# An algebraic closure for barycentric algebras and convex sets

Gábor Czédli and A. B. Romanowska

Abstract. Let $A$ be an algebra (of an arbitrary finitary type), and let $\boldsymbol{\gamma}$ be a binary term. A pair $(a, b)$ of elements of $A$ will be called a *$\boldsymbol{\gamma}$-eligible pair*, if for each $x$ in the subalgebra generated by $\{a, b\}$ such that $x$ is distinct from $a$ there exists an element $y$ in $A$ such that $b = xy\boldsymbol{\gamma}$. We say that $A$ is a *$\boldsymbol{\gamma}$-closed algebra*, if for each $\boldsymbol{\gamma}$-eligible pair $(a, b)$ there is an element $c$ with $b = ac\boldsymbol{\gamma}$. We call $A$ a *closed algebra*, if it is $\boldsymbol{\gamma}$-closed for all binary terms $\boldsymbol{\gamma}$ that do not induce a projection.

Let $T$ be a unital subring of the field of real numbers. Equipped with all the binary operations $(x, y) \mapsto (1-p)x + py$, $p \in T$ and $0 < p < 1$, $T$ becomes a *mode*, that is, an idempotent algebra in which any two term functions commute. In fact, the mode $T$ is a (generalized) *barycentric algebra*. Let $\mathcal{Q}(T)$ denote the quasivariety generated by this mode.

Our main theorem asserts that each mode of $\mathcal{Q}(T)$ extends to a minimal closed cancellative mode, which is unique in a reasonable sense. In fact, we prove a slightly stronger statement. As corollaries, we obtain a purely algebraic description of the usual topological closure of convex sets, and we exemplify how to use the main theorem to show that certain *open* convex sets are not isomorphic.

## 1. Introduction and motivation

**1.1. The initial idea.** The "classical" convex sets, which are the convex subsets of the real affine space $\mathbb{R}^n$, are easily described by algebraic tools as follows. The open unit interval $\{x \in \mathbb{R} : 0 < x < 1\}$ of real numbers will be denoted by $I^o(\mathbb{R})$. For $q \in I^o(\mathbb{R})$, let $\underline{q}$ denote the binary *barycentric operation* $(\mathbb{R}^n)^2 \to \mathbb{R}^n$ with $(x, y) \mapsto xy\underline{q}$, where $xy\underline{q}$ stands for $(1-q)x + qy$. With the notation $\underline{I}^o(\mathbb{R}) = \{\underline{q} : q \in I^o(\mathbb{R})\}$, the nonempty convex subsets of $\mathbb{R}^n$ are exactly the subalgebras of $\big(\mathbb{R}^n; \underline{I}^o(\mathbb{R})\big)$.

Our primary goal and the original motivation of the present work were to describe the topological closure of a convex set by algebraic methods, in the language of the barycentric algebra $\big(\mathbb{R}^n; \underline{I}^o(\mathbb{R})\big)$. For an illustration we will use the real affine plane $\mathbb{R}^2$. The initial idea is quite simple. First of all, we need an algebraic property that characterizes topologically closed convex sets. Let $A$ be a subalgebra of $\big(\mathbb{R}^n; \underline{I}^o(\mathbb{R})\big)$. Let $p \in I^o(\mathbb{R})$, and let $a$ and $b$ be distinct

2010 *Mathematics Subject Classification*: Primary 08A99, secondary 52A01.

*Key words and phrases*: mode, barycentric algebra, medial groupoid, entropic groupoid, entropic algebra, convex set, algebraic closure, closed algebra.

This research was supported by the NFSR of Hungary (OTKA), grant numbers K77432 and K83219, by TÁMOP-4.2.1/B-09/1/KONV-2010-0005, and by the Warsaw University of Technology under grant number 504G/1120/0054/000. Part of the work on this paper was conducted during the visit of the second author to Iowa State University, Ames Iowa, in Summer 2010.

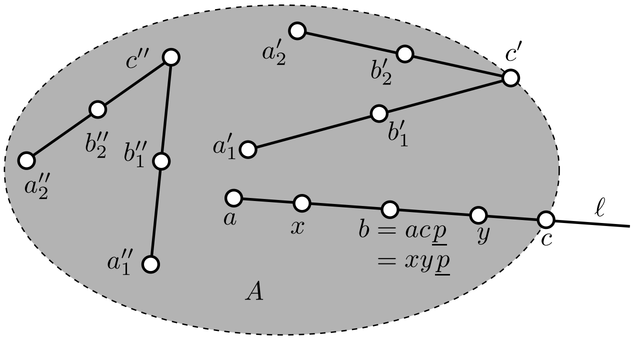


FIGURE 1. Some $\underline{p}$-eligible pairs of $A$ in case $p = 1/2$

points of $A$; see Figure 1, where $A \subseteq \mathbb{R}^2$ and $p = 1/2$. Let $\ell$ denote the line through $a$ and $b$. Then there is a unique point $c$ on $\ell$ such that $b = ac\underline{p}$. Note that we do not assume that $c \in A$. Roughly speaking, our intention is to call the pair $(a, b)$ a $\underline{p}$-eligible pair of $A$ if all the elements of the (left closed and right open) line segment $\ell[b, c)$ belong to $A$. More precisely, $(a, b)$ is called a *$\underline{p}$-eligible pair* of $A$ if for each $x \in \ell(a, b]$, there exists an $y = y(x, a, b, \underline{p})$ in $A$ such that $xy\underline{p} = b$. Equivalently, $(a, b)$ is $\underline{p}$-eligible iff for every $x$ that belongs to the subalgebra generated by $\{a, b\}$ but distinct from $a$, there exists an $y \in A$ such that $xy\underline{p} = b$.

Let $E_p(A)$ denote the set of $\underline{p}$-eligible pairs of $A$. By our intuition (or by Corollary 2.5), $A$ is topologically closed iff for each $(a, b) \in E_p(A)$, there exists an element $c \in A$ such that $b = ac\underline{p}$. This is an *algebraic description* of topologically closed convex sets. Furthermore, to obtain a closure of $A$, we have to add, to each $(a, b) \in E_p(A)$, a (possibly new) element $c$ with the property $b = ac\underline{p}$. Of course, we want to obtain a subalgebra of $\big(\mathbb{R}^n; \underline{I}^o(\mathbb{R})\big)$ since otherwise we could add the same element for all $(a, b) \in E_p(A)$. Hence, if $(a_1', b_1'), (a_2', b_2') \in E_p(A)$ "aim at the same direction", then they need the same element $c'$, which is either a new element (like $c'$ in Figure 1), or an old element (like $c''$ for $(a_1'', b_1'')$ and $(a_2'', b_2'')$ in the figure).

Therefore, we will show that the $p$-eligible pairs constitute a subalgebra $E_p(A)$ of $A^2$ and the relation "aiming in the same direction" is an "internally definable" congruence relation $\sim_p$ on $E_p(A)$. Finally, we will show that the quotient algebra $E_p(A)/{\sim_p}$ does not depend on $p$, $\big(A; \underline{I}^o(\mathbb{R})\big)$ has a natural embedding into $E_p(A)/{\sim_p}$, and $E_p(A)/{\sim_p}$ is isomorphic to the topological closure of $A$.

