CDW-independent subsets in distributive lattices

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Abstract. A subset X of a lattice L with 0 is called *CDW-independent* if (1) it is CD-independent, i.e., for any $x, y \in X$, either $x \le y$ or $y \le x$ or $x \wedge y = 0$ and (2) it is weakly independent, i.e., for any $n \in \mathbf{N}$ and $x, y_1, \dots, y_n \in X$ the inequality $x \le y_1 \vee \dots \vee y_n$ implies $x \le y_i$ for some i. A maximal CDW-independent subset is called a CDW-basis. With combinatorial examples and motivations in the background, the present paper points out that any two CDW-bases of a finite distributive lattice have the same number of elements. Moreover, if a lattice variety $\mathcal{V}$ contains a nondistributive lattice then there exists a finite lattice L in $\mathcal{V}$ such that L has CDW-bases X and Y with $|X| \ne |Y|$.

The classical notion of independent subsets of (semimodular or modular) lattices has many applications ranging from von Neumann's coordinatization theory to combinatorial aspects via matroid theory. Some other notions of independence were introduced in [2] and [3], and there was a decade witnessing an intensive study of weak independence, cf. Lengvárszky's [8] and his other papers. Recently, the result of [2] has been successfully applied to combinatorial problems, cf. [4], Barát, Hajnal and Horváth [1], Horváth, Németh and Pluhár [6], and Pluhár [11]. Interestingly enough, many subsets occurring in these combinatorial applications [1], [4], [6] and [11], and also in E. K. Horváth, G. Horváth, Németh and Szabó [7], and Lengvárszky [9] and [10], enjoy another property, which has recently been named CD-independence in [5]. The present paper is motivated by the observation that a lot of subsets occurring in the combinatorial papers [1], [4], [6], [7], [9], [10] and [11] are both CD-independent and weakly independent. In fact, instead of [2], the main result of the present short note would also be applicable in most of these papers.

Now, let us recall resp. introduce the basic definitions. Let L be a lattice with 0. A subset H of L is called *weakly independent* iff for all $h, h_1, \dots h_n \in H$ which satisfy $h \le h_1 \vee \dots \vee h_n$ there exists an $i \in \{1, \dots, n\}$ such that $h \le h_i$. A subset X of L will be called *CD-independent* if for any $x, y \in X$, either $x \le y$ or $y \le x$ or $x \wedge y = 0$. In other words, if any two elements of X either form a Chain (i.e., they are Comparable) or they are Disjoint; the initials explain our terminology. Subsets which are both CD-independent and weakly independent will be called *CDW-independent subsets*. Maximal CDW-independent subsets of

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L are called *CDW-bases* of L. Similarly, maximal weakly independent resp. CD-independent subsets are called *weak bases* resp. *CD-bases*. Any two weak bases of a finite distributive lattice have the same number of elements, cf. [2], and the same is true for CD-bases, cf. [5].

Now, before formulating an analogous result for CDW-bases, we present some easy examples. (For other examples cf. [4], [6] and [11].) Let L be a finite distributive lattice, let $J_0(L)$ stand for the set of join-irreducible elements of L, and let C be a maximal chain in L. Then $J_0(L)$ is a weak basis but it is not CD-independent (and therefore not CDW-independent) in general. In virtue of [2], C is a weak basis and therefore it is a CDW-basis, however, it is not a CD-basis in general.

The goal of the present paper is to establish the following theorem. A lattice variety is said to be trivial if it is the class of one element lattices.

Main Theorem.

(1) *Any two CDW-bases of a finite distributive lattice have the same number of elements.*

(2) *Let $\mathcal{V}$ be a nontrivial variety of lattices such that for any finite $L \in \mathcal{V}$ and any two CDW-bases X and Y of L we have $|X| = |Y|$. Then $\mathcal{V}$ is the variety of distributive lattices.*

Proof. Let X be a CDW-basis of a finite distributive lattice L. We can assume that $|L| \geq 3$. Since X is weakly independent, [2] implies that $|X| \leq \ell(L) + 1$, where $\ell(L)$ denotes the length of L. The reverse inequality, $|X| \geq \ell(L) + 1$, will be proved via induction on the size of L.

The first case is when $1 \in X$. Let $u = \bigvee(X \setminus \{1\})$. Since X is weakly independent, $u \neq 1$. Since $X \cup \{v\}$ would be CDW-independent for any $v \in [u, 1] \setminus \{u, 1\}$, the maximality of X gives that u is a coatom. It is easy to see that $X' := X \cap \downarrow u$ is a CDW-basis of the principal ideal $\downarrow u$. Hence the induction hypothesis yields $|X| = 1 + |X'| \geq 1 + \ell(\downarrow u) + 1 = \ell(L) + 1$.

The second case is when $1 \notin X$. Let n denote the number of maximal elements of X, and let $\{a_1, \ldots, a_n\}$ be the set of these maximal elements. If we had $n = 1$ then $X \cup \{1\}$ would be CDW-independent again, a contradiction. Hence $n \geq 2$. If we had $a_1 \vee \cdots \vee a_n < 1$ then $X \cup \{1\}$ would be CDW-independent, a contradiction. So, $a_1 \vee \cdots \vee a_n = 1$. Since $a_1 \parallel a_i$ for $i \in \{2, \cdots, n\}$, the CD-independence of X gives $a_1 \wedge a_i = 0$. Now let $b_1 = a_1$ and $b_2 = a_2 \vee \cdots \vee a_n$. Then $b_1 \vee b_2 = 1$ and $b_1 \wedge b_2 = (a_1 \wedge a_2) \vee \cdots \vee (a_1 \wedge a_n) = 0$.

