A COVER-PRESERVING EMBEDDING OF SEMIMODULAR LATTICES INTO GEOMETRIC LATTICES

GÁBOR CZÉDLI AND E. TAMÁS SCHMIDT

Abstract. Extending former results by G. Grätzer and E. W. Kiss (1986) and M. Wild (1993) on *finite* (upper) semimodular lattices, we prove that each semimodular lattice L *of finite length* has a cover-preserving embedding into a geometric lattice G of the same length. The number of atoms of our G equals the number of join-irreducible elements of L.

1. Introduction

Semimodularity, which is the lattice theoretic counterpart of exchange property, is one of the most important links between combinatorics and lattice theory. A particular interest is deserved by geometric lattices, originally called matroids. It was shown in G. Grätzer and E. W. Kiss [5] that each finite (upper) semimodular lattice L has a cover-preserving embedding into a finite geometric lattice.

For a lattice K, the set of non-zero join-irreducible elements and the set of atoms of K will be denoted by $J(K)$ and $A(K)$, respectively. The length of L, that is $\sup\{n : L \text{ has an } (n+1)\text{-element chain}\}$, will be denoted by $\ell(L)$. Our aim is to give an easy-to-understand construction of a lattice $G(L)$ for each semimodular lattice L of finite length such that the following statement holds.

Theorem 1. *Let L be a semimodular lattice of finite length. Then $G = G(L)$ is a geometric lattice such that L is a cover-preserving sublattice of G, $|J(L)| = |A(G)|$, and $\ell(L) = \ell(G)$.*

For the sake of emphasis, the above formulation is a bit redundant. Indeed, $\ell(L) = \ell(G)$ implies that the sublattice L is a cover-preserving sublattice and, in addition, $\{0_G, 1_G\} \in L$. Theorem 1 trivially implies the following statement.

Corollary 2. *Semimodular lattices of finite length are characterized as cover-preserving sublattices of geometric lattices of finite length.*

Background, notation, and terminology. For general information on semimodular lattices the reader is referred to G. Grätzer [3] and M. Stern [8]. We use the terminology and notation of G. Grätzer [4]. The Glossary of Notation of [4] is available as a pdf file at

`http://mirror.ctan.org/info/examples/Math_into_LaTeX-4/notation.pdf`

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2. Construction

For the rest of the paper, let L be a fixed semimodular lattice of finite length $\ell(L) = h(1)$. Following the convention of, say, P. Crawley and R. P. Dilworth [1] or G. Grätzer [3] and [4], we assume that L is non-empty. Let $H(L)$ denote $J(L)\setminus A(L)$, the set of "high" join-irreducible elements. Insert a new element x' into L for each $x \in H(L)$ such that $x' \neq y'$ for $x \neq y$. Extend the original order by $0 \prec x' \prec x$ for every $x \in H(L)$; this way we obtain $P = (P; \leq)$. The construction of P is depicted in Figure 1; the black-filled elements stand for $J(L)$ while the grey-filled ones are the new elements.

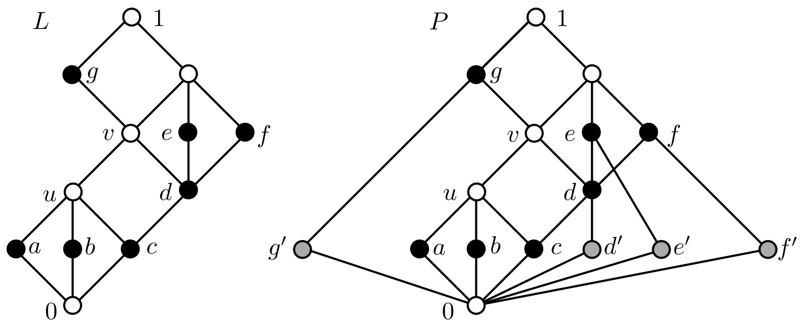

Figure 1. An example of L and the corresponding P

Although P is a lattice, it is not semimodular in general. Hence we consider P as a partial join-semilattice $P = (P; \vee_P)$. Loosely speaking, $\vee_P$ will be the largest extension of $\vee_L$ to P such that $P = (P; \vee_P)$ is a "semimodular partial join-semilattice". The exact definition of $\vee_P$ is the following.

- If $x, y \in P$ are comparable or $\{x, y\} \subseteq L$, then $x \vee_P y$ is defined, and it has the usual meaning.
- If $x, y \in P \setminus L$ and $x \neq y$, then $x \vee_P y$ is undefined.
- Suppose that $x \in L$, $y \in P \setminus L$, and $x \parallel y$. Then $y = z'$ for a unique $z \in H(L)$ and $x \vee_P y$ is defined iff $x \vee_L z$ covers x in L; if $x \vee_P y$ is defined, then it equals $x \vee_L z$, so it is the supremum of $\{x, y\}$.
- Suppose that $x \in P \setminus L$, $y \in L$, and $x \parallel y$. Then $x \vee_P y$ is defined iff $y \vee_P x$ is defined according to the previous case; if $x \vee_P y$ is defined, then $x \vee_P y = y \vee_P x$.

For example, in Figure 1, $u \vee_P d' = v$, $c \vee_P d' = d$, and $g \vee_P f' = 1$, while $b \vee_P d'$ and $d' \vee_P e'$ are undefined.

Let us call a non-empty subset I of P *an ideal of* P iff

- I is an order-ideal, that is, $x \in I$, $y \in P$ and $y \leq x$ imply $y \in I$, and
- I is closed with respect to $\vee_P$, that is, if $x, y \in I$ and $x \vee_P y$ is defined, then $x \vee_P y \in I$.

Since the intersection of ideals is an ideal again, the ideals of P form a complete lattice $\mathcal{I}(P) = (\mathcal{I}(P), \subseteq)$.

Let I be an ideal of P. Then the largest element of $I \cap L$, that is $\bigvee(I \cap L)$, is called the *trunk* of I. It is denoted by $\operatorname{trunk}(L)$. The set

$$\operatorname{branch}(I) := I \setminus {\downarrow}\operatorname{trunk}(I) = \{x \in I : x \not\leq \operatorname{trunk}(I)\}$$

is called the *branch* of I. Clearly, $I = {\downarrow}\operatorname{trunk}(I) \cup \operatorname{branch}(I)$. Hence the trunk and the branch determine the ideal. Let

$$\langle a; S\rangle_{\mathrm{id}}$$

denote the ideal whose trunk and branch are a and S, respectively. For example, for P in Figure 1, $\langle e; \{g'\}\rangle_{\mathrm{id}} = \{0, c, d', e', d, e, g'\}$ is an ideal. Notice that the trunk and the branch cannot be independently chosen. For example, if d is the trunk of an ideal I, then neither $\{d'\}$ nor $\{e', g'\}$ can be the branch of I. Hence none of the notations $\langle d; \{d'\}\rangle_{\mathrm{id}}$ and $\langle d; \{e', g'\}\rangle_{\mathrm{id}}$ is allowed in case of Figure 1.

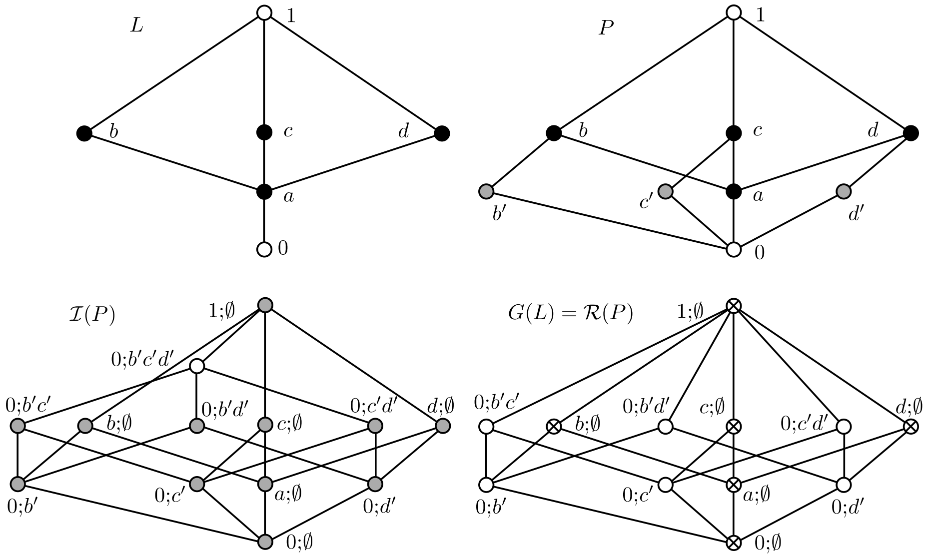

FIGURE 2. An example of $\mathcal{I}(P)$ and $G(L) = \mathcal{R}(P)$

For an ideal $I = \langle a; S\rangle_{\mathrm{id}}$ of P, we define the *rank of* I as follows:

$$r(I) := h(\operatorname{trunk}(I)) + |\operatorname{branch}(I)| = h(a) + |S|.$$

In general, $r(I)$ is a cardinal number. If $r(I)$ is finite, then I is said to be of *finite rank*. We say that $I \in \mathcal{I}(P)$ is a *trimmed ideal* iff

$$\text{for all } J \in \mathcal{I}(P),\ I < J \text{ implies } r(I) < r(J).$$

For example, $\langle 0; \emptyset\rangle_{\mathrm{id}} = 0_{\mathcal{I}(P)}$ and $\langle 1; \emptyset\rangle_{\mathrm{id}} = 1_{\mathcal{I}(P)}$ are always trimmed ideals. Since $1_{\mathcal{I}(P)}$ is of finite rank, every trimmed ideal is of finite rank. We will show that the set $\mathcal{R}(P)$ of all trimmed ideals of $\mathcal{I}(P)$ forms a complete meet-subsemilattice of $\mathcal{I}(P)$.

Hence $\mathcal{R}(P) = (\mathcal{R}(P), \leq)$ is a lattice. It is the geometric lattice $G = G(L)$ we intended to construct. We will prove that $A(\mathcal{R}(P)) = \{\langle a; \emptyset\rangle_{\mathrm{id}} : a \in A(L)\} \cup \{\langle 0; \{b'\}\rangle_{\mathrm{id}} : b \in H(L)\}$ and $\varphi\colon x \mapsto \langle x; \emptyset\rangle_{\mathrm{id}}$ is a cover-preserving $L \to \mathcal{R}(P)$ lattice embedding.

The construction of $G(L) = \mathcal{R}(P)$ is illustrated in Figure 2. To save space in the figure, the ideals $\langle u; \{x'_1, x'_2, \ldots, x'_n\}\rangle_{\rm id}$ and $\langle u; \emptyset\rangle_{\rm id}$ are denoted by $u;x'_1x'_2\ldots x'_n$ and $u;\emptyset$, respectively. The trimmed ideals of $\mathcal{I}(P)$ are represented by grey-filled circles, while the cross-filled circles of $\mathcal{R}(P)$ show how L is embedded in $\mathcal{R}(P)$.

3. Proofs and auxiliary statements

Our proof uses a lot of ideas of G. Grätzer and E. W. Kiss [5]; the influence of [5] will be detailed in the last section.

Let us call an order-ideal J of P a *semi-ideal* if $J \cap L$ is closed with respect to join, that is, if $J \cap L$ is a lattice ideal of L. Then $\mathrm{trunk}(J) = \bigvee(L \cap J)$, which belongs to J, and $\mathrm{branch}(J) = J \setminus {\downarrow}\mathrm{trunk}(J)$ are meaningful even for semi-ideals, and we still have $J = {\downarrow}\mathrm{trunk}(J) \cup \mathrm{branch}(J)$. Let the notation $\langle a; S\rangle_{\rm si}$ stand for the semi-ideal with trunk a and branch S. Every ideal is a semi-ideal. For $a \in L$ and $S \subseteq P$, the notation $\langle a; S\rangle_{\rm si}$ is permitted, that is a and S are the trunk and the branch of the same semi-ideal, iff $S \subseteq (P \setminus L) \setminus {\downarrow}a$.

We can extend the definition of r to semi-ideals J in the natural way: $r(J) = h(\mathrm{trunk}(J)) + |\mathrm{branch}(J)|$. The least ideal including J, that is the *ideal generated by* J, will be denoted by J^*.

Lemma 3. *For any semi-ideal J of P, $r(J) \geq r(J^*)$.*

Proof. The semimodularity of L implies that, for any $x \in H(L)$ and $u, v \in L$,

$$\text{if } 0 \neq u \leq v \text{ and } u \vee_P x' \text{ is defined, then } v \vee_P x' \text{ is also defined.} \tag{1}$$

Hence, for $a \in L$ and $S \subseteq P$, $\langle a; S\rangle_{\rm si}$ is an ideal of P if and only if

$$S \subseteq (P \setminus L) \setminus {\downarrow}a, \text{ and } a = 0 \text{ or } a \vee_P x' \text{ is undefined for all } x' \in S. \tag{2}$$

Let $J = \langle a; S\rangle_{\rm si}$ be a semi-ideal, and let $\langle b; T\rangle_{\rm id}$ stand for J^*. Clearly, $a \leq b$. We prove the lemma by induction on $n = h(b) - h(a)$. We will assume that $a > 0$, for otherwise J is an ideal and $J^* = J$.

Let $n = 0$, that is, $a = b$. If J is not $\vee_P$-closed, then (1) yields an element $x' \in S$ such that $e = a \vee_P x'$ is defined. Since $e \in J^*$, we obtain $a < e \leq b$, which contradicts $a = b$. Hence J is $\vee_P$-closed, so the lemma follows from $J = J^*$.

Assume that $n > 0$. Since J is not $\vee_P$-closed, (1) implies the existence of an element $x' \in S$ such that $e = a \vee_P x'$ is defined. The definition of $\vee_P$ yields that $h(e) = h(a) + 1$, whence $h(b) - h(e) = n - 1$. Let $W = S \setminus {\downarrow}e$, and consider the semi-ideal $K = \langle e; W\rangle_{\rm si}$. Since $x' \in S \setminus W$ implies $|S| \geq 1 + |W|$, we conclude $r(J) = h(a) + |S| \geq h(a) + 1 + |W| = h(e) + |W| = r(K)$. The induction hypothesis gives $r(K) \geq r(K^*)$, so $r(J) \geq r(K^*)$. Finally, we conclude from $J \subset K \subseteq J^*$ that $J^* = K^*$. □

Lemma 4. *For any $I, J \in \mathcal{I}(P)$, we have*

$$r(I) + r(J) \geq r(I \vee J) + r(I \wedge J). \tag{3}$$

Proof. For a semi-ideal Y and $x \in H(L)$, let $w_Y(x') = 1$ if $x' \in \mathrm{branch}(Y)$, and let $w_Y(x') = 0$ otherwise. Notice that $w_Y(x') = 1$ iff $x' \in Y$ and $x \not\leq \mathrm{trunk}(Y)$. The following formula, in which $\sum$ denotes the sum of *cardinal numbers*, is obvious:

$$r(Y) = h(\mathrm{trunk}(Y)) + \sum_{x \in H(L)} w_Y(x'). \tag{4}$$

Let $I = \langle a; S\rangle_{\mathrm{id}}$ and $J = \langle b; T\rangle_{\mathrm{id}}$ be ideals of P. Then $\operatorname{trunk}(I \wedge J) = a \wedge b$. Consider $W = \big((I \cup J) \setminus L\big) \setminus {\downarrow}(a \vee b)$ and the semi-ideal $K = \langle a \vee b; W\rangle_{\mathrm{si}}$. Since $I \cup J \subseteq K \subseteq I \vee J$, we conclude that $K^* = I \vee J$. Therefore, by Lemma 3, formula (3) will clearly follow from

$$r(I) + r(J) \geq r(K) + r(I \wedge J). \tag{5}$$

The semimodularity of L yields that $h(a) + h(b) \geq h(a \vee b) + h(a \wedge b)$. So, by (4), formula (5) will follow from

$$w_I(x') + w_J(x') \geq w_K(x') + w_{I\wedge J}(x'), \quad \text{for any } x \in H(L). \tag{6}$$

Denoting $\big(w_I(x'), w_J(x'), w_K(x'), w_{I\wedge J}(x')\big)$ by $\vec{w}$, we prove (6) by excluding the following four "wrong" cases.

Case $\vec{w} = (1, 0, 1, 1)$: then $x' \in I \wedge J \subseteq J$ and $w_J(x') = 0$ imply $x \leq b \leq a \vee b$, which contradicts $w_K(x') = 1$.

Case $\vec{w} = (0, 1, 1, 1)$: excluded by symmetry.

Case $\vec{w} \in \{(0, 0, 0, 1), (0, 0, 1, 1)\}$: then $x' \in I \wedge J$ and $x \nleq a \wedge b$ yields that $x \nleq a$ or $x \nleq b$, and $x' \in I$ and $x' \in J$. This means that $w_I(x') = 1$ or $w_J(x') = 1$, a contradiction.

Case $\vec{w} \in \{(0, 0, 1, 0), (0, 0, 1, 1)\}$: then $x \nleq a \vee b$ and $x' \in W \subseteq I \cup J$. Hence we have either $x' \in I$ and $x \nleq a$, contradicting $w_I(x') = 0$, or $x' \in J$ and $x \nleq b$, contradicting $w_I(x') = 0$. $\square$

Lemma 5. *Any chain of $\mathcal{R}(P)$ is of length at most $\ell(L)$.*

Proof. Since $1_{\mathcal{R}(P)} = \langle 1; \emptyset\rangle_{\mathrm{id}}$ is of rank $\ell(L)$, the statement is evident. $\square$

Lemma 6. *$\mathcal{R}(P) = (\mathcal{R}(P), \subseteq)$ is a complete lattice, a complete meet-subsemilattice of $\mathcal{I}(P)$. Moreover, $\ell(\mathcal{R}(P)) = \ell(L)$.*

Proof. Assume that $I_1 = \langle a_1; S_1\rangle_{\mathrm{id}}$ and $I_2 = \langle a_2; S_2\rangle_{\mathrm{id}}$ belong $\mathcal{R}(P)$. We have to show that $I := I_1 \wedge I_2 = I_1 \cap I_2 = \langle a; S\rangle_{\mathrm{id}}$ is a trimmed ideal of $\mathcal{I}(P)$.

Suppose that $x' \in S$. Then $x \nleq a = a_1 \wedge a_2$, whence $x \nleq a_j$ for some $j \in \{1, 2\}$. Since $x' \in S \subseteq I \subseteq I_j$, we obtain $x' \in S_j$. Hence $S \subseteq S_1 \cup S_2$, whence $n := r(I)$ is finite.

By way of contradiction, we suppose that I is not trimmed. Then the set $M = \{X \in \mathcal{I}(P) : I < X \text{ and } r(X) \leq n\}$ is not empty. Observe that M satisfies the descending chain condition, that is, $X_0 > X_1 > X_2 > \cdots$ is impossible, if $X_i \in M$ for all $i \in \mathbf{N}_0$. Indeed, the branch of X_0 has at most $r(X_0) \leq n$ elements. So we can can keep the trunk and decreasing the branch only in at most n steps. Hence $\operatorname{trunk}(X_0) > \operatorname{trunk}(X_{n+1})$. Similarly, $r(X_{n+1}) \leq n$ implies $\operatorname{trunk}(X_{n+1}) > \operatorname{trunk}(X_{2(n+1)})$, and so on. This way we obtain an infinite decreasing sequence $\operatorname{trunk}(X_0) > \operatorname{trunk}(X_{n+1}) > \operatorname{trunk}(X_{2(n+1)}) > \operatorname{trunk}(X_{3(n+1)}) > \cdots$ in L, which is a contradiction.

Due to the descending chain condition, we can choose a minimal element J in M. Remember that $I < J$ and $r(J) \leq r(I) = n$. Let $K_j = I_j \wedge J$ for $j \in \{1, 2\}$. Clearly, $I \leq K_j \leq J$. Notice that $r(K_j) \geq r(J)$, because otherwise $r(K_j) < r(J) \leq r(I)$ would imply $I < K_j < J$, contradicting the minimality of J in M. Using $r(K_j) \geq r(J)$ and Lemma 4, we obtain

$$r(I_j) + r(J) \geq r(I_j \vee J) + r(K_j) \geq r(I_j \vee J) + r(J),$$

whence $r(I_j) \geq r(I_j \vee J)$. This excludes $I_j < I_j \vee J$, since I_j is trimmed. Hence $I_j = I_j \vee J$, that is, $J \leq I_j$, for $j \in \{1,2\}$. So, $J \leq I_1 \wedge I_2 = I$, which contradicts $J \in M$.

We have seen that $\mathcal{R}(P)$ is closed with respect to binary meets (intersections), whence Lemma 5 yields that $\mathcal{R}(P)$ is closed with respect to arbitrary meets. Hence $\mathcal{R}(P) = (\mathcal{R}(P), \subseteq)$ is a complete lattice.

Finally, Lemma 5 gives that $\ell(\mathcal{R}(P)) \leq \ell(L)$. The reverse inequality follows from the fact $\{\langle c; \emptyset\rangle_{\mathrm{id}} : c \in C\}$ is a $|C|$-element chain of $\mathcal{R}(P)$ for any chain C of L. $\square$

Lemma 7. *If $I, J \in \mathcal{I}(P)$ and $I \prec J$, then $r(J) \leq r(I) + 1$*

Proof. Let $I = \langle a; S\rangle_{\mathrm{id}}$ and $J = \langle b; T\rangle_{\mathrm{id}}$. We have to consider two cases.

First, assume that $a = b$. Then $S \subset T$. Fix an element t in $T \setminus S$. Let $U = S \cup \{t\}$ and consider the semi-ideal $K = \langle a; U\rangle_{\mathrm{si}}$. Since $I \subset K \subseteq J$ and $I \prec J$, we have $K^* = J$. (We notice but do not use that $K^* = K$.) Hence Lemma 3 yields that

$$r(J) = r(K^*) \leq r(K) = h(a) + |U| = h(a) + |S| + 1 = r(I) + 1.$$

The second case is $a < b$. Select an element $c \in L$ with $a \prec c \leq b$, let $U = S \setminus \downarrow c$, and consider the semi-ideal $K = \langle c; U\rangle_{\mathrm{si}}$. From $I \subset K \subseteq J$ we conclude $K^* = J$ again, whence

$$\begin{aligned} r(J) = r(K^*) \leq r(K) &= h(c) + |U| = h(a) + 1 + |U| \\ &\leq h(a) + 1 + |S| = r(I) + 1. \end{aligned}$$

$\square$

If there is a danger of confusion, the covering relation of $\mathcal{I}(P)$ and that of $\mathcal{R}(P)$ will be denoted by $\prec_{\mathcal{I}(\mathrm{P})}$ and $\prec_{\mathcal{R}(\mathrm{P})}$, respectively.

Lemma 8. *If $I \prec_{\mathcal{R}(\mathrm{P})} J$ for $I, J \in \mathcal{R}(P)$, then $r(J) = r(I) + 1$.*

Proof. Let $I \prec_{\mathcal{R}(\mathrm{P})} J$, and let $a = \mathrm{trunk}(I)$. Fix an element X of

$$M = \{Z \in \mathcal{I}(P) : I < Z \leq J\}$$

in the following way. If $\mathrm{trunk}(Z) = a$ for some $Z \in M$, then let $X = I \cup \{x'\}$ where $x' \in \mathrm{branch}(Z) \setminus \mathrm{branch}(I)$. If $\mathrm{trunk}(Z) > a$ for all $Z \in M$, then take a minimal element c in $\{\mathrm{trunk}(Z) : Z \in M\}$, and let $X = \bigcap\{Z \in M : \mathrm{trunk}(Z) = c\}$. Clearly, $I \prec_{\mathcal{I}(\mathrm{P})} X \leq J$ in both cases.

Since I is trimmed, we have $r(X) = r(I) + 1$ by Lemma 7.

Consider the set

$$F = \{Z \in \mathcal{I}(P) : X \leq Z \leq J \text{ and } r(Z) = r(I) + 1\}.$$

It is not empty, for $X \in F$. By way of contradiction, suppose $\langle c_0; U_0\rangle_{\mathrm{id}} < \langle c_1; U_1\rangle_{\mathrm{id}} < \langle c_2; U_2\rangle_{\mathrm{id}} < \cdots$ is an infinite ascending chain in F. Since $c_0 \leq c_1 \leq c_2 \leq \cdots$ and $\ell(L)$ is finite, there is a k such that $c_k = c_{k+1} = c_{k+2} = \cdots$. Then $U_k \subset U_{k+1} \subset U_{k+2} \subset \cdots$. This is a contradiction, for $|U_i| \leq r(\langle c_i; U_i\rangle_{\mathrm{id}}) = r(I) + 1$ for all i.

Since F satisfies the ascending chain condition, we can choose a maximal element K in F. Since $I < X \leq K \leq J$ and $I \prec_{\mathcal{R}(\mathrm{P})} J$, it suffices to show that K is a trimmed ideal. Indeed, this would imply $K = J$ and $r(J) = r(K) = r(I) + 1$.

Consider an arbitrary ideal $Y \in \mathcal{I}(P)$ with $K < Y$; we have to show that $r(K) < r(Y)$. From $I < K \leq J \wedge Y$ and $I \in \mathcal{R}(P)$ we infer $r(J \wedge Y) \geq r(I) + 1 = r(K)$, while $J \in \mathcal{R}(P)$ gives $r(J \vee Y) \geq r(J)$. Using these two inequalities and Lemma 4, we obtain

$$r(J) + r(Y) \geq r(J \vee Y) + r(J \wedge Y) \geq r(J \vee Y) + r(K) \geq r(J) + r(K). \tag{7}$$

Hence $r(K) \leq r(Y)$, and it suffices to exclude $r(K) = r(Y)$. Assume that $r(K) = r(Y)$. Then all the inequalities in (7) are equalities, and the last of them gives $r(J \vee Y) = r(J)$. Hence $J \in \mathcal{R}(P)$ implies $J \not< J \vee Y$, whence we obtain $Y \leq J$. Therefore, $K < Y \leq J$, and the maximality of K gives $r(Y) \neq r(I) + 1 = r(K)$, a contradiction. $\square$

Lemma 9. *$\mathcal{R}(P)$ is a semimodular lattice.*

Proof. Let $I, J, K \in \mathcal{R}(P)$ with $I \prec_{\mathcal{R}(\mathrm{P})} J$, $I \leq K$, and $J \nleq K$. Then $I = J \wedge K$. Lemmas 8 and 4 imply

$$r(I) + 1 + r(K) = r(J) + r(K) \geq r(J \vee K) + r(I).$$

Hence $1 + r(K) \geq r(J \vee K)$. If there was an $X \in \mathcal{R}(P)$ with $K < X < J \vee K$, then the definition of trimmed ideals would give $r(J \vee K) \geq r(X) + 1 \geq r(K) + 1 + 1$, a contradiction. Hence $K \prec_{\mathcal{R}(\mathrm{P})} J \vee K$, proving the semimodularity of $\mathcal{R}(P)$. $\square$

Proof of Theorem 1. We know from Lemma 9 that $G(L) = \mathcal{R}(P)$ is a semimodular lattice. Lemma 5 implies that, for each $X \subseteq \mathcal{R}(P)$, there is a finite subset Y of X such that $\bigvee X = \bigvee Y$. Hence all the elements of $\mathcal{R}(P)$ are compact, and each element of $\mathcal{R}(P)$ is the join of finitely many elements of $J(\mathcal{R}(P))$.

It follows from Lemma 8 that, for any $I \in \mathcal{R}(P)$, $r(I)$ is the usual height of I in $\mathcal{R}(P)$. Applying this observation to $1_{\mathcal{R}(P)} = \langle 1; \emptyset \rangle_{\mathrm{id}}$ and using Lemma 6, we infer that $\ell(\mathcal{R}(P)) = \ell(L)$. Since $\langle x; \emptyset \rangle_{\mathrm{id}}$ is always a trimmed ideal, $\varphi \colon L \to \mathcal{R}(P)$, $x \mapsto \langle x; \emptyset \rangle_{\mathrm{id}}$ is clearly a lattice embedding. It is cover-preserving, since $\ell(\mathcal{R}(P)) = \ell(L)$.

Clearly,

$$B = \{\langle a; \emptyset \rangle_{\mathrm{id}} : a \in A(L)\} \cup \{\langle 0; \{b'\} \rangle_{\mathrm{id}} : b \in H(L)\}$$

is a subset of $A(\mathcal{R}(P))$ with $|B| = |J(L)|$.

Next, we will show that $J(\mathcal{R}(P)) \subseteq B$. Let $I = \langle a; S \rangle_{\mathrm{id}} \in J(\mathcal{R}(P))$. Since

$$I = \langle a; \emptyset \rangle_{\mathrm{id}} \vee \bigvee_{x' \in S} \langle 0; \{x'\} \rangle_{\mathrm{id}} \tag{8}$$

holds in $\mathcal{R}(P)$, we conclude that either $|S| = 1$ and $a = 0$, or $a \neq 0$ and $|S| = 0$. We have $I \in B$ in the first case, so assume that $a \neq 0$ and $|S| = 0$. We can also assume that $h(a) > 1$, for otherwise I is again in B. If a belonged to $H(L)$, then it would have a unique lower cover $b \in L$ and $I = \langle b; \emptyset \rangle_{\mathrm{id}} \vee \langle 0; \{a'\} \rangle_{\mathrm{id}}$ would contradict $I \in J(\mathcal{R}(P))$. Hence a is a non-zero join-reducible element in L, but this is a contradiction again, for this property is preserved by φ and $I = \varphi(a)$.

This shows that $J(\mathcal{R}(P)) \subseteq B$. Finally, $J(\mathcal{R}(P)) \subseteq B \subseteq A(\mathcal{R}(P)) \subseteq J(\mathcal{R}(P))$, whence $A(\mathcal{R}(P)) = J(\mathcal{R}(P))$ completes the proof. $\square$

4. Historical comments

Our construction is motivated by the Grätzer-Kiss Embedding Theorem stating that each finite semimodular lattice has a cover-preserving embedding into a finite geometric lattice. Grätzer and Kiss start from the lattice of certain ideals of an appropriate P (which is larger and more complicated than our P). Our semi-ideals are exactly their ideals. The Grätzer-Kiss lattice $E_{\mathrm{gk}}(L)$ given in [5] (see

M. Stern [8], too) also consists of the trimmed ideals of P. Finally, they derive that $E_{\rm gk}(L)$ does the job from their very general results on pseudo rank functions defined on arbitrary finite lattices.

We could not use their general results and the corresponding auxiliary statements, for our L is not assumed to be finite. Developing similar but necessarily more complicated results for the infinite case would not have been economic. Hence we have borrowed from [5] only as much as necessary. Our approach gives no direct references to Theorems 8 and 10 and several lemmas of [5], for this would not help the reader in the present environment. However, the *proofs* of these statements are included in our approach.

When restricted to finite L, our $G(L)$ resembles $E_{\rm gk}(L)$ in the sense that both are relatively easy to visualize. If D is a finite distributive lattice, then so is $G(D)$, and $G(D)$ is the smallest distributive geometric (that is, Boolean) lattice including D as a cover-preserving sublattice; this follows easily from $|J(D)| = \ell(D)$. The Grätzer-Kiss lattice does not have this property. Indeed, $|A(E_{\rm gk}(L))| = \sum_{x\in J(L)} h(x)$, which implies that, for the three-element chain C_3, $E_{\rm gk}(C_3) = M_3$. Notice at this point that no construction can preserve modularity, for M. Hall and R. P. Dilworth [6] constructed a finite modular lattice that is not a sublattice of any modular geometric lattice, see also Cor. IV.5.22 of G. Grätzer [3].

M. Wild [9], using the toolkit of matroid theory, gave a very short proof of the Grätzer-Kiss Embedding Theorem. The finite geometric lattice $E_{\rm dw}(L)$ of [9] has the property that $|A(E_{\rm dw}(L))| = |J(L)| = |A(G(L))|$. The great merit of Wild's approach is that his proof is very short; this is due to the fact that he defines $E_{\rm dw}(L)$ in terms of matroid theory and uses powerful tools from this theory.

Next, we mention two old embedding theorems. Although they were put into the shade by P. Pudlák and J. Tůma [7], they are quite relevant here.

D. T. Finkbeiner [2] embedded an arbitrary finite lattice into a semimodular lattice. Even if finally we could not use his method, [2] gave us some ideas how to develop Grätzer and Kiss' method further.

The Dilworth Embedding Theorem states that each finite lattice L can be embedded in a finite geometric lattice. It was M. Wild [9] who noticed that the *proof* of this theorem, see pages 125–131 in P. Crawley and R. P Dilworth [1], yields a cover-preserving embedding, provided L is semimodular.

After translating M. Wild's matroid theoretic proof to the language of lattice theory, we can see that [1] and [9] produce the same lattice $E_{\rm dw}(L)$. In effect, $E_{\rm dw}(L)$ consists of the "trimmed" members of the Boolean lattice of all subsets of $J(L)$ according to an appropriate rank function. Opposed to this Boolean lattice, our $\mathcal{I}(P)$ is usually not even semimodular but it reflects more properties of L. This is why our construction and that of Grätzer-Kiss are easier to visualize for lattice theorists.

Finally, we mention that the best cover-preserving embedding is not known yet. Indeed, if L_2 is the lattice of Figure 2, then $G(L_2) = E_{\rm dw}(L_2)$ and $|E_{\rm gk}(L_2)| > |G(L_2)| = 12$. However, L_2 is clearly a cover-preserving sublattice of the ten-element geometric lattice $C_2 \times M_3$.

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University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY 6720

E-mail address: czedli@math.u-szeged.hu

URL: http://www.math.u-szeged.hu/∼czedli/

Mathematical Institute of the Budapest University of Technology and Economics, Műegyetem rkp. 3, H-1521 Budapest, Hungary

E-mail address, E. T. Schmidt: schmidt@math.bme.hu

URL: http://www.math.bme.hu/∼schmidt/