**1.2. Connections with Universal Algebra and Mode Theory.** The barycentric algebra $\big(\mathbb{R}^n; \underline{I}^o(\mathbb{R})\big)$ is a particular case of *modes*, to be defined soon. Hence, the initial idea described above is closely connected with Mode Theory, see [16]. However, as it will be made clear below, our work also has a lot of connections with basic problems of Universal Algebra. Notice at this point that the present paper is self-contained modulo any standard book on Universal Algebra, see the easy-to-reach Burris and Sankappanavar [1] for example.

We elaborate the basic definition for general algebras, not only for barycentric ones. Even if we cannot show interesting examples other than barycentric algebras at the moment, there might be some.

To achieve more generality, we replace $\mathbb{R}$ by a unital subring $T$ of $\mathbb{R}$ such that $T \neq \mathbb{Z}$. This leads to several kinds of difficulties. First, a single step of "closing the $\underline{p}$-eligible pairs", that is, passing from $A$ to $E_p(A)/{\sim_p}$, does not seem to be sufficient to obtain a closed algebra in general. Thus we form a directed union of the "$\lambda$-step $\underline{p}$-closures", where $\lambda$ ranges in a set of ordinal numbers. Second, we have to use $\underline{p}$ for all $p \in I^o(T)$, possibly continuously many, since $E_p(A)/{\sim_p}$ may depend on the choice of $p$. This makes the directed union a bit more complicated.

To increase generality even further, we do not restrict the investigation to subalgebras of the generalized barycentric algebra $\big(T^n; \underline{I}^o(T)\big)$. We want to consider more algebras in the variety $\mathcal{V}(T)$ generated by $\big(T; \underline{I}^o(T)\big)$. However, we cannot consider all members of $\mathcal{V}(T)$, since several quasi-identities (like the cancellativity laws for the barycentric operations), which hold in $\big(T; \underline{I}^o(T)\big)$ but not in all members of $\mathcal{V}(T)$, are needed in the construction of the 1-step $\underline{p}$-closure $E_p(A)/{\sim_p}$. Because of these quasi-identities, it is reasonable to consider only the members of an appropriate subquasivariety $\mathcal{H}(T)$ of $\mathcal{V}(T)$. This subquasivariety cannot be too large since otherwise the 1-step $\underline{p}$-closure construction will not work. On the other hand, $\mathcal{H}(T)$ should contain $\big(T; \underline{I}^o(T)\big)$, and it should also contain the closure of any of its members.

It is not clear at first sight if there exists an appropriate $\mathcal{H}(T)$. In fact, the treatment of $\mathcal{H}(T)$ occupies a large part of the paper. This part can be interesting for experts of Universal Algebra, because of the following. We need certain quasi-identities that make the 1-step $\underline{p}$-closure construction, $E_p(A)/{\sim_p}$, possible. Unfortunately, the quotient algebra $E_p(A)/{\sim_p}$ does not inherit the abovementioned properties, and we also have to construct $E_q(E_p(A)/{\sim_p})/{\sim_q}$ (and infinitely many further quotient algebras) to form a directed union. Thus, to define $\mathcal{H}(T)$, we stipulate infinitely many appropriately chosen further quasi-identities. These quasi-identities will imply that $\mathcal{H}(T)$ is closed under "repeated" 1-step closures. The exact, necessarily technical definition of $\mathcal{H}(T)$ is postponed to Section 5.

The notation $\mathcal{H}(T)$ comes from "hereditary" since 1-step $\underline{p}$-closures inherit the quasi-identities defining $\mathcal{H}(T)$ in Subsection 5.3. This notation also comes from "hypercubic" identities, to be introduced in Section 5, which are responsible for the desired properties of $\mathcal{H}(T)$. Actually, in terms of Universal Algebra, Lemma 5.1 asserts that the hypercubic identities hold in all clones as heterogeneous algebras. However, we do not study this aspect of these identities.

**1.3. The rest of the motivation.** The initial idea was strongly motivated by the fact that $\big(A; \underline{I}^o(\mathbb{R})\big)$ was embedded into $\big(\mathbb{R}^n; \underline{I}^o(\mathbb{R})\big)$, which is already closed. (Unless otherwise stated, "closed" and "closure" are always understood in our new, algebraic sense. Sometimes, for emphasis, we say that "our closure" or add that "in our sense".) As detailed in Section 11, the book [16] gives analogous embeddings not only for finite dimension and not only for $T = \mathbb{R}$. Hence, we were motivated by the fact that, in many cases, we knew the existence of a closed generalized barycentric algebra $\big(B; \underline{I}^o(T)\big)$ that includes $\big(A; \underline{I}^o(T)\big)$, and we could take the smallest closed subalgebra including $\big(A; \underline{I}^o(T)\big)$. However, this definition of the closure of $\big(A; \underline{I}^o(T)\big)$ would not be satisfactory since we also want to prove *uniqueness*.

To enlighten the situation with an analogous classical construction, assume that $f \in \mathbb{Q}[x]$ is an irreducible polynomial of degree $n$. We are interested in the splitting field of $f$, that is, the smallest field $K$ over which $f$ decomposes into the product of linear factors. Since $f$ decomposes this way in $\mathbb{C}[x]$ by the fundamental theorem of classical algebra, the existence of $K$ is obvious. However, to prove its uniqueness, we have to construct $K$ without referring to $\mathbb{C}$, and then we can prove the uniqueness by chasing the steps of the construction.

Similarly, we give an intrinsic definition of our closure. Then, by chasing the steps of the construction, we are able to prove the desired uniqueness.

Yet another ingredient of our motivation is the isomorphism problem of convex sets as barycentric algebras. This problem is surely easier for *closed* convex sets. Exercise 2.6 illustrates how to benefit from the uniqueness mentioned above in case of *open* convex sets.

**1.4. Outline.** Most of the preliminaries are given in Section 2. Theorem 2.3, the main theorem about the existence and uniqueness of our closure, is also formulated in this section. The most important related statements and an example for the isomorphism problem, Exercise 2.6, are also presented there. Although Theorem 2.3 depends on some technical definitions to be defined afterwards, the location of the postponed definitions is always given.

Sections 3–8, which represent the majority of the present paper, are devoted to the proof of the main theorem. Section 3 proves some basic properties of generalized barycentric algebras. In Section 4, the rudiments of aiming congruences are developed for the (easy) case when $1/6$ belongs to our ring $T$.

Section 5 contains a lot of complicated computations with terms. This is why we discontinue the Polish notation for terms (but only in this section), and develop a powerful parametric notational toolkit in Subsection 5.1. In Subsection 5.2, we introduce some term compositions. We use it in Lemma 5.1 to show that every clone satisfies a family of certain identities, which we call hypercubic identities. Based on hypercubic compositions, Subsection 5.3 defines some quasi-identities and the quasivariety $\mathcal{H}(T)$, see (5.11). Benefiting from the parametric notation, Subsection 5.4 is devoted to some congruences defined by hypercubic compositions.