Now let $L_i = \downarrow b_i$ and $X_i = X \cap L_i$ for $i \in \{1, 2\}$. Since (b_1, b_2) is a pair of complementary elements, it is well known that L is (isomorphic to) the direct product of L_1 and L_2. According to this direct decomposition, the ith component of an element $y \in L$ will be denoted by $y^{(i)}$. Then $y = y^{(1)} \vee y^{(2)}$ and $y^{(1)} \wedge y^{(2)} = 0$. From now on, the role of b_1 and that of b_2 will be symmetric.

If $x \in X$ then $x \leq 1 = a_1 \vee \cdots \vee a_n$, whence $x \in X_1 \cup X_2$ by the weak independence of X. This means that $X = X_1 \cup X_2$. On the other hand, $X_1 \cap X_2 \subseteq L_1 \cap L_2 = \{0\}$. Hence $X_1 \cap X_2 = \{0\}$, for every *maximal* CDW-independent subset contains 0.

Now we are going to show that, say, X_1 is a CDW-basis of L_1; then, by symmetry, X_i will be a CDW-basis of L_i for every $i \in \{1, 2\}$. Clearly, X_1 is CDW-independent.

Suppose, by way of contradiction, that X_1 is not a maximal CDW-independent subset of L_1. Choose an element $c \in L_1 \setminus X_1$ such that $X_1' := X_1 \cup \{c\}$ is CDW-independent. Since $c_1 \wedge y = 0$ for any $y \in L_2$, $X' := X \cup \{c\}$ is CD-independent. Now, to show that it is weakly independent, which will be a contradiction, let us assume that $h, h_1, \ldots, h_k \in X'$ such that

$$h \le h_1 \vee \cdots \vee h_k \,. \tag{1}$$

Notice that $0 \in \{h^{(1)}, h^{(2)}\}$ and $0 \in \{h_t^{(1)}, h_t^{(2)}\}$ for $t \in \{1, \ldots, k\}$. The direct decomposition tells us that (1) is equivalent to the conjunction of the following two formulas:

$$h^{(1)} \le h_1^{(1)} \vee \cdots \vee h_k^{(1)}$$
$$h^{(2)} \le h_1^{(2)} \vee \cdots \vee h_k^{(2)} \,.$$

We have to find an $i \in \{1, \ldots, k\}$ such that $h \le h_i$, i.e., $h^{(1)} \le h_i^{(1)}$ and $h^{(2)} \le h_i^{(2)}$. If $h^{(2)} = 0$ then the CDW-independence of X_1' yields an i with $h^{(1)} \le h_i^{(1)}$ while $h^{(2)} = 0 \le h_i^{(2)}$. Now consider the case $h^{(1)} = 0$. Then $0 = c^{(2)}$ implies that $X_2 \cup \{c^{(2)}\}$ is CDW-independent, whence we can find an i with $h^{(2)} \le h_i^{(2)}$ while $h^{(1)} = 0 \le h_i^{(1)}$.

Finally, having seen that X_i is a CDW-basis of $L_i = {\downarrow} b_i$, the induction hypothesis gives

$$|X| = |X_1| + |X_2| - |X_1 \cap X_2| = |X_1| + |X_2| - 1 \ge \ell(L_1) + 1 + \ell(L_2) + 1 - 1 = \ell(L) + 1.$$

This proves part (1) of the theorem.

Let $\mathcal{D}$ denote the variety of distributive lattices, and let $\mathcal{V}$ be another nontrivial lattice variety, distinct from $\mathcal{D}$. It is well known that $\mathcal{D} \subset \mathcal{V}$. Hence, to prove part (2) of the theorem, it suffices to show that $\mathcal{V}$ contains a lattice L such that L has CDW-bases of different sizes.

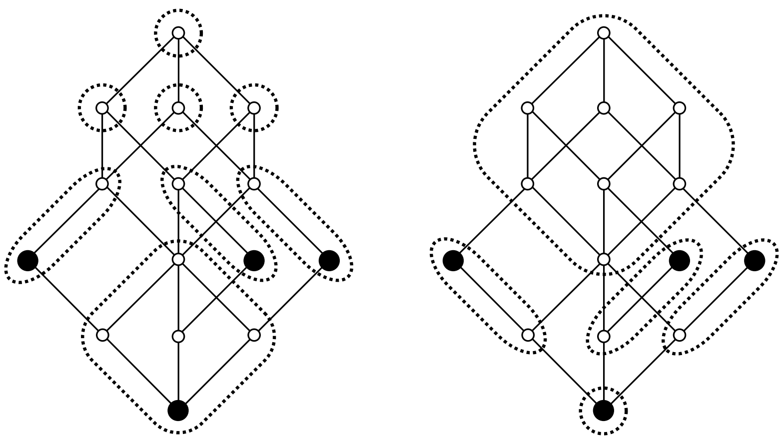

FIGURE 1. L with congruences α and β

Since $\mathcal{V} \not\subseteq \mathcal{D}$, either N_5 or M_3 belongs to $\mathcal{V}$. If $N_5 \in \mathcal{V}$ then let $L = N_5$; this works, for L contains two maximal chains of different sizes, and these chains are CDW-bases. If $M_3 \in \mathcal{V}$ then let L be the lattice depicted in Figure 1. The black-filled enlarged elements constitute a four-element CDW-basis while every maximal chain is a six-element CDW-basis. Now, to show that L belongs to $\mathcal{V}$, consider two congruences, α and β, of L as shown in the figure. Then $\alpha \cap \beta$ is the least congruence, so L is a subdirect product of L/α and L/β. Since L/α is the direct cube of the two-element lattice, it is in $\mathcal{V}$. Finally, $L/\beta \cong M_3 \in \mathcal{V}$, and we conclude that $L \in \mathcal{V}$. □

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