of contradiction, let us assume that (6.3) fails for some $I, J \in \boldsymbol{\mathcal{H}}$. Then $I \cap J = [0_I \vee 0_J, 1_I \wedge 1_J]_L$ contains an eye e. If $0_I \vee 0_J \notin \{0_I, 0_J\}$, then $0_I \vee 0_J$ is join-reducible. If $0_I \vee 0_J \in \{0_I, 0_J\}$, then $0_I \vee 0_J$ is meet-reducible. Hence, in both cases, $0_I \vee 0_J \notin \mathrm{Ji}\, L \cap \mathrm{Mi}\, L$. Dually, $1_I \wedge 1_J \notin \mathrm{Ji}\, L \cap \mathrm{Mi}\, L$. Therefore, $e \notin \{0_I \vee 0_J, 1_I \wedge 1_J\}$. Hence, $0_I \vee 0_J < e < 1_I \wedge 1_J$, and we conclude that $0_I \vee 0_J \leq e_* < a < e^* \leq 1_I \wedge 1_J$. This yields that $a \in I' \cap J'$. We obtain $b \in I' \cap J'$ similarly, which is a contradiction since $I' \cap J'$ is a chain. This proves (6.3).

Next, assume that $\boldsymbol{\mathcal{H}}'$ satisfies 3.1(ii), and let $(I, J) \in \mathcal{E}\big(\boldsymbol{\mathcal{H}}\big)$. Then $(I', J') \in \mathcal{E}(\boldsymbol{\mathcal{H}}')$ by (6.2), whence $I' \cap J'$ is a chain. So is $I \cap J$ by (6.3). Hence, $\boldsymbol{\mathcal{H}}$ also satisfies 3.1(ii).

As a preparation for 3.1(iii), assume that K' is an arbitrary rectangular interval of L'. Equivalently, see 6.1(vii), we assume that K is a rectangular interval of L. We claim that

$$\begin{gathered} K' \text{ is the full slimming sublattice of } K \\ \text{with respect to (the restriction of) } D. \end{gathered} \tag{6.4}$$

It suffices to show that if K contains an eye $e \in L - L'$, then $\{e_*, e^*\} \subseteq K'$. But this is easy: if $0_K \le e \le 1_K$, then $0_K < e < 1_K$ since $e \in \operatorname{Ji} L \cap \operatorname{Mi} L$, $0_K \notin \operatorname{Mi} L$, and $1_K \notin \operatorname{Ji} L$. Hence, $0_K \le e_* < e < e^* \le 1_K$, indeed.

Since D' is a restriction of D, either of (6.1) and (6.4) yields that

$$\mathrm{C_l}(K', D') = \mathrm{C_l}(K, D) \text{ and } \mathrm{C_r}(K', D') = \mathrm{C_r}(K, D). \tag{6.5}$$

It follows from (2.13), (2.14) and (2.15) that, for every rectangular lattice R and $F \in \operatorname{Dgr}(R)$,

$$\begin{gathered} \operatorname{UBnd}(R, F) = \operatorname{Bnd}(R, F) - \bigl(R - \operatorname{Mi} R\bigr) \text{ and} \\ \operatorname{LBnd}(R, F) = \operatorname{Bnd}(R, F) - \bigl(R - \operatorname{Ji} R\bigr). \end{gathered} \tag{6.6}$$

Furthermore, it follows from (6.4) that $K' - \operatorname{Mi} K' = K - \operatorname{Mi} K$ and $K' - \operatorname{Ji} K' = K - \operatorname{Ji} K$. This together with (6.5) and (6.6) yields that

$$\operatorname{UBnd}(K', D') = \operatorname{UBnd}(K, D) \quad \text{and} \quad \operatorname{LBnd}(K', D') = \operatorname{LBnd}(K, D). \tag{6.7}$$

Next, assume that $\boldsymbol{\mathcal{H}}$ is a patchwork system for D. We have already seen that $\boldsymbol{\mathcal{H}}'$ satisfies 3.1(i) and 3.1(ii). Let $(I', J') \in \mathcal{E}(\boldsymbol{\mathcal{H}}')$. Since $(I, J) \in \mathcal{E}\bigl(\boldsymbol{\mathcal{H}}\bigr)$ by (6.2), we have that, say, $I \cap J \subseteq \operatorname{UBnd}(I, D) \cap \operatorname{LBnd}(J, D)$. Hence, using (6.7), we obtain that $I' \cap J' \subseteq I \cap J \subseteq \operatorname{UBnd}(I, D) \cap \operatorname{LBnd}(J, D) = \operatorname{UBnd}(I', D') \cap \operatorname{LBnd}(J', D')$. This shows that $\boldsymbol{\mathcal{H}}'$ also satisfies 3.1(iii) for D', so it is a patchwork system for D'.

Conversely, assume that $\boldsymbol{\mathcal{H}}'$ is a patchwork system for D'. We have already seen that $\boldsymbol{\mathcal{H}}$ satisfies 3.1(i) and 3.1(ii). Let $(I, J) \in \mathcal{E}\bigl(\boldsymbol{\mathcal{H}}\bigr)$. Since $(I', J') \in \mathcal{E}(\boldsymbol{\mathcal{H}}')$ by (6.2), we have that $I' \cap J'$ is a chain and, say, $I' \cap J' \subseteq \operatorname{UBnd}(I', D') \cap \operatorname{LBnd}(J', D')$. Consequently, using (6.3) and (6.7), we obtain that $I \cap J = I' \cap J' \subseteq \operatorname{UBnd}(I', D') \cap \operatorname{LBnd}(J', D') = \operatorname{UBnd}(I, D) \cap \operatorname{LBnd}(J, D)$. Hence, $\boldsymbol{\mathcal{H}}$ satisfies 3.1(iii) for D. Thus it is a patchwork system for D. This completes the proof of (viii).

Next, armed with (viii), we derive (i). Assume that L is patchwork-reducible. Then there is a $D \in \operatorname{Dgr}(L)$ and there is a nontrivial patchwork system $\boldsymbol{\mathcal{H}}$ for D. Let L'_D be the full slimming sublattice of L with respect to D. We conclude from (viii) that $\boldsymbol{\mathcal{H}}' = \boldsymbol{\mathcal{H}}'_D$ is a nontrivial patchwork system for (the diagram restricted to) L'_D. Hence, L'_D is patchwork-reducible, and so is L' since $L' \cong L'_D$ by Lemma 4.1.

Conversely, assume that L' is patchwork-reducible. Hence, there is a D' in $\operatorname{Dgr}(L')$ such that there is a nontrivial patchwork system $\boldsymbol{\mathcal{H}}'$ for D'. Let $F \in \operatorname{Dgr}(L)$, and take the full slimming sublattice L_0 of L determined by F. Clearly, F^{mir} would determine the same full slimming sublattice L_0. The restriction of F to L_0 is denoted by F_0. We know from Lemma 4.1 (and Remark 4.2) that there exists a lattice isomorphism $\varphi\colon L' \to L_0$. After replacing F by F^{mir} if necessary, we obtain from Lemma 4.7 that $\varphi\colon (L', D') \to (L_0, F_0)$ is a directed diagram isomorphism. Since 3.1(iii) is based on concepts preserved by this sort of isomorphisms, $\varphi(\boldsymbol{\mathcal{H}}') = \{\varphi(I) : I \in \boldsymbol{\mathcal{H}}'\}$ is a nontrivial patchwork system for F_0. Hence, 6.1(viii) yields a nontrivial patchwork system for F, proving that L is patchwork-reducible. This proves (i).

To prove the "if" part of (v), we assume that L is GC-decomposable; we have to show that so is L'. By the assumption, there are a proper ideal I and a proper filter F of L such that $L = I \cup F$ and $C := I \cap F$ is a chain. Let $I' := I \cap L'$, $F' := F \cap L'$, and $C' := C \cap L' = I' \cap F'$. Clearly, $L' = I' \cup F'$. Suppose, for a contradiction, that $C' = \varnothing$. Then C contains an eye e since $C \neq \varnothing$. Using that C' is empty and $\{e_*, e^*\} \subseteq L'$, we infer that $\{e_*, e^*\} \cap C = \varnothing$. It follows from $e_* < e \in I$ and $e_* \notin C$ that $e_* \in I - F$. Dually, we obtain that $e^* \in F - I$. Using $L = I \cup F$, we have that $a = a_e \in I$ or $a \in F$. However, $a \in I$ gives that $e^* = e \vee a \in I$, contradicting $e^* \in F - I$, while $a \in F$ gives that $e_* = e \wedge a \in F$, contradicting $e_* \in I - F$. This contradiction yields that C' is nonempty, indeed. So C' is a chain since $C' \subseteq C$. Since 0_L and 1_L are not eyes, they belong to L', and $1_{L'} = 1_L$ and $0_{L'} = 0_L$. Hence, if $1_{L'}$ belonged to I', then $1_L \in I$ would contradict $I \neq L$. Therefore, $1_{L'} \notin I'$ shows that I' is a proper ideal of L'. Working with $0_{L'} = 0_L$ dually, we obtain that F' is a proper filter of L'. Thus L' is GC-decomposable. This proves the "if" part of (v).

To prove the "only if" part of (v), we next assume that L' is GC-decomposable, and we have to show that so is L. By the assumption, there are $u, v \in L'$ such that $L' = [0, v]_{L'} \cup [u, 1]_{L'}$, $0 < u \leq v < 1$, and $C' := [u, v]_{L'} = [0, v]_{L'} \cap [u, 1]_{L'}$ is a chain. Define $I := [0, v]_L$ and $F := [u, 1]_L$. Then I and F are proper subsets of L since $u \neq 0$ and $v \neq 1$. To get a contradiction, suppose that $C := [u, v]_L = I \cap F$ contains an eye. Then $e \in [u, v]_L - \{u, v\}$ since $e \in L - L'$. Hence, $e_*, e^* \in [u, v]_L$ implies that the 4-cell S is included in the chain $C' = [u, v]_{L'}$, a contradiction. Thus we conclude that $C = C'$, whence C is a chain in L. To get another contradiction, suppose that $L \neq I \cup F$. Then there is an eye e such that $e \notin I = [0, v]_L$ and $e \notin F = [u, 1]_L$. Hence, $e^* \notin [0, v]_L$ and $e_* \notin [u, 1]_L$. Using $e_* = a \wedge b$ and $e^* = a \vee b$, we conclude that $\{a, b\} \not\subseteq [0, v]_L$ and $\{a, b\} \not\subseteq [u, 1]_L$. In fact, it is more reasonable to write $\{a, b\} \not\subseteq [0, v]_{L'}$ and $\{a, b\} \not\subseteq [u, 1]_{L'}$ since $a, b \in L'$. On the other hand, $\{a, b\} \subseteq L' = [0, v]_{L'} \cup [u, 1]_{L'}$. Therefore, we have that, say, $a \in [0, v]_{L'} - [u, 1]_{L'}$ and $b \in [u, 1]_{L'} - [0, v]_{L'}$. It follows from $e_* \notin [u, 1]_L$, $e_* < b$ and $b \in [u, 1]_{L'}$ that $e_* < e_* \vee u \leq b$. This together with $e_* \prec b$ implies that $e_* \vee u = b$. We know that u belongs to $[0, v]_{L'}$. Since $e_* \leq a \in [0, v]_{L'}$, we have that e_* also belongs to $[0, v]_{L'}$. Therefore, $b = e_* \vee u \in [0, v]_{L'}$, which is a contradiction. Thus $L = I \cup F$, and L is GC-decomposable. This proves (v). ☐

7. Getting rid of diagrams

The fact that many of our concepts depends (at least formally) on the diagram chosen causes a lot of inconvenience. The aim of this section is to get rid of this difficulty by proving Proposition 3.2. The following lemma is not surprising.

Lemma 7.1. *Let I be a rectangular interval of a planar semimodular lattice L, and let $D \in \mathrm{Dgr}(L)$. Assume that $x \in I - \mathrm{C_r}(I, D)$. Then the right neighbor of x (in L, with respect to D) exists, and it belongs to I.*

Proof. Let C_0 and C_1 be maximal chains in $\downarrow 0_I$ and $\uparrow 1_I$, respectively. Clearly, there are a unique $s \in \mathrm{C_l}(I, D)$ and a unique $t \in \mathrm{C_r}(I, D)$ such that $h(s) = h(t) = h(x)$. If we had that $x \in \mathrm{C_r}(L, D)$, then the left-right dual of (5.3), applied to I and also to L, would imply that x and t are mutually on the right of each other, whence (5.2) would yield that $x = t \in \mathrm{C_r}(I, D)$, a contradiction. Hence, $x \notin \mathrm{C_r}(L, D)$. Therefore, in virtue of (5.4), the (unique) right neighbor y of x makes sense. Moreover, the left-right dual of (5.3) together with $x \notin \mathrm{C_r}(I, D)$ implies that t is strictly on the right of x. Consequently, y is on the left of t, whence y is on the left of the maximal chain $C_0 \cup \mathrm{C_r}(I, D) \cup C_1$. On the other hand, s is on the left of x by (5.3), which yields that s is on the left of y. This gives that y is on the right of $C_0 \cup \mathrm{C_l}(I, D) \cup C_1$. Finally, $y \in I$ follows by (5.6). $\square$

Proof of Proposition 3.2. In order to show that (i) $\Rightarrow$ (ii), we assume (i). Let $D \in \mathrm{Dgr}(L)$ be a diagram witnessing that 3.1(iii) holds. Consider a pair $(I, J) \in \mathcal{E}(\boldsymbol{\mathcal{H}})$. By the assumptions, $I \cap J$ is a chain and, say, $I \cap J \subseteq \mathrm{UBnd}(I, D) \cap \mathrm{LBnd}(J, D)$. Let $F \in \mathrm{Dgr}(L)$ be another diagram. We already know from Lemma 5.5 that $I \cap J \subseteq \mathrm{Bnd}(I, F) \cap \mathrm{Bnd}(J, F)$. If we have an element $x \in I \cap J$ such that $x \in \mathrm{Bnd}(I, F) - \mathrm{UBnd}(I, F)$, then x has at least two covers within I by (2.14), applied to F, but this contradicts (2.13), applied to D. Hence, $I \cap J \subseteq \mathrm{UBnd}(I, F)$. Similarly, if we have an element $x \in I \cap J$ such that $x \in \mathrm{Bnd}(J, F) - \mathrm{LBnd}(J, F)$, then x has at least two lower covers within J by (2.15), applied to F, but this contradicts (2.13), applied to D. Hence, $I \cap J \subseteq \mathrm{LBnd}(J, F)$. Thus $I \cap J \subseteq \mathrm{UBnd}(I, F) \cap \mathrm{LBnd}(J, F)$, which means that (ii) holds.

Next, to show that (ii) $\Rightarrow$ (iv), we assume (ii). Let $D \in \mathrm{Dgr}(L)$. By the I-J symmetry, we can assume that $I \cap J \subseteq \mathrm{UBnd}(I, D) \cap \mathrm{LBnd}(J, D)$. Suppose, for a contradiction, that $I \cap J \not\subseteq \mathrm{C_{ur}}(I, D)$ and $I \cap J \not\subseteq \mathrm{C_{ul}}(I, D)$. Then there are $x, y \in I \cap J$ such that $x \in \mathrm{C_{ul}}(I, D) - \mathrm{C_{ur}}(I, D)$ and $y \in \mathrm{C_{ur}}(I, D) - \mathrm{C_{ul}}(I, D)$. Since $I \cap J$ is a chain, we can assume by left-right symmetry that $x \leq y$. Using (2.12) and Lemma 4.9(ii), we obtain that $1_I = w_l(I, D) \vee w_r(I, D) \leq x \vee y = y \leq 1_I$, which gives that $y = 1_I \in \mathrm{C_{ul}}(I, D)$, a contradiction. This shows that

$$I \cap J \subseteq \mathrm{C_{ul}}(I, D) \quad \text{or} \quad I \cap J \subseteq \mathrm{C_{ur}}(I, D). \tag{7.1}$$

The dual argument yields that

$$I \cap J \subseteq \mathrm{C_{ll}}(J, D) \quad \text{or} \quad I \cap J \subseteq \mathrm{C_{lr}}(J, D). \tag{7.2}$$

We can assume that the disjunction "or" is an exclusive disjunction both in (7.1) and (7.2) since otherwise the desired 3.2(iiib) for D would trivially hold. Hence, by the left-right symmetry and keeping the targeted 3.2(iiib) in mind, we can suppose for a contradiction that

$$\begin{gathered} I \cap J \subseteq \mathrm{C_{ul}}(I, D),\ I \cap J \not\subseteq \mathrm{C_{ur}}(I, D), \\ I \cap J \subseteq \mathrm{C_{ll}}(J, D), \text{ and } I \cap J \not\subseteq \mathrm{C_{lr}}(J, D). \end{gathered} \tag{7.3}$$

Firstly, we assume that there is a $u \in (I \cap J) - \{1_I, 0_J\}$. Then $u \notin \mathrm{C_r}(I, D)$ since $\mathrm{C_{ul}}(I, D) \cap \mathrm{C_r}(I, D) = \{1_I\}$. Similarly, $u \notin \mathrm{C_r}(J, D)$ since $\mathrm{C_{ll}}(J, D) \cap \mathrm{C_r}(J, D) = \{0_J\}$. Hence, by Lemma 7.1, the right neighbor v of u with respect to D exists, and it belongs to $I \cap J$. However, then $u \parallel v$ together with $u, v \in I \cap J$ is a contradiction since $I \cap J$ is a chain.

Secondly, we assume that there is no such u. By (7.3), we can select $x, y \in I \cap J$ such that $x \in \mathrm{C_{ul}}(I, D) - \mathrm{C_{ur}}(I, D)$ and $y \in \mathrm{C_{ll}}(J, D) - \mathrm{C_{lr}}(J, D)$. Notice that $x \notin \mathrm{C_r}(I, D)$ since $I \cap J \subseteq \mathrm{UBnd}(I, D)$, and $y \notin \mathrm{C_r}(J, D)$ since $I \cap J \subseteq \mathrm{LBnd}(J, D)$. If we had $x \geq y$, then $u := x$ (or $u := y$) would lead to the previous case. Hence, we assume that $x < y$. If we had $x \neq 0_J$ or $y \neq 1_I$, then $u := x$ or $u := y$ would again lead to the previous case. Hence, $x = 0_J$ and $y = 1_I$. We can also assume that $0_J = x \prec y = 1_I$ since otherwise, using the convexity of $I \cap J$, we could choose a $u \in I \cap J \cap [x, y] - \{x, y\}$, which would lead to the previous case again. Let z

be the unique atom of J that belongs to $\mathrm{C_r}(J, D)$. Similarly, let t be the unique coatom of I that belongs to $\mathrm{C_r}(I, D)$.

Extend $\mathrm{C_r}(I, D) \cup \{s \in \mathrm{C_l}(J, D) : s \geq y = 1_I\}$ to a maximal chain C of L. Since $h(0_J) = h(1_I) - 1 = h(t) \in \mathrm{C_r}(I, D) \subseteq C$ and $0_J = x \notin \mathrm{C_r}(I, D)$, we obtain from the left-right dual of (5.3) that 0_J is strictly on the left of t. Hence, (5.1) yields that 0_J is strictly on the left of C. Similarly, $1_I = y \notin \mathrm{C_r}(J, D)$ together with the left-right dual of (5.3) gives that z is strictly on the right of $1_I = y \in C$, whence (5.1) yields that z is strictly on the right of C. However, then $0_J \prec z$ contradicts (2.5). Thus (7.3) leads to a contradiction, proving (ii) $\Rightarrow$ (iv).

The implication (iii) $\Rightarrow$ (i) is evident. So is (iv) $\Rightarrow$ (iii) since L is planar. □

8. Patch lattices

We are not in the position of proving Theorem 3.4 yet. However, some of its parts will be needed in the next sections. Now we prove these parts.

Lemma 8.1. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) *of Theorem 3.4 hold.*

Proof. (ii) $\Rightarrow$ (iii) is obvious.

By Lemma 6.1, it suffices to prove the implication (iii) $\Rightarrow$ (iv) only for *slim* semimodular lattices. Hence, assume that L is a slim semimodular lattice and (iii) holds. Let $D \in \mathrm{Dgr}(L)$. We know from (2.4) that there is a double irreducible element in $\mathrm{C_l}(L, D) - \{0, 1\}$. In fact, there is a smallest one since $\mathrm{C_l}(L, D)$ is a chain; we denote it by a. Let b_0 denote the smallest element of $\mathrm{C_r}(L, D) - {\downarrow}a$, and let $(b_0)_*$ be the unique lower cover of b_0 that belongs to $\mathrm{C_r}(L, D)$. Let $c := a \vee b_0$. From semimodularity and $a = a \vee (b_0)_*$, we obtain that $a \prec c$. Since $\mathrm{C_l}(L, D)$ is a chain and $a \in \mathrm{C_l}(L, D)$ has exactly one cover in L, we conclude that $c \in \mathrm{C_l}(L, D)$. Let b be the largest element of $\mathrm{C_r}(L, D) \cap {\downarrow}c$. Then $c = a \vee b$ since $b_0 \leq b \leq c$ and $c = a \vee b_0$.

Assume that $z_1, z_2 \in {\downarrow}c \cap {\uparrow}b = [b, c]$. By (2.10), there are $x_i \in \mathrm{C_l}(L, D)$ and $y_i \in \mathrm{C_r}(L, D)$ such that $z_i = x_i \vee y_i$, for $i \in \{1, 2\}$. By the definition of b and $z_i \leq c$, we know that $y_i \leq b$. Hence, $z_i = z_i \vee b = x_i \vee b$, for $i \in \{1, 2\}$. Since $x_1, x_2 \in \mathrm{C_l}(L, D)$ are comparable, so are z_1 and z_2. This together with $b \leq c$ shows that ${\downarrow}c \cap {\uparrow}b$ is a chain. Next, consider an arbitrary $z \in L$; we want to show that $z \in {\downarrow}c \cup {\uparrow}b$. By (2.10), $z = x \vee y$ for some $x \in \mathrm{C_l}(L, D)$ and $y \in \mathrm{C_r}(L, D)$. We can assume that $y < b$ since otherwise $z \in {\uparrow}b$. Then we can assume that $c < x$ since otherwise $z = x \vee y \leq c \vee b = c$ would mean that $z \in {\downarrow}c$. Therefore, $b \leq c < x \leq x \vee y = z$, that is, $z \in {\uparrow}b$. This shows that $L = {\downarrow}c \cup {\uparrow}b$.

Thus, by (iii), either ${\downarrow}c = L$ or ${\uparrow}b = L$. But $(b_0)_* < b_0 \leq b$ excludes the latter, so ${\downarrow}c = L$, which means that $c = 1$. This shows that the smallest (and therefore every) doubly irreducible element on the left boundary is a coatom. In particular, there is exactly one left weak corner with respect to D; it is a coatom and it will be denoted by $w_l(D)$. Similarly, there is exactly one right weak corner $w_r(D)$. Since L is glued sum indecomposable, $\mathrm{C_l}(L, D) \cap \mathrm{C_r}(L, D) = \{0, 1\}$. This yields that $w_l(D) \neq w_r(D)$. Hence, $w_l(D) \vee w_r(D) = 1$ since they are coatoms. If $w_l(D) \wedge w_r(D) = 0$, then (iv) is clear.

To get a contradiction, suppose that ${\downarrow}w_l(D)$ is not a chain. By Lemma 2.3, ${\downarrow}w_l(D)$ has a glued some indecomposable component A. Obviously, $\mathrm{C_l}(A, D) \cap \mathrm{C_r}(A, D) = \{0_A, 1_A\}$. Hence, (2.4) yields an element $s \in \mathrm{C_l}(A, D) - \mathrm{C_r}(A, D)$ such that s is doubly irreducible within A. It is obvious by Lemma 2.3 that $\mathrm{C_r}(A, D) \subseteq$

$\mathrm{C_r}(\downarrow w_l(D), D)$ and, taking (5.5) into account,

$$\mathrm{C_l}(A, D) \subseteq \mathrm{C_l}(\downarrow w_l(D), D) \subseteq \mathrm{C_l}(L, D).$$

Hence, we conclude that $s \in \mathrm{C_l}(L, D) - \mathrm{C_r}(\downarrow w_l(D), D)$. Moreover, s is doubly irreducible also within $\downarrow w_l(D)$. Let s^* denote the unique cover of s in

$$\mathrm{C_l}(\downarrow w_l(D), D) \subseteq \mathrm{C_l}(L, D).$$

Evidently, s is join-irreducible not only in $\downarrow w_l(D)$ but also in L. Since $w_l(D)$ is the only doubly irreducibly element (that is, a weak left corner) on the left boundary of L and $s < 1_A \leq w_l(D)$, we conclude that s is meet-reducible in L. Therefore, s has a cover $s' \in L - \downarrow w_l(D)$. Notice that $s' \neq s^*$. Hence, (5.3) yields that s' is strictly on the right of s^*, and we obtain from (5.1) that s' is strictly on the right of the maximal chain $\mathrm{C_l}(L, D) = \mathrm{C_l}(\downarrow w_l(D), D) \cup \{1\}$. If s' was on the left of the maximal chain $C := \mathrm{C_r}(\downarrow w_l(D)) \cup \{1\}$, then (5.6) (with 0 and $w_l(D)$ acting as u and v, respectively) would imply that $s' \in \downarrow w_l(D)$. Therefore

$$s' \text{ is strictly on the right of } C. \tag{8.1}$$

Let $t \in \mathrm{C_r}(A, D) \subseteq \mathrm{C_r}(\downarrow w_l(D), D) \subseteq C$ be the unique element with $h(t) = h(s)$. Then $s \neq t$ since $s \notin \mathrm{C_r}(A, D)$. It follows from (5.3) that s is strictly on the left of t. Hence, (5.1) gives that s is strictly on the left of C. However, this fact together with (8.1) and $s \prec s'$ contradicts (2.5), proving that $\downarrow w_l(D)$ is a chain.

Therefore, $\downarrow w_l(D) \subseteq \mathrm{C_l}(L, D)$ and, similarly, $\downarrow w_r(D) \subseteq \mathrm{C_r}(L, D)$. Combining this with the glued sum indecomposability of L, we conclude that $w_l(D) \wedge w_r(D) \in \mathrm{C_l}(L, D) \cap \mathrm{C_r}(L, D) = \{0, 1\}$. This gives the desired $w_l(D) \wedge w_r(D) = 0$. Thus (iv) holds, proving the implication (iii) $\Rightarrow$ (iv).

The implication (iv) $\Rightarrow$ (v) is evident.

Assume (v). Then L has two coatoms whose meet is 0, whence (vi) follows easily from Lemma 5.4. This proves that (v) $\Rightarrow$ (vi).

To show (vi) $\Rightarrow$ (ii), take a fixed planar diagram of L. Let a and b be the leftmost and the rightmost coatoms of L, respectively. Assume that I is an ideal and F is a filter of L such that $L = I \cup F$ and $I \cap F \neq \varnothing$. We have to show that $L \in \{I, F\}$. If $a, b \in F$, then $F = L$ since $0 = a \wedge b \in F$. If $a, b \in I$, then $I = L$ since $1 = a \vee b \in I$. Therefore, since $\{a, b\} \subseteq L = I \cup F$, we can assume that, say, $a \in I$ and $b \in F$. Consider the smallest element of $I \cap F$. Clearly, it is 0_F. If $0_F = 0$, then $F = L$. Hence, we can assume that $0 < 0_F$. Since $0_F \leq a$ would lead to the contradiction $0 < 0_F \leq a \wedge b = 0$, we conclude that $0_F \not\leq a$. Hence, $1 = a \vee 0_F \in I$, implying that $I = L$. Thus (vi) $\Rightarrow$ (ii). ☐

9. Some properties of patch intervals

The lemmas of this section formulate some properties of patch intervals, also called patches, of L. Eventually, these properties will be easy consequences of Theorem 3.6. However, we have to prove them now since they will be used in the proof of Theorem 3.6.

Lemma 9.1. *Let I and J be patches of a slim semimodular lattice L such that $0_J \in I - \mathrm{UBnd}(I) = I - \{1_I, w_l(I), w_r(I)\}$. Then $I \subseteq J$ or $J \subseteq I$. (By Lemma 4.9, the choice of $D \in \mathrm{Dgr}(L)$ is irrelevant.)*

Proof. Assume that $J \not\subseteq I$. Then $\{w_l(J), w_r(J)\} \not\subseteq I$ since otherwise $1_J = w_l(J) \vee w_r(J) \in I$ and the convexity of I would imply that $J \subseteq I$. Let, say, $w_l(J) \notin$

I. Applying Lemma 5.1 to $0_J < w_l(J)$, we obtain an element $x \in \mathrm{UBnd}(I) = \mathrm{UBnd}(I, D)$ such that $0_J \leq x \leq w_l(J)$. In fact, we have that $0_J < x < w_l(J)$ by the assumptions. There are four cases to consider.

Case 1. Assume that $x = w_r(I)$. Since $0_J < x = w_r(I)$ and $[0_I, w_r(I)] = \mathrm{C_{lr}}(I)$ is a chain by (2.11) and (2.12), 0_J has a unique cover y_1 in $\mathrm{C_{lr}}(I) \subseteq \mathrm{C_r}(I)$. By (2.14), 0_J has another cover $y_0 \in I$, which is strictly on the left of y_1. Since $y_1 \in [0_J, w_r(I)] \subset [0_J, w_l(J)] \subseteq \mathrm{C_l}(J)$, (2.14) yields that 0_J has a cover $y_2 \in \mathrm{C_r}(J)$, which is strictly on the right of y_1. Their position shows that y_0, y_1 and y_2 are three distinct covers of 0_J. Thus the present case is excluded by (2.1).

Case 2. Assume that $x = w_l(I)$ and $w_r(J) \in I$. Then $w_l(I) < w_l(J)$. This together with $w_l(J) \ngeq w_r(J)$ give that $w_l(I) \ngeq w_r(J)$. So $w_l(I) < w_l(I) \vee w_r(J) \in I$ yields that $w_l(I) \vee w_r(J) = 1_I$. Since $J \nsubseteq I$, we know that $1_J \neq 1_I$. But $1_I = w_l(I) \vee w_r(J) \leq w_l(J) \vee w_r(J) = 1_J$, so $w_r(J) \leq 1_I < 1_J$. Combining this with $w_r(J) \prec 1_J$, we obtain that $w_r(J) = 1_I$. Hence, $w_l(I) \leq w_l(J) \wedge 1_I = w_l(J) \wedge w_r(J) = 0_J$ and $0_J \leq x = w_l(I)$ give that $0_J = w_l(I)$, contradicting the assumptions of the lemma. Thus this case is excluded again.

Case 3. Assume that $x = w_l(I)$ and $w_r(J) \notin I$. Again, we know that $w_l(I) < w_l(J)$. Applying Lemma 5.1 to $0_J < w_r(J)$, we obtain a $y \in \mathrm{UBnd}(I) = \{w_l(I), w_r(I), 1_I\}$ such that $0_J \leq y \leq w_r(J)$. If we had that $y \in \{w_l(I), 1_I\}$, then $w_l(I) \leq y < w_r(J)$ together with $w_l(I) < w_l(J)$ would give that $w_l(I) \leq w_l(J) \wedge w_r(J) = 0_J \in I$, implying $0_J \in \{w_l(I), 1_I\}$, a contradiction. Hence, $y = w_r(I)$, and we have that $w_r(I) \leq w_r(J)$. By the definition of x and y, we know that $w_l(I) \in [0_J, w_l(J)]$ and $w_r(I) \in [0_J, w_r(J)]$. Hence, $0_J \leq w_l(I) \wedge w_r(I) \leq w_l(J) \wedge w_r(J) = 0_J$, that is, $0_J = w_l(I) \wedge w_r(I) = 0_I$. This and $1_J = w_l(J) \vee w_r(J) \geq w_l(I) \vee w_r(I) = 1_I$ yields that $I \subseteq J$, as desired.

Case 4. Assume that $x = 1_I$. Applying Lemma 5.1 to $0_J < w_r(J)$ again, we obtain a $y \in \mathrm{UBnd}(I) = \{w_l(I), w_r(I), 1_I\}$ such that $0_J \leq y \leq w_r(J)$. The possibility $y \in \{w_l(I), w_r(I)\}$ belongs, apart from notation and left-right symmetry, to the scope of the previous three cases. Hence, we can assume that $y = 1_I$. However, then $0_J \leq x \wedge y \leq w_l(J) \wedge w_r(J) = 0_J$ implies that $0_J = x \wedge y = 1_I \wedge 1_I = 1_I$, contradicting the assumptions of the lemma. So this case is excluded. ☐

Lemma 9.2. *Let I and J be maximal patches of a* planar *semimodular lattice L. If they have the same top, then they coincide. Moreover, 0_I is the intersection of all lower covers of 1_I.*

Proof. Let us fix a planar diagram D of L, and keep Lemma 4.9(ii) in mind. Alternatively, no matter how D is fixed since the concept of a (maximal) patch interval does not depend on D; this fact is due to (ii) or (iii) of Lemma 8.1, which clearly do not depend on D. With respect to D, let a and b be the leftmost and the rightmost lower covers of $1_I = 1_J$, respectively, and let $u := a \wedge b$. Then a and b are the leftmost coatom and the rightmost coatom of $K := [u, 1_I]$. By the (iv) $\Leftrightarrow$ (v) part of Lemma 8.1, we conclude that $K \in \mathrm{Patch}(L)$. Since $w_l(I), w_r(I), w_l(J), w_r(J) \in K$ by Lemma 5.4, $0_I = w_l(I) \wedge w_r(I)$ and 0_J also belong to K. Hence, $I, J \subseteq K$. Therefore, $I, J \in \mathrm{MaxPatch}(L)$ yields that $I = K = J$. We have also obtained that $0_I = 0_K = a \wedge b$. In virtue of Lemma 5.4, this proves the second part. ☐

Lemma 9.3. *Let I and J be maximal patches of a slim semimodular lattice L. Then either $I = J$, or I and J are disjoint, or $I \cap J$ is a chain.*

Proof. Let $D \in \mathrm{Dgr}(L)$ be fixed. Let $a, b \in I \cap J$ such that $a \parallel b$; we have to show that $I = J$. By Lemma 9.2, this is clear if $1_I = 1_J$. Suppose, by way of contradiction, that $1_I \neq 1_J$. Then, say, $1_I \ngeq 1_J$. Lemma 5.1, applied to I and $a, b \leq 1_J$, yields elements $a', b' \in \{w_l(I), w_r(I), 1_I\}$ such that $a \leq a' < 1_J$ and $b \leq b' < 1_J$.

If $1_I \in \{a', b'\}$, then $1_I \leq 1_J$. Otherwise, if we had $a' = b' \in \{w_l(I), w_r(I)\}$, then a and b would belong to the same chain (in I) by (2.11) and (2.12), which would contradict $a \parallel b$. Hence, $\{a', b'\} = \{w_l(I), w_r(I)\}$, which gives that $1_I = w_l(I) \vee w_r(I) = a' \vee b' \leq 1_J$. Hence, in all cases, $1_I \leq 1_J$. So $1_I < 1_J$ since they are distinct. Therefore, the convexity of J, $a \in J$, and $a \leq 1_I < 1_J$ yield that $1_I \in J - \{1_J\}$.

Assume first that $1_I \in \mathrm{Bnd}(J)$; then $1_I \in \mathrm{Bnd}(J) - \{1_J\} = \mathrm{LBnd}(J)$. Let, say, $1_I \in \mathrm{C_{ll}}(J)$. Then $a \parallel b$ belong to the same chain $\mathrm{C_{ll}}(J)$ of J by (2.11) and (2.12), a contradiction. Hence, 1_I is in the interior of J, whence its lower covers, $w_l(I)$ and $w_r(I)$, belong to J by (2.6) and (2.8). Consequently, $0_I = w_l(I) \wedge w_r(I) \in J$. Hence, $0_J \leq 0_I < 1_I < 1_J \in J$ yields that $I \subset J$, contradicting $I, J \in \mathrm{MaxPatch}(L)$. □

Lemma 9.4. *Let L be a slim semimodular lattice with a fixed $D \in \mathrm{Dgr}(L)$, and let $I, J \in \mathrm{MaxPatch}(L)$ such that $|I \cap J| = 1$. Then, up to I-J and left-right symmetries, either $I \cap J = \{w_r(I, D)\} = \{w_l(J, D)\}$, or $I \cap J = \{1_I\} = \{0_J\}$.*

The direct square $\mathbf{3}^2$ of the three-element chain shows that both cases can occur.

Proof of Lemma 9.4. Let x denote the unique element of $I \cap J$. There are several cases to consider.

Case 1. Assume that $x \in \{0_I, 1_I, 0_J, 1_J\}$. Firstly, let $x \in \{1_I, 1_J\}$, say, $x = 1_I$. Since I contains all lower covers of x by Lemma 9.2 but none of these lower covers are in J since $|I \cap J| = 1$, we conclude that $x = 0_J$, as desired. Secondly, let $x \in \{0_I, 0_J\}$, say, $x = 0_I$. By (2.1) and the definition of a patch lattice, x has exactly two covers in L, and both covers of x belong to I. Since none of these covers can belong to J by $|I \cap J| = 1$, we obtain that $x = 1_J$, as desired.

Case 2. To get a contradiction, suppose that $x \in \mathrm{int}(I) \cup \mathrm{int}(J)$. Say, $x \in \mathrm{int}(J)$. Then, by (2.6) and (2.8), all upper covers of x belong to J. But none of them can belong to the singleton set $I \cap J$, whence we obtain $x = 1_I$. By the previous case, this implies that $0_J = x \in \mathrm{int}(J)$, which is a contradiction.

Case 3. Suppose, for a contradiction, that $x \in \mathrm{LBnd}(J) - \{0_J, w_l(J, D), w_r(J, D)\}$, or the same holds for I. Then x has exactly two covers , x_1 and x_2, in J by (2.1) and (2.14). Since $\{x_1, x_2\} \cap I = \varnothing$ by $|I \cap J| = 1$, (2.1) and the convexity of I imply that $x = 1_I$. Hence, the first case we considered gives that $x = 0_J$, a contradiction.

Case 4. Assume that $x \in \{w_l(I), w_r(I), w_l(J), w_r(J)\}$, where $w_l(I)$ stands for $w_l(I, D)$, etc. We can also assume that $x \in \{w_l(I), w_r(I)\} \cap \{w_l(J), w_r(J)\}$ since otherwise the situations belongs to the scope of one of the previous cases. To complete the proof, we have to exclude that $x = w_l(I) = w_l(J)$ or $x = w_r(I) = w_r(J)$. To get a contradiction, suppose that, say, $x = w_l(I) = w_l(J)$. We know from Lemma 9.2 that 1_I is distinct from 1_J. Clearly, both of them cover x, whence they are the only covers of x by (2.1). Let, say, 1_I be on the left of 1_J. It follows from semimodularity that $S = \{x, 1_I, 1_J, 1_I \vee 1_J\}$ is a 4-cell with $\mathrm{C_l}(S) = \{x, 1_I, 1_I \vee 1_J\}$ and $\mathrm{C_r}(S) = \{x, 1_J, 1_I \vee 1_J\}$. Let C_0 and C_1 be maximal chains of $\downarrow x$ and $\uparrow(1_I \vee 1_J)$, respectively. Let $W = C_0 \cup \mathrm{C_l}(S) \cup C_1$ and $E = C_0 \cup \mathrm{C_r}(S) \cup C_1$. Since $w_r(I)$ is strictly on the right of $w_l(I) = x \in E$ and $h(w_r(I)) = h(w_l(I))$, we conclude from the left-right dual of (5.1) that $w_r(I)$ is strictly on the right of E. Using $1_J \in E$ and $h(1_J) = h(x) + 1 = h(1_I)$ similarly, we obtain from (5.1) that 1_I is strictly on the left of E. Thus (2.5) applies to T and $w_r(I) \prec 1_I$, and we obtain a contradiction. □

Lemma 9.5. *For a slim semimodular lattice L with a fixed $D \in \mathrm{Dgr}(L)$, let $I, J \in \mathrm{MaxPatch}(L)$ such that at least one of the following two conditions holds:*

(i) $|I \cap J| \geq 3$;
(ii) $J \cap \mathrm{int}(I, D)$ *is nonempty, or* $I \cap \mathrm{int}(J, D)$ *is nonempty.*

Then $I = J$.

Proof. To prove part (i) by way of contradiction, we suppose that $I \neq J$ but $|I \cap J| \geq 3$. We have that $I \parallel J$ since they are maximal patches. Let x be the least element of $I \cap J$. Since $I \cap J$ is a chain by Lemma 9.3 and $|I \cap J| \geq 3$, $x \notin \mathrm{UBnd}(I) = \mathrm{UBnd}(I, D)$ and $x \notin \mathrm{UBnd}(J)$. It follows from Lemma 9.1 that $x \notin \{0_I, 0_J\}$. First we consider the case when x is meet-reducible. Then, by (2.1), x has exactly two covers. Both of these covers belongs to I, either since $x \in \mathrm{Bnd}(I) - \mathrm{UBnd}(I)$ and (2.14) applies, or since $x \in \mathrm{int}(I)$ and (2.8) together with (2.6) says so. By the same reason, both covers of x belongs to J. But this is impossible since $I \cap J$ is a chain. Therefore, x is in $\mathrm{Mi}\, L$, whence also in $\mathrm{Mi}\, I \cap \mathrm{Mi}\, J$. This, $x \notin \mathrm{UBnd}(I)$, $x \notin \mathrm{UBnd}(J)$, and (2.14) yield that $x \in \mathrm{int}(I) \cap \mathrm{int}(J)$. By (2.6) and (2.8), all lower covers of x are in $I \cap J$. This contradicts the choice of x.

To prove part (ii) by way of contradiction, we suppose that $x \in \mathrm{int}(I) \cap J$ and $I \neq J$. By Lemma 9.4, $|I \cap J| \neq 1$. Hence, $|I \cap J| = 2$ by part (i). Since $I \cap J$ is a convex sublattice, it is of the form $\{x, y\}$, where either $x \prec y$, or $y \prec x$.

Assume first that $x \prec y$. If y belonged to $\mathrm{Ji}\, I$, which equals $\mathrm{LBnd}(I) - \{0_I\}$ by (2.10) and (2.13), then x would belong to $\mathrm{LBnd}(I)$ by (2.12), which would contradict $x \in \mathrm{int}(I)$. Hence, y is join-reducible in I and $y \in \mathrm{int}(I) \cup \{1_I\}$. Consequently, y has at least two lower covers in I. All lower covers (taken in L) of y belong to I either since $y = 1_I$ and Lemma 9.2 applies, or since $y \in \mathrm{int}(I)$ and (2.8) together with (2.6) applies. Since $|I \cap J| = 2$, y has only one lower cover (namely, x) in J. That is, $y \in \mathrm{Ji}\, J = \mathrm{LBnd}(J) - \{0_J\}$ by (2.10) and (2.13). Hence, $x \in \mathrm{Bnd}(J) - \mathrm{UBnd}(J)$ by (2.12), and x has exactly two upper covers in J by (2.14) and (2.1). Both of these upper covers belong also to I by (2.6) and (2.8) since x is in the interior of I. Therefore, $I \cap J$ has at least three distinct elements, x and its upper covers, which contradicts part (i) of the present lemma.

Secondly, we assume that $y \prec x$. All lower covers of x belong to I by (2.6) and (2.8). Hence, y is the only lower cover of x in J since otherwise $|I \cap J| \geq 3$ would contradict part (i) of the present lemma. Consequently, $x \in \mathrm{Ji}\, J = \mathrm{LBnd}(J) - \{0_J\}$ by (2.10) and (2.13). Hence, $y \in \mathrm{Bnd}(J) - \mathrm{UBnd}(J)$ by (2.12). Moreover, y has exactly two upper covers in J (and also in L) by (2.14) combined with (2.1). These upper covers of y are x and, say, x'. Since $x \in \mathrm{int}(I)$, either $y \in \mathrm{int}(I)$, or $y \in \mathrm{Bnd}(I) - \mathrm{UBnd}(I)$. In both cases, either by (2.6) and (2.8), or by (2.14) combined with (2.1), $x, x' \in I$. Hence, x, x' and y are three distinct elements of $I \cap J$, which contradicts part (i) again. □

10. Proving the main results and their corollaries

Proof of Proposition 2.4. Part (i) is [6, Lemma 22]. To prove part (ii), observe that if we add forks to a fixed diagram, then the left and the right weak corners, and also the principal filters they determine, do not change. Hence, there is a $D \in \mathrm{Dgr}(L)$ such that $G \cong \uparrow w_l(L,D) \times \uparrow w_r(L,D)$. Therefore, part (ii) follows from Lemma 4.9. Part (iii) is included in (the last sentence of) [6, Theorem 11]. Finally, the existence in part (iv) follows from Proposition 2.1 and Lemma 6.1(ii), while Lemma 4.1 yields the uniqueness. □

Proof of Theorem 3.6. First we deal with the particular case when L is a glued sum indecomposable *slim* semimodular lattice. Fix a planar diagram D of L. Since $S \subseteq [0_S, 1_S] \in \mathrm{Patch}(L)$ holds for all covering squares S of L, we conclude that 3.1(i) holds in MaxPatch(L). So does 3.1(ii) by Lemma 9.3. To show 3.1(iii), assume that $(I,J) \in \mathcal{E}\big(\mathrm{MaxPatch}(L)\big)$. Then $1 \le |I \cap J| \le 2$ by Lemma 9.5. Since 3.1(iii) clearly holds by Lemma 9.4 if $|I \cap J| = 1$, we assume that $|I \cap J| = 2$. Then $I \cap J$ is of the form $\{x \prec y\}$ since it is a convex sublattice. Lemma (5.5) or Lemma 9.5 yields that $x, y \in \mathrm{Bnd}(I,D) \cap \mathrm{Bnd}(J,D)$.

Suppose, for a contradiction, that $y \notin \{1_I, 1_J\}$. Then $y \in \mathrm{LBnd}(I,D) \cap \mathrm{LBnd}(J,D)$ and x belongs to both $\mathrm{Bnd}(I,D) - \mathrm{UBnd}(I,D)$ and $\mathrm{Bnd}(J,D) - \mathrm{UBnd}(J,D)$. Hence, by (2.14), x has a cover $y_1 \in I - \{y\}$, and it also has a cover $y_2 \in J - \{y\}$. We have that $y_1 \in I - J$ and $y_2 \in J - I$ since y is the largest element of $I \cap J$. Hence, y, y_1 and y_2 are three distinct covers of x, which contradicts (2.1).

Therefore, up to the I-J symmetry, we can assume that $y = 1_J$. This, $x \prec y$, and $\{x,y\} \subseteq \mathrm{Bnd}(I,D) \cap \mathrm{Bnd}(J,D)$ imply that $I \cap J = \{x,y\} \subseteq \mathrm{UBnd}(J,D)$. If we had $y = 1_I$, then Lemma 9.2 would yield that $I = J$, contradicting $(I,J) \in \mathcal{E}\big(\mathrm{MaxPatch}(L)\big)$. Hence, taking $\{x,y\} \subseteq \mathrm{Bnd}(I,D) \cap \mathrm{Bnd}(J,D)$ into account, y belongs to $\mathrm{Bnd}(I,D) - \{1_I\} = \mathrm{LBnd}(I,D)$, which gives that $I \cap J = \{x,y\} \subseteq \mathrm{LBnd}(I,D)$. This proves that MaxPatch(L) satisfies 3.1(iii). Thus Theorem 3.6 holds for the slim case.

Next, we drop the assumption that L is slim. Let L' be the full slimming sublattice of L with respect to a fixed planar diagram D. By Lemma 6.1(iv), L' is a glued sum indecomposable slim semimodular lattice. If we consider the intervals I as pairs of elements $(0_I, 1_I)$, then MaxPatch(L) and MaxPatch(L') become the same by Lemma 6.1(vii). Hence, the already proven slim case of the theorem together with Lemma 6.1(viii) completes the proof. □

Proof of Theorem 3.4. We already know from Lemma 8.1 that (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) ⇔ (vi). Moreover, (iv) ⇒ (vii) follows from Theorem 2.4. Hence, it suffices to show that (vii) ⇒ (v), (v) ⇒ (i) and (i) ⇒ (iii).

Assume that (vii) holds. Let F be a diagram of the four-element rectangular lattice S from which a diagram D of L is obtained first by adding forks, and then by adding eyes. Then $w_l(S,F)$ is the leftmost coatom of F, $w_r(S,F)$ is its rightmost coatom, and their intersection is 0. They remain the leftmost and the rightmost coatoms of the actual diagram, respectively, if we add forks and eyes. Furthermore, the least element of the lattice does not change. Hence, $w_l(S,F)$ and $w_r(S,F)$ will become the leftmost coatom and the rightmost coatom of D, and $w_l(S,F) \wedge w_r(S,F) = 0_S = 0_L$. This shows that (vii) ⇒ (v) holds.

Assume that (v) holds. Then there exists a diagram $D \in \mathrm{Dgr}(L)$ such that for the (unique) coatoms $a \in \mathrm{C_l}(L, D)$ and $b \in \mathrm{C_r}(L, D)$, we have that $a \wedge b = 0$. Notice that a and b are the leftmost coatom and the rightmost coatom with respect to D, respectively. Let $\boldsymbol{\mathcal{H}}$ be a patchwork system for L. By Proposition 3.2, it is a patchwork system for the diagram D. Let c be the right neighbor of a in D; it is a coatom. Let $S = [a \wedge c, a \vee c = 1]$; it is a 4-cell of D by [6, Lemma 13]. Hence, it is a covering square, and it is a subset of some $I \in \boldsymbol{\mathcal{H}}$ by 3.1(i). Therefore, there is an $I \in \boldsymbol{\mathcal{H}}$ such that $a \in I$ and $1_I = 1_L$. Similarly, there is a $J \in \boldsymbol{\mathcal{H}}$ such that $b \in J$ and $1_J = 1_L$. To get a contradiction, suppose that $I \neq J$. Then $(I, J) \in \mathcal{E}\big(\mathrm{MaxPatch}(L)\big)$ since $1_L \in I \cap J$ shows that $I \cap J$ is nonempty. Hence, 3.1(iii) yields that $1 \in I \cap J \subseteq \mathrm{LBnd}(I, D) \cup \mathrm{LBnd}(J, D)$, which is a contradiction since 1_K is never on the lower boundary of a rectangular interval K. Hence, $I = J$. Since $0_L = a \wedge b \in I$ and $1_L \in I$, we obtain that $\boldsymbol{\mathcal{H}} = \{I\}$ by Remark 3.3(i). This proves the implication (v) $\Rightarrow$ (i).

Next, to show that (i) implies (iii), assume that (iii) fails. We have to show that (i) also fails. We can assume that L is glued sum indecomposable since otherwise (i) fails by definition. Fix a diagram $D \in \mathrm{Dgr}(L)$. By the assumption, there are a proper ideal I and a proper filter F such that $L = I \cup F$, and $C := I \cap F$ is a chain. We assume that I and F are chosen so that $|C|$ is minimal. By (2.6), I and F are planar lattices, and they are clearly semimodular. Let $C = [a, b] = [0_F, 1_I]$. We conclude that $a < b$ since otherwise $a = b$ would be comparable with all elements of L, contradicting the glued sum indecomposability of L. Since I and F are proper subsets, $I \neq C \neq F$ and $|I|, |F| \geq 3$. To get a contradiction, suppose that there is an $x \in I - \{0, b\}$ such that $I = {\downarrow}x \cup {\uparrow}x$. Then $a \in {\uparrow}x$ would imply that $F \subseteq {\uparrow}x$, whence $L = {\uparrow}x \cup {\downarrow}x$ would contradict the glued sum indecomposability of L. Therefore $a \in {\downarrow}x - {\uparrow}x$, that is, $a < x < b$. Since $I = {\downarrow}x \cup {\uparrow}x = {\downarrow}x \cup [x, b]$ and $[x, b] \subseteq F$, we can replace I by ${\downarrow}x$ in the original decomposition. Then $C = [a, b]$ is replaced by $[a, x]$, which contradicts the minimality of $|C|$. Hence, there is no x with $I = {\downarrow}x \cup {\uparrow}x$. This together with $|I| \geq 3$ implies that I is glued sum indecomposable. We have not used semimodularity, so F is also glued sum indecomposable by duality.

Let a_1 and b_1 be the unique elements of $C = [a, b]$ such that $a \prec a_1$ and $b_1 \prec b$. Since F is glued sum indecomposable, $a = 0_F$ has a cover a_2 distinct from a_1. The glued sum indecomposability of I yields that $b = 1_F$ has a lower cover b_2 distinct from b_1. Since $C = I \cap F$ is a chain containing a_1, we obtain that $a_2 \notin C$. But $a_2 \in F$, whence $a_2 \in F - I$. The dual consideration shows that $b_2 \in I - F$. If a belonged to $\mathrm{int}(I, D)$, then (2.8) together with (2.6) would imply that $a_2 \in I$, a contradiction. Hence, $a \in \mathrm{Bnd}(I, D)$. Dually, we obtain that $b \in \mathrm{Bnd}(F, D)$. Without loss of generality, we can assume that $a \in \mathrm{C_l}(I, D)$. Since $\mathrm{C_l}(I, D)$ is a maximal chain in I, we have that $\{x \in \mathrm{C_l}(I, D) : a \leq x\}$ is a maximal chain in $[a, 1_I] = [a, b] = C$. But C is itself a chain, whence

$$C = \{x \in \mathrm{C_l}(I, D) : a \leq x\} \subseteq \mathrm{C_l}(I, D). \tag{10.1}$$

Since now we cannot assume that $b \in \mathrm{C_r}(F, D)$, lattice duality yields only that $C \subseteq \mathrm{C_l}(F, D)$ or $C \subseteq \mathrm{C_r}(F, D)$. However, we claim that

$$C \subseteq \mathrm{C_l}(I, D) \cap \mathrm{C_r}(F, D). \tag{10.2}$$

In view of the previous observation, it suffices to exclude that $C \subseteq \mathrm{C_l}(F, D)$. Suppose, for a contradiction, that $C \subseteq \mathrm{C_l}(F, D)$, and keep (10.1) in mind. Let

$x \in \mathrm{C_l}(L, D)$ be the unique element with $h(x) = h(b)$. We obtain from (5.3) that x is on the left of b. On the other hand, $x \in I$ or $x \in F$, and $b \in C \subseteq \mathrm{C_l}(I, D) \cap \mathrm{C_l}(F, D)$. Hence, (5.3) (applied to I or F) yields that b is on the left of x. Using (5.2), we conclude that $b = x \in \mathrm{C_l}(L, D)$. Hence, we obtain from (5.5) that $\mathrm{C_l}(I, D) \subseteq \mathrm{C_l}(L, D)$. Let $E_1 := E \cap \uparrow b$. It is a maximal chain in $\uparrow b$. Therefore, since $\mathrm{C_l}(I, D)$ is a maximal chain in $\downarrow b = I$, we obtain that $E := \mathrm{C_l}(I, D) \cup E_1$ equals $\mathrm{C_l}(L, D)$. Notice that E is a maximal chain in L. Let $W := \mathrm{C_r}(I) \cup E_1$; it is also a maximal chain of L.

Let $y \in \mathrm{C_r}(I, D)$ denote the unique element with $h(y) = h(a)$. Since $a \in \mathrm{C_l}(I) - \{0_I, 1_I\}$ and I is glued sum indecomposable, $a \notin \mathrm{C_r}(I, D)$. Hence, (5.3) yields that a is strictly on the left of y. So we obtain from (5.1) that

$$a \text{ is strictly on the left of } W. \tag{10.3}$$

Trivially (or it follows from (5.1) and (5.3)), we have that a_2 is on the right of $E = \mathrm{C_l}(L, D)$. If a_2 was on the left of W, then (5.6), applied for $(0, b)$ instead of (u, v), would imply that $a_2 \in I$, which contradicts $a_2 \in F - I$. Therefore, a_2 is strictly on the right of W. This together with (10.3) and $a \prec a_2$ contradicts (2.5). Thus (10.2) is proved.

The restriction of D to I and F will be denoted by D_I and D_F, respectively. By Theorem 3.6 and Proposition 3.2, MaxPatch(I) and MaxPatch(F) are patchwork systems for D_I and D_F, respectively. Let $\boldsymbol{\mathcal{H}} := \mathrm{MaxPatch}(I) \cup \mathrm{MaxPatch}(F)$; we claim that it is a patchwork system for D.

To get a contradiction, suppose that there is a covering square $S = \{u \wedge v, u, v, u \vee v\}$ such that $S \not\subseteq I$ and $S \not\subseteq F$. Then, say, $u \in F - I$ and $v \in I - F$. Extend C to a maximal chain $C^\bullet$ of L. Clearly, $u \vee v \in F - I$. Since $h(a) = h(0_F) \le h(u) = h(v) \le h(1_I) = h(b)$, the chain $C = [a, b]$ has a unique element x such that $h(x) = h(u) = h(v)$. Furthermore, $v \notin F$ gives that $v < b$, whence $h(v) + 1 \le h(b)$. Consequently, $h(u \vee v) = h(v) + 1 \le h(b)$, and there is an element $y \in C$ with $h(y) = h(u \vee v)$.

Using that $v \in I$ and $x \in C \subseteq \mathrm{C_l}(I)$, (5.3) yields that v is on the right of $x \in C^\bullet$. This fact, $v \ne x$, and (5.1) yield that v is strictly on the right of $C^\bullet$. Since $u \vee v \ne y \in C \subseteq \mathrm{C_r}(F, D)$, the left-right dual of (5.3) yields that $u \vee v$ is strictly on the left of y. Hence, (5.1) yields that $u \vee v$ is strictly on the left of $C^\bullet$. Thus $v \prec u \vee v$ contradicts (2.5). This proves that each covering square is either a subset of I or a subset of F. This implies that 3.1(i) holds for $\boldsymbol{\mathcal{H}}$.

Next, assume that $(J, K) \in \mathcal{E}(\boldsymbol{\mathcal{H}})$. If

$$J, K \in \mathrm{MaxPatch}(I) \text{ or } J, K \in \mathrm{MaxPatch}(F),$$

then 3.1(ii) and 3.1(iii) clearly hold for (J, K). Hence, we can also assume that $J \in \mathrm{MaxPatch}(I)$ and $K \in \mathrm{MaxPatch}(F)$. Since $J \cap K \subseteq I \cap F = C$ and C is a chain, 3.1(ii) holds for (J, K).

Using that $\mathrm{int}(J, D) \subseteq \mathrm{int}(I, D)$ by (2.9) and $C \subseteq \mathrm{Bnd}(I, D)$ by (10.2), we obtain that $\mathrm{int}(J, D) \cap C = \varnothing$. Hence, $J \cap C = \big(\mathrm{int}(J, D) \cap C\big) \cup \big(\mathrm{Bnd}(J, D) \cap C\big) = \mathrm{Bnd}(J, D) \cap C \subseteq \mathrm{Bnd}(J, D)$. Assume for a contradiction that $J \cap C \not\subseteq \mathrm{Mi}\, J$. Then there is an $x \in J \cap C$ with at least two covers in J. All these covers belong to C since F is a filter. This is a contradiction since C is chain. Consequently, $J \cap C \subseteq \mathrm{Mi}\, J$. Combining this with $J \cap C \subseteq \mathrm{Bnd}(J, D)$ and (2.14), we obtain that $J \cap C \subseteq \mathrm{UBnd}(J, D)$. This together with $J \cap K \subseteq C$ yields that $J \cap K \subseteq J \cap K \cap C \subseteq J \cap C \subseteq \mathrm{UBnd}(J, D)$. Dualizing the above argument (in particular, replacing (2.14) by (2.15)) we obtain that $J \cap K \subseteq J \cap K \cap C \subseteq K \cap C \subseteq \mathrm{LBnd}(K, D)$. Hence, 3.1(iii) (with D) holds. Therefore, $\boldsymbol{\mathcal{H}}$ is a patchwork system for D. Since $|\boldsymbol{\mathcal{H}}| = |\mathrm{MaxPatch}(I) \cup \mathrm{MaxPatch}(F)| = |\mathrm{MaxPatch}(I)| + |\mathrm{MaxPatch}(F)| \geq 1 + 1 = 2$, we conclude that (i) fails. This completes the proof of the implication (i) $\Rightarrow$ (iii). □

Proof of Corollary 3.5. Since gluing preserves semimodularity (see, for instance, C. Herrmann [16] or [3, Lemma 6.1]), the statement follows from (iii) $\Leftrightarrow$ (iv) of Theorem 3.4. □

Proof of Corollary 3.8. As mentioned before Corollary 3.7, only the second part needs a proof. Since non-chain intervals of length 2 are atomistic, all we have to show is that if I is an interval of length greater than 2, then I is not atomistic. Otherwise, let $\{a_1, \dots, a_n\}$ be a maximal independent system of atoms of I. Then n is the length of I and these atoms generate a Boolean sublattice B of length n, see G. Grätzer [11, Theorem IV.2.5] or [12, Theorem 381]. This is a contradiction since B is not planar for $n \geq 3$. □

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