Section 6 proves some basic properties of the quasivariety $\mathcal{H}(T)$. Section 7 is devoted to one-step closures, while multi-step closures are treated in Section 8. Also, Section 8 completes the proof of the main theorem.

The statement of the main theorem simplifies a lot when $T$ is more or less a field; this is analyzed in Section 9. In Section 10, we compare our closure with the classical topological closure. Finally, Section 11 relates our achievements with some results of the reference book [16] on modes.

## 2. Preliminaries and the main results

**2.1. Modes and barycentric algebras.** Let $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ be $m$-ary and $n$-ary term functions of an algebra $A$, respectively. (Except for Section 5, we will follow the tradition of, say, [16] by using the *Polish notation* $x_1 \dots x_m \boldsymbol{\gamma}$ rather than $\boldsymbol{\gamma}(x_1, \dots, x_m)$.) If for any system $(x_{ij} : 1 \le i \le m \text{ and } 1 \le j \le n)$ of elements of $A$ we have

$$x_{11} \dots x_{m1} \boldsymbol{\gamma}\, x_{12} \dots x_{m2} \boldsymbol{\gamma} \cdots x_{1n} \dots x_{mn} \boldsymbol{\gamma}\, \boldsymbol{\delta} =$$
$$x_{11} \dots x_{1n} \boldsymbol{\delta}\, x_{21} \dots x_{2n} \boldsymbol{\delta} \cdots x_{m1} \dots x_{mn} \boldsymbol{\delta}\, \boldsymbol{\gamma},$$

then we say that $\boldsymbol{\gamma}$ commutes with $\boldsymbol{\delta}$. An algebra is called *entropic*, if any two of its (not necessarily distinct) operations (equivalently, any two term functions) commute with each other. An algebra is *idempotent* if each singleton subset is a subalgebra. Idempotent entropic algebras are called *modes*, see [16]. The main classes of modes include the class of affine modules over commutative unital rings and that of subreducts (subalgebras of reducts) of affine modules. These are the classes of modes that interest us in this paper.

As a general assumption for the *whole paper*,

$$T \text{ is } \underline{\textit{always}} \text{ a subring of } \mathbb{R} \text{ such that } \mathbb{Z} \subset T \subseteq \mathbb{R}. \tag{2.1}$$

The open unit interval $\{r \in T : 0 < r < 1\}$ will be denoted by $I^o(T)$. The assumption $\mathbb{Z} \subset T$ yields that $I^o(T) \ne \emptyset$. For each $p \in I^o(T)$, we consider a binary operation symbol $\underline{p}$, and let $\underline{I}^o(T) = \{\underline{p} : p \in I^o(T)\}$. Defining the so-called *barycentric operation*

$$\underline{p} \colon T^2 \to T, \text{ where } (x, y) \mapsto xy\underline{p} := (1-p)x + py,$$

for all $p \in I^o(T)$, we obtain an algebra $(T; \underline{I}^o(T))$. The name of the operations comes from the fact that the barycenter $z$ of a two-body system with weight $(1-p)$ in the point $x$ and weight $p$ in the point $y$ is given by the barycentric operation, namely, $z = xy\underline{p}$.

The members of the variety generated by the algebra $(T; \underline{I}^o(T))$ are modes. In what follows, they will be called *barycentric algebras* (rather then generalized barycentric algebras). Note, however, that originally barycentric algebras were defined as members of the variety generated by the algebra $(\mathbb{R}; \underline{I}^o(\mathbb{R}))$ (see [10], [12, Ch. 2], [13]), and then more generally as members of the varieties generated by $(F; \underline{I}^o(F))$, each for a subfield $F$ of the field $\mathbb{R}$ (see [16, Chs. V, VII]). Modes and barycentric algebras have been studied intensively in the monographs [12] and [16]. See also the bibliography of [16], and many later papers including, e.g., [8], [9], [11], and [15]. The equational theory of barycentric algebras over a field is well-understood, see [16, Chs. V and VII], and over the field of reals Neumann [10], and then Ignatov [3].

For brevity, barycentric algebras will often be referred to as modes. We say that a mode $A = (A; \underline{I}^o(T))$ is *cancellative* if the cancellation laws

$$xy\underline{p} = xz\underline{p} \Rightarrow y = z \quad \text{and} \quad yx\underline{p} = zx\underline{p} \Rightarrow y = z$$

hold for all $x, y, z \in A$ and $p \in I^o(T)$. Consider the following classes of modes:

$$\begin{aligned}
\mathcal{V}(T) &= \text{ the variety generated by } (T; \underline{I}^o(T)),\\
\mathcal{C}(T) &= \{A \in \mathcal{V}(T) : A \text{ is cancellative}\},\\
\mathcal{Q}(T) &= \text{ the quasivariety generated by } (T; \underline{I}^o(T)),\\
\mathcal{H}(T) &= \text{the quasivariety to be defined in Section 5.}
\end{aligned}$$

Since quasivarieties are defined by quasi-identities (also called universally quantified Horn sentences) and the cancellation laws are quasi-identities that clearly hold in $(T; \underline{I}^o(T))$, it follows that $\mathcal{C}(T)$ is a quasivariety and $\mathcal{Q}(T) \subseteq \mathcal{C}(T) \subseteq \mathcal{V}(T)$. The quasivariety $\mathcal{H}(T)$ will be an economical technical tool to formulate and prove our main theorem.

As usual, $\mathbb{N} = \{1, 2, 3, \ldots\}$ will stand for the set of positive integers, and $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. The proofs of the following two statements are postponed to the next section. The following lemma is well-known in the particular case when $T$ happens to be a field, see [16, 3.7.14 and 7.6.3]. (For the field $\mathbb{R}$, see also Neumann [10] and Ignatov [3].) For a stronger statement, see Lemma 3.1 later.

**Lemma 2.1** (The canonical example)**.** *For $n \in \mathbb{N}$, let $X$ be a nonempty convex subset of the real space $\mathbb{R}^n$. Then $(X; \underline{I}^o(T))$, with the usual meaning of the barycentric operations, belongs to $\mathcal{Q}(T)$.*

For the particular case when $T$ is a subfield of $\mathbb{R}$, the following statement follows from the characterization of free barycentric algebras, see [16, Lemma 5.8.2] and Neumann [10]. For free subreducts of affine modules see also [11].

**Lemma 2.2.** *For each nontrivial binary term $\gamma$ of $\mathcal{V}(T)$ that is not a projection in $\mathcal{V}(T)$, there exists a unique $p \in I^o(T)$ such that $\mathcal{V}(T)$ satisfies the identity $x_1x_2\gamma = x_1x_2\underline{p}$. Conversely, if $p \in I^o(T)$, then $\underline{p}$ is not a projection in $\mathcal{V}(T)$.*

Unless otherwise stated, we will not consider terms that induce a projection in $\mathcal{V}(T)$. Motivated by this lemma, we will use the following notational convention for $\underline{\Gamma} \subseteq \underline{I}^o(T)$ and $\Delta \subseteq I^o(T)$:

$$\Gamma := \{p \in I^o(T) : \underline{p} \in \underline{\Gamma}\}, \qquad \underline{\Delta} = \{\underline{p} : p \in \Delta\}.$$

Note that, for technical reasons, in subscripts we always write $\Gamma$ instead of $\underline{\Gamma}$.

**2.2. Identities and quasi-identities in barycentric algebras.** Keeping (2.1) in mind, the following identities clearly hold in $\mathcal{V}(T)$ for all $p, q \in I^o(T)$:

$$xx\underline{p} = x \qquad \text{(idempotence)},$$
$$xy\underline{p} = yx\underline{1-p} \qquad \text{(skew commutativity)},$$
$$xy\underline{p}\,uv\underline{p}\,\underline{q} = xu\underline{q}\,yv\underline{q}\,\underline{p} \qquad \text{(entropic law)}.$$

These identities and the cancellation laws will be our main working tools. (Note that, in general, many other identities hold in $\mathcal{V}(T)$ that are not consequences of the above three.) We will use $=^i$, $=^{sc}$ and $=^e$ to indicate that the idempotence, the skew commutativity and the entropic law is used, respectively. Furthermore, $=^e$ is also used to refer to the fact that any two term functions commute in $\mathcal{V}(T)$. Similarly, if an equality follows from, say, formula (3.2), then we write $=^{(3.2)}$ instead of $=$. An analogous convention applies for the sign $\iff$ of logical equivalence.

Note that there is a very trivial algorithm to decide if a given identity holds in $\mathcal{V}(T)$. Namely, we can perform some very elementary calculations in $T$ at the high school level. For example, in the case of the entropic law,

$$\begin{aligned} xy\underline{p}\,uv\underline{p}\,\underline{q} &= (1-q)\big((1-p)x+py\big) + q\big((1-p)u+pv\big) = \\ &= (1-p)(1-q)x + p(1-q)y + (1-p)qu + pqv, \end{aligned}$$

and we obtain the same expression for $xu\underline{q}\,yv\underline{q}\,\underline{p}$. The present paper contains some identities that are much more complex than the entropic law. Unless there is an elegant way, we will not present their trivial but often very tedious proofs of the above kind. If the reader wants to see, without spending hours on boring trivialities, why these identities, namely, (4.8), (9.1), (9.2), (9.5), (9.10) and (9.12), hold in $\mathcal{V}(T)$, he or she can resort to any sort of computer algebra, including the Maple worksheet available at the authors' web site.

One of the advantages of Polish notation is that we do not have to use parentheses. However, sometimes we put superfluous parentheses in long terms as reference points. These parentheses should be disregarded. For example, $(ab\underline{r})\,(b)\underline{h}\,(de\underline{q})\,\underline{w}$ is the same term as $ab\underline{r}\,b\underline{h}\,de\underline{q}\,\underline{w}$, but the former one makes it easier to reference certain subterms of (9.5) later.

**2.3. Closed algebras.** Let $A$ be an algebra (of an arbitrary finitary type). For $a, b \in A$, the subalgebra generated by $\{a, b\}$ is denoted by $\langle a, b\rangle$, and $(a, b\rangle$ will stand for $\langle a, b\rangle \setminus \{a\}$. For a binary term $\boldsymbol{\gamma}$ in the language of $A$, a pair $(a, b) \in A^2$ will be called a $\boldsymbol{\gamma}$-*eligible pair* if for each $x \in (a, b\rangle$ there exists an element $y = y(x) \in A$ such that $b = xy\boldsymbol{\gamma}$. A $\boldsymbol{\gamma}$-eligible pair $(a, b)$ is called $\boldsymbol{\gamma}$-*closed* if there is an element $c$ with $b = ac\boldsymbol{\gamma}$. If $\boldsymbol{\gamma}$ induces the first projection on $A$, then $b = ac\boldsymbol{\gamma}$ is possible only when $a = b$. If $\boldsymbol{\gamma}$ induces the second projection, then all pairs are $\boldsymbol{\gamma}$-closed. Hence, we always assume that $\boldsymbol{\gamma}$ does not induce a projection on $A$. If all $\boldsymbol{\gamma}$-eligible pairs are $\boldsymbol{\gamma}$-closed, then $A$ is called a $\boldsymbol{\gamma}$-*closed algebra*. Let $\underline{\Gamma}$ be a set of binary terms, none of them inducing a projection on $A$. Then $A$ is called $\underline{\Gamma}$-closed if it is $\boldsymbol{\gamma}$-closed for all $\boldsymbol{\gamma} \in \underline{\Gamma}$.

Finally, we call $A$ a *closed algebra* if it is $\boldsymbol{\gamma}$-closed for all binary terms $\boldsymbol{\gamma}$ not inducing a projection on $A$.

While an easy transfinite induction shows that each algebra $A$ is a subalgebra of a closed algebra $B$, we are interested only in cases where $B$ has some nice properties. In this paper, we are interested in modes, whence Lemma 2.2 allows us to deal with subsets of $\underline{I}^o(T)$ instead of arbitrary sets $\underline{\Gamma}$ of binary terms.

**2.4. The main result and some related statements.** If $A$ is a submode (that is, a subalgebra) of a mode $B$, then $B$ is called an *extension* of $A$. Two extensions, $B_1$ and $B_2$, of $A$ are said to be *isomorphic over* $A$, if there is an isomorphism $B_1 \to B_2$ that acts identically on $A$. We say that a closed extension $B$ of $A$ is *the closure* of $A$, if $B$ is a *minimal* closed extension of $A$, that is, if there is no closed submode $X$ of $B$ such that $A \subseteq X \subset B$. The notion of a $\boldsymbol{\gamma}$-closure and, for a subset $\Gamma$ of $I^o(T)$, that of a $\underline{\Gamma}$-closure are defined analogously.

**Theorem 2.3** (Main Theorem)**.** *Keeping* (2.1) *in mind, let* $\Gamma$ *be a nonempty subset of* $I^o(T)$*. Then the following three statements hold.*

(i) $\mathcal{Q}(T) \subseteq \mathcal{H}(T) \subseteq \mathcal{C}(T)$.
(ii) *If* $1/6 = 6^{-1}$ *belongs to* $T$*, then* $\mathcal{H}(T) = \mathcal{C}(T)$.
(iii) *Each* $A \in \mathcal{H}(T)$ *has a* $\underline{\Gamma}$*-closure in* $\mathcal{H}(T)$*, which is uniquely determined up to isomorphism over A. In particular, each* $A \in \mathcal{H}(T)$ *has a unique closure in* $\mathcal{H}(T)$.

The quasivariety $\mathcal{H}(T)$ will be defined in Subsection 5.3, see (5.11). The unique $\Gamma$-*closure* of $A$ will be denoted by

$$K_\Gamma^\infty(A),$$

and it will be constructed in Section 8. The construction will trivially imply that $|K_\Gamma^\infty(A)| \le |A| + |\Gamma|$ holds for every $A \in \mathcal{V}(T)$. The *closure* of $A$, which is the $\underline{I}^o(T)$-closure of $A$, is denoted by

$$K_{\mathrm{all}}^\infty(A).$$

The situation with Theorem 2.3 is more pleasant if $T$ happens to be a field. Then somewhat more can be stated, see Section 9, and a shorter proof would be possible based on already known results, see Section 11.

To present an example, we will prove the following statement.

**Proposition 2.4.** *Assume that* $S$ *is a subfield of* $\mathbb{R}$ *such that* $T \subseteq S$*. Let* $n \in \mathbb{N}$*, and let* $H \subseteq \mathbb{R}^n$ *be a nonempty convex set. The usual topological closure of* $H$ *is denoted by* $H^{tc}$*. Consider the barycentric algebras* $\big(H \cap S^n; \underline{I}^o(T)\big)$ *and* $\big(H^{tc} \cap S^n; \underline{I}^o(T)\big)$*. Then both of these algebras belong to* $\mathcal{Q}(T)$*, and*

$$\big(H^{tc} \cap S^n; \underline{I}^o(T)\big) \text{ is isomorphic to } K_{\mathrm{all}}^\infty\big(H \cap S^n; \underline{I}^o(T)\big) \text{ over } H \cap S^n.$$

*Moreover, for every* $p \in I^o(T)$, $K^\infty_{\rm all}\big(H \cap S^n; \underline{I}^o(T)\big)$ *equals the* one-step $\underline{p}$-closure $K^{(1)}_p\big(H \cap S^n; \underline{I}^o(T)\big)$, *to be defined in* (7.1).

**Corollary 2.5.** *A convex subset* $H$ *of* $\mathbb{R}^n$ *is topologically closed iff* $\big(H; \underline{I}^o(\mathbb{R})\big)$ *is closed in our sense iff there is a* $p \in I^o(\mathbb{R})$ *such that* $\big(H; \underline{I}^o(\mathbb{R})\big)$ *is* $\underline{p}$*-closed.*

Theorem 2.3 and Proposition 2.4 are useful in studying the isomorphism classes of convex sets in algebraic sense. In particular, they can be used to extend some results of [9] from closed polygons to open ones. To keep the size of the paper limited, here we demonstrate the power of our results by solving only the following exercise in Section 10.

**Exercise 2.6.** Denote 1/2 by $h$. (Here $h$ comes from "half".) Consider the set $C = \{(x, y) \in \mathbb{Q}^2 : x^2 + y^2 < 1\}$ of all rational points of the open unit *circle* and the set $D = \{(x, y) \in \mathbb{Q}^2 : 0 < x,\ 0 < y \text{ and } x + y < 1\}$ of all rational points of the open *triangle* with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Are the groupoids $(C; \underline{h})$ and $(D; \underline{h})$ isomorphic?

These groupoids are *commutative binary modes*, that is, idempotent, commutative and entropic (or medial) groupoids, see e.g. Ježek and Kepka [4] and [5], and [16], [8], and [14]. An easier version of Exercise 2.6, with $\mathbb{R}$ instead of $\mathbb{Q}$ and the unit square instead of a triangle, has been raised for students of the first author for several years.

Proposition 2.4 and our closure are not as trivial as they may look at the first sight. As opposed to Proposition 2.4, next we state that even if our closure makes sense for a subset of $\mathbb{R}^n$, it can be very far from the topological closure. The isomorphism in Lemma 2.7 will be understood in $\mathcal{V}(T)$. The subfield of $\mathbb{R}$ generated by $T$ will be denoted by $\langle T\rangle_{\rm field}$.

**Lemma 2.7.** *Let* $S$ *be a subfield of* $\mathbb{R}$ *such that* $T \subseteq S$ *and the degree of the field extension* $S|\langle T\rangle_{\rm field}$ *is at least* $\aleph_0$. *Let* $C = \big(\{(x, y) \in S^2 : x^2 + y^2 < 1\}; \underline{I}^o(T)\big)$, *which is a submode of* $\big(\mathbb{R}^2; \underline{I}^o(T)\big)$. *Then* $C^{tc} \cap S^2 \cong K^\infty_{\rm all}(C)$ *by Proposition 2.4. However,* $\big(\mathbb{R}^2; \underline{I}^o(T)\big)$ *(in fact, even* $\big(S^2; \underline{I}^o(T)\big)$*) has another submode* $B$ *such that* $B \cong C$ *but* $B^{tc} \cap S^2$ *is* not *isomorphic with* $K^\infty_{\rm all}(B)$.

## 3. Three easy proofs

This section is devoted to Lemmas 2.1 and 2.2. Actually, Lemma 2.1 follows in two different ways. First, it is a particular case of a stronger statement, Lemma 3.1 below. Although Lemma 3.1 is known from [16, Thm. 5.8.6 and Lemma 7.6.3] for the particular case when $F = T$ is a subfield of $\mathbb{R}$, our approach is entirely different from [16]. Let ${}_FF^n$ denote the $n$-dimensional affine space over $F$, which is the full idempotent reduct of the $n$-dimensional vector space (denoted the same way). By a *convex subset* of ${}_FF^n$ we mean a subset closed with respect to all the barycentric operations $\underline{p}$, $p \in I^o(F)$.

**Lemma 3.1.** *Let $F$ be an arbitrary subfield of $\mathbb{R}$ such that $T \subseteq F$. Let $n \in \mathbb{N}$, and let $X$ be a nonempty convex subset of the space ${}_F F^n$. Then $(X; \underline{I}^o(T))$, with the usual meaning of the barycentric operations, belongs to $\mathcal{Q}(T)$.*

*Proof.* Let $G$ be the subfield of $F$ generated by $T$. First, we intend to show that $(G; \underline{I}^o(T))$ belongs to $\mathcal{Q}(T)$. For $b \in T \setminus \{0\}$, let $\frac{1}{b} \cdot T := \{a/b : a \in T\}$. It is well-known that $G = \bigcup_{b \in T\setminus\{0\}} \frac{1}{b} \cdot T$. This union is a directed one, since, for any $b, c \in T$, we have $\frac{1}{b} \cdot T \subseteq \frac{1}{bc} \cdot T$ and $\frac{1}{c} \cdot T \subseteq \frac{1}{bc} \cdot T$. Observe that, for every $b \in T \setminus \{0\}$, $\big(\frac{1}{b} \cdot T; \underline{I}^o(T)\big) \cong \big(T; \underline{I}^o(T)\big)$, since $\frac{1}{b} \cdot T \to T$, $x \mapsto b \cdot x$ is clearly an isomorphism. So, $(G; \underline{I}^o(T))$ is a directed union of certain subalgebras that are isomorphic to $\big(T; \underline{I}^o(T)\big)$. Since quasi-identities are clearly preserved by forming directed unions, we conclude that $\big(G; \underline{I}^o(T)\big)$ belongs to $\mathcal{Q}(T)$.

Next, consider $F$ as a vector space[1] ${}_G F$ over $G$ in the natural way. Let $\kappa$ denote the dimension of ${}_G F$ over $G$. Then ${}_G F$ is embeddable in the $\kappa$-th direct power of the vector space ${}_G G$. So, we can assume that ${}_G F$ is a subspace of ${}_G G^\kappa$. Since $\underline{p}$ is a term of the vector space ${}_G F$ for all $p \in I^o(T)$, we get that $(F; \underline{I}^o(T))$ is a subalgebra of $(G; \underline{I}^o(T))^\kappa$. Since $\mathcal{Q}(T)$ is closed with respect to subalgebras and direct products, the statement follows from $\big(G; \underline{I}^o(T)\big) \in \mathcal{Q}(T)$. □

The reader may be interested in the following geometric argument.

*Proof of Lemma 2.1.* Let $F = \{r/s : r \in T,\ 0 \neq s \in T\}$, the field of fractions (also called quotient field) of $T$. For $S \subseteq F^k$, let $S^\perp$ denote $\{\vec{x} \in F^k : \vec{s} \perp \vec{x}$ for all $\vec{s} \in S\}$, where $\vec{s} \perp \vec{x}$ means that $s_1 x_1 + \cdots + s_k x_k = 0$. Then $S^\perp$ is a subspace of the vector space ${}_F F^k$. It is well-known from Linear Algebra that (even for finite dimensional vector spaces over arbitrary fields)

$$(S^\perp)^\perp \text{ equals } {}_F\langle S\rangle, \text{ the subspace spanned by } S. \tag{3.1}$$

(For example, (3.1) is the corollary to Theorem 4.3.2 in Herstein [2], or see formula (3.2.5) in van Lint [18].)

Let $\boldsymbol{\gamma}$ be a $k$-ary term in $\mathcal{V}(T)$, distinct from a projection. A trivial induction on the length of $\boldsymbol{\gamma}$ shows that there are $a_1, \ldots, a_k \in T$ such that $a_1 + \cdots + a_k = 1$ and

$$x_1 \ldots x_k \boldsymbol{\gamma} = a_1 x_1 + \cdots + a_k x_k \tag{3.2}$$

holds for all $x_1, \ldots, x_k \in \mathbb{R}$. Next, let $\boldsymbol{\gamma}_i$ and $\boldsymbol{\delta}_i$ be $k$-ary terms in $\mathcal{V}(T)$. It follows from (3.2) that there are $a_{i1}, \ldots, a_{ik} \in T$ such that, with the notation $\vec{a}_i = (a_{i1}, \ldots, a_{ik})$ and $\vec{x} = (x_1, \ldots, x_k)$, for all $\vec{x} \in \mathbb{R}^k$ we have

$$x_1 \ldots x_k \boldsymbol{\gamma}_i = x_1 \ldots x_k \boldsymbol{\delta}_i \iff \vec{a}_i \perp \vec{x}. \tag{3.3}$$

Keeping the previous notation, let $\chi$ be a quasi-identity of the form

$$\boldsymbol{\gamma}_1 = \boldsymbol{\delta}_1 \wedge \cdots \wedge \boldsymbol{\gamma}_m = \boldsymbol{\delta}_m \Rightarrow \boldsymbol{\gamma}_0 = \boldsymbol{\delta}_0.$$

[1] Alternatively, we could consider affine spaces in this paragraph.

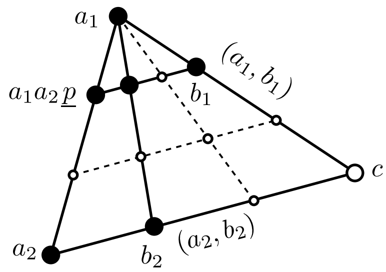


FIGURE 2. The aiming congruence $\sim_p$ in case $p = 1/3$

We get from (3.3) that $\chi$ holds in $(T; \underline{I}^o(T))$ iff, for all $\vec{x} \in T^k$,

$$\vec{x} \in \{\vec{a}_1 \dots, \vec{a}_m\}^\perp \Rightarrow \vec{x} \perp \vec{a}_0. \tag{3.4}$$

Now, assume that $\chi$ holds in $\big(T; \underline{I}^o(T)\big)$. Since each $\vec{x} \in F^k$ is of the form $\frac{1}{r} \cdot \vec{y}$ for some $r \in T$ and $\vec{y} \in T^k$, we easily conclude from (3.4) that, for all $\vec{x} \in F^k$, $\vec{x} \in \{\vec{a}_1 \dots, \vec{a}_m\}^\perp$ implies $\vec{x} \perp \vec{a}_0$. This clearly means that $\vec{a}_0 \in (\{\vec{a}_1, \dots, \vec{a}_m\}^\perp)^\perp$, understood in $F^k$. Therefore, using (3.1), we infer that $\vec{a}_0 \in {}_F\langle\{\vec{a}_1, \dots, \vec{a}_m\}\rangle$. But then $\vec{a}_0 \in {}_{\mathbb{R}}\langle\{\vec{a}_1, \dots, \vec{a}_m\}\rangle =^{(3.1)} (\{\vec{a}_1, \dots, \vec{a}_m\}^\perp)^\perp$, understood in $\mathbb{R}^k$. This means that (3.4) holds for all $\vec{x} \in \mathbb{R}^k$. Hence, resorting to (3.3) again, we conclude that $\chi$ holds in $(\mathbb{R}; \underline{I}^o(T))$. Thus, $(\mathbb{R}; \underline{I}^o(T))$ belongs to $\mathcal{Q}(T)$, and so does the subalgebra $(X; \underline{I}^o(T))$ of its direct power. $\square$

*Proof of Lemma 2.2.* If $\boldsymbol{\gamma}$ is not a projection in $\mathcal{V}(T)$, then it is not a projection in $(T; I^o(T))$. Hence, with the notation of (3.2), $\{0, 1\} \cap \{a_1, a_2\} = \emptyset$. Therefore, $x_1x_2\boldsymbol{\gamma} = x_1x_2\underline{a_2}$ holds in $(T; I^o(T))$, and so it holds in $\mathcal{V}(T)$ as well. The uniqueness and the converse are evident. $\square$

## 4. The aiming congruence when $1/6 \in T$

To motivate our idea, fix a $p \in I^o(T)$. Consider the mode $\big(X; \underline{I}^o(T)\big)$ from Lemma 2.1. For $(a, b) \in X^2$ and $c \in X$, we say that the pair (or vector) $(a, b)$ *aims* at $c$ with respect to $p$ if $b = ac\underline{p}$. Roughly speaking, we will define a relation $\sim_p$ on $X^2$ such that $(a_1, b_1) \sim_p (a_2, b_2)$ iff $(a_1, b_1)$ and $(a_2, b_2)$ aim at the same point with respect to $p$, see Figure 2. This relation, called an *aiming congruence*, will play a crucial role. We will not assume that $c \in X$, so we need an exact "inner description" of $\sim_p$ that does not rely on $c$. Moreover, and this is the main source of difficulty, we deal with an arbitrary $A \in \mathcal{H}(T)$ or, wherever it is feasible, with an arbitrary $A \in \mathcal{C}(T)$. Note that the final plan with $\sim_p$ is to add the same $c$ to the mode for all $\underline{p}$-eligible pairs $(a, b)$ "aiming" at the same non-existing element such that, in the enlarged mode, $(a, b)$ should aim at $c$.

For $A \in \mathcal{C}(T)$, $p \in I^o(T)$ and $(x_1, x_2), (x_3, x_4) \in A^2$, we define

$$(x_1, x_2) \sim_p (x_3, x_4) \text{ iff } x_1x_4\underline{p} = x_1x_3\underline{p}x_2\underline{p}. \tag{4.1}$$

See Figure 2 with $(x_1,x_2,x_3,x_4) := (a_1,b_1,a_2,b_2)$ for illustration, but remember that now we are in $A \in \mathcal{C}(T)$ rather than in the real space. For later use, we formulate the definition of $\sim_p$ in a slightly different form, too. We define two quaternary terms, $f_p^L$ and $f_p^R$, as follows:

$$x_1x_2x_3x_4 f_p^L := x_1x_4\underline{p} \text{ and } x_1x_2x_3x_4 f_p^R := x_1x_3\underline{p}x_2\underline{p}. \tag{4.2}$$

The superscripts come from "left" and "right", respectively. Note that

$$(x_1,x_2) \sim_p (x_3,x_4) \text{ iff } x_1x_2x_3x_4 f_p^L = x_1x_2x_3x_4 f_p^R. \tag{4.3}$$

In other words, we will say that $\sim_p$ is the "*LR-equalizer relation*" of the pair $(f_p^L, f_p^R)$. (The terminology LR is intended to express that the left part, $(x_1,x_2)$ of $(x_1,x_2,x_3,x_4)$ is "equalized" with the right part, $(x_3,x_4)$.) In the following lemma, we do not assume that $1/6 \in T$.

**Lemma 4.1.** *Let $A \in \mathcal{V}(T)$ and $p \in I^o(T)$. Then $\sim_p$ is a reflexive and compatible relation on $A^2$. If it is a congruence relation and $A \in \mathcal{C}(T)$, then the quotient mode $A^2/{\sim_p}$ belongs to $\mathcal{C}(T)$.*

*Proof.* Since $uv\underline{p} =^i uu\underline{p}v\underline{p}$,

$$\mathcal{V}(T) \text{ satisfies the identity } uvuv f_p^L = uvuv f_p^R, \tag{4.4}$$

and the reflexivity of $\sim_p$ follows from (4.3).

Next, assume that $(x_1,x_2) \sim_p (x_3,x_4)$, $(y_1,y_2) \sim_p (y_3,y_4)$, and $q \in I^o(T)$. Then $x_1x_2x_3x_4 f_p^L = x_1x_2x_3x_4 f_p^R$ and $y_1y_2y_3y_4 f_p^L = y_1y_2y_3y_4 f_p^R$, and

$$\begin{aligned} x_1y_1\underline{q}\, x_2y_2\underline{q}\, x_3y_3\underline{q}\, x_4y_4\underline{q}\; f_p^L &=^e x_1x_2x_3x_4 f_p^L\, y_1y_2y_3y_4 f_p^L\, \underline{q} = \\ x_1x_2x_3x_4 f_p^R\, y_1y_2y_3y_4 f_p^R\, \underline{q} &=^e x_1y_1\underline{q}\, x_2y_2\underline{q}\, x_3y_3\underline{q}\, x_4y_4\underline{q}\; f_p^R. \end{aligned} \tag{4.5}$$

Hence, $(x_1y_1\underline{q}, x_2y_2\underline{q}) \sim_p (x_3y_3\underline{q}, x_4y_4\underline{q})$. This means that $(x_1,x_2)(y_1,y_2)\underline{q} \sim_p (x_3,x_4)(y_3,y_4)\underline{q}$. So, $\sim_p$ is a compatible relation. For later reference, notice that the same argument shows that, for any pair $(g^L, g^R)$ of terms on $2^{k+1}$ variables,

$$\text{the LR-equalizer relation of } (g^L, g^R) \text{ is a compatible relation on } A^{2^k}. \tag{4.6}$$

Finally, assume that $A \in \mathcal{C}(T)$ and $\sim_p$ is a congruence relation on $A^2$. Then the quotient mode $A^2/{\sim_p}$ makes sense and belongs to $\mathcal{V}(T)$. The elements of $A^2/{\sim_p}$ are the $\sim_p$-blocks denoted by $(x,y)^{\sim_p}$, where $(x,y) \in A^2$. Let $(a,b)$, $(c_1,d_1)$ and $(c_2,d_2)$ belong to $A^2$, and assume that $(a,b)^{\sim_p}(c_1,d_1)^{\sim_p}\underline{q} = (a,b)^{\sim_p}(c_2,d_2)^{\sim_p}\underline{q}$. This means that $(ac_1\underline{q}, bd_1\underline{q}) \sim_p (ac_2\underline{q}, bd_2\underline{q})$. This gives the second equation in the following formula:

$$\begin{aligned} ab\underline{p}\, c_1d_2\underline{p}\,\underline{q} &=^e ac_1\underline{q}\, bd_2\underline{q}\,\underline{p} = ac_1\underline{q}\, ac_2\underline{q}\,\underline{p}\, bd_1\underline{q}\,\underline{p} =^e aa\underline{p}\, c_1c_2\underline{p}\,\underline{q}\, bd_1\underline{q}\,\underline{p} \\ &=^i a\, c_1c_2\underline{p}\,\underline{q}\, bd_1\underline{q}\,\underline{p} \;=^e ab\underline{p}\, c_1c_2\underline{p}d_1\underline{p}\,\underline{q}. \end{aligned}$$

Hence, by the cancellativity of $\underline{q}$, we infer that $c_1d_2\underline{p} = c_1c_2\underline{p}d_1\underline{p}$. This means that $(c_1,d_1) \sim_p (c_2,d_2)$, that is $(c_1,d_1)^{\sim_p} = (c_2,d_2)^{\sim_p}$. This proves one of the cancellation laws while the other one follows by left-right symmetry (or by skew commutativity). Thus, $A^2/{\sim_p} \in \mathcal{C}(T)$. □

**Lemma 4.2.** *If $1/6 = 6^{-1}$ belongs to $T$, $p \in I^o(T)$, and $A \in \mathcal{C}(T)$, then the relation $\sim_p$ is a congruence on $A^2$.*

Notice that $1/6$ is in $T$ iff both $1/2$ and $1/3$ are in $T$.

*Proof of Lemma 4.2.* In what follows, like in Exercise 2.6, we will write $\underline{h}$ instead of $\underline{1/2}$. Note that $\underline{h} \in \underline{I}^o(T)$ is a commutative operation by skew commutativity. To show that the following auxiliary identity

$$(x_1 y_2 \underline{p})\,(x_2 y_1 \underline{p})\underline{h} = (x_1 x_2 \underline{p}\, y_1 \underline{p}\,)(x_2 x_1 \underline{p}\, y_2 \underline{p})\underline{h} \tag{4.7}$$

holds in $\mathcal{V}(T)$, let us compute:

$$\begin{aligned} x_1 x_2 \underline{p}\, y_1 \underline{p}\, x_2 x_1 \underline{p}\, y_2 \underline{p}\ \underline{h} &=^e x_1 x_2 \underline{p}\, x_2 x_1 \underline{p}\ \underline{h}\, y_1 y_2 \underline{h}\ \underline{p} \\ &=^e x_1 x_2 \underline{h}\, x_2 x_1 \underline{h}\ \underline{p}\, y_1 y_2 \underline{h}\ \underline{p} =^{sc} x_1 x_2 \underline{h}\, x_1 x_2 \underline{h}\ \underline{p}\, y_1 y_2 \underline{h}\ \underline{p} \\ &=^i x_1 x_2 \underline{h}\, y_1 y_2 \underline{h}\ \underline{p} =^{sc} x_1 x_2 \underline{h}\, y_2 y_1 \underline{h}\ \underline{p} =^e x_1 y_2 \underline{p}\, x_2 y_1 \underline{p}\ \underline{h}, \end{aligned}$$

as required.

To prove the symmetry of $\sim_p$, assume that $(x_1, y_1) \sim_p (x_2, y_2)$. Then $x_1 y_2 \underline{p} = x_1 x_2 \underline{p}\, y_1 \underline{p}$ and the cancellativity of $\underline{h}$ together with (4.7) imply that $x_2 y_1 \underline{p} = x_2 x_1 \underline{p}\, y_2 \underline{p}$, that is, $(x_2, y_2) \sim_p (x_1, y_1)$. Hence, $\sim_p$ is symmetric.

In what follows,

$$\underline{t} \text{ will stand for } \underline{1/3}.$$

Note that $t$ comes from "<u>t</u>hird". Observe that the following identity

$$(x_1 y_2 \underline{p})\,(x_1 x_3 \underline{p} y_1 \underline{p}\,)\underline{h}\,(x_2 y_3 \underline{p})\ \underline{t} = (x_1 x_2 \underline{p} y_1 \underline{p})\,(x_1 y_3 \underline{p})\ \underline{h}\,(x_2 x_3 \underline{p} y_2 \underline{p})\ \underline{t} \tag{4.8}$$

holds in $\mathcal{V}(T)$. To prove the transitivity of $\sim_p$, assume that, for $(x_i, y_i) \in A^2$, $(x_1, y_1) \sim_p (x_2, y_2)$ and $(x_2, y_2) \sim_p (x_3, y_3)$. Then, by definition, the first and the third parenthesized subterms on both sides of (4.8) give the same elements. Hence, applying the cancellativity of $\underline{t}$ and then the cancellativity of $\underline{h}$, we conclude that the second parenthesized subterms on both sides are also equal. This means that $(x_1, y_1) \sim_p (x_3, y_3)$. Thus, $\sim_p$ is transitive □

## 5. Hypercubic compositions and the quasivariety $\mathcal{H}(T)$

**5.1. Notation for complicated terms.** We will consider *pairs of terms*. The final purpose is to define congruences as LR-equalizers of these pairs. Since no specific property of $\mathcal{V}(T)$ will be used in the proof of Lemma 5.1, we will consider terms of an arbitrary fixed type at the beginning.

We are going to deal with pairs of high complexity. Hence, it is reasonable to introduce an appropriate shorthand notation. This is not only a question of brevity. The classical "parametric" expressions $\sum(x_i : i \in I)$ and $\prod(x_j : j \in J)$ (also used in subscripted form) have the advantage that, beside being short, they allow certain manipulations with their parameters. It is not rare that an argument would be quite difficult to find or follow with the parameter-free $x_1 + \cdots + x_n$ and $y_1 \ldots y_m$ technique. Our situation is the same, so we need a parametric notation for arbitrary operations. Since it would be very unusual to write $(x_i : i \in I) \sum$, not to mention the nonsense $x_i : i \in I \sum$, in this section (and only here) we use the *classical* $\boldsymbol{\gamma}(x_0, x_1)$ *notation* rather than the Polish notation $x_0 x_1 \boldsymbol{\gamma}$.

Let $f$ be a pair of terms. The first (left) and the second (right) component of a pair $f$ will be denoted by $f^L$ and $f^R$, respectively, that is, $f = (f^L, f^R)$. We always assume that $f^L$ and $f^R$ have the *same arity*, which is called the arity of the pair $f$. If $k$ denotes this arity and $z$ is a $k$-tuple, then $f(z)$ will stand for the pair $\big(f^L(z), f^R(z)\big)$.

Let $\mathbf{2}$ denote the two-element (ordered) set $\{0, 1\}$ with $0 < 1$, and let $\mathbf{2}_R^L = \{L, R\}$ with $L < R$ . The elements of $\mathbf{2}^k$ or $\mathbf{2}_R^{L\,k}$ will be treated as strings, and both comma and juxtaposition will mean concatenation. For example, if $u = 01 \in \mathbf{2}^2$, then each of $x_{1,u}$ and $x_{1u}$ is $x_{101}$. Unless otherwise stipulated by commas or range specifications, the ordering of the variables is always the lexicographic one. Our self-explanatory notational system is exemplified as follows. If $z = (z_u : u \in \mathbf{2}^3)$, then

$$\begin{aligned} z &= (z_{000}, z_{001}, z_{010}, z_{011}, z_{100}, z_{101}, z_{110}, z_{111}), \\ z^L &= (z_{000}, z_{001}, z_{010}, z_{011}) = (z_{0v} : v \in \mathbf{2}^2) \text{ is the \textit{left part} of } z, \\ z^R &= (z_{100}, z_{101}, z_{110}, z_{111}) = (z_{1v} : v \in \mathbf{2}^2) \text{ is the \textit{right part} of } z. \end{aligned}$$

For $z = (z_v : v \in 2^n)$, $z = (z_{v0} : v \in 2^{n-1})$ and $z = (z_{v1} : v \in 2^{n-1})$ are called the *even part* and the *odd part* of $z$, respectively. The range specifications determine the ordering of variables as follows:

$$\begin{aligned} \big((y_{ij} : i \in \mathbf{2}) : j \in \mathbf{2}\big) &= (y_{00}, y_{10}, y_{01}, y_{11}), \\ \big((y_{ij} : j \in \mathbf{2}) : i \in \mathbf{2}\big) &= (y_{00}, y_{01}, y_{10}, y_{11}). \end{aligned}$$

Furthermore, let us see some more complex examples, where $z = (z_u : u \in \mathbf{2}^n)$:

$$\begin{aligned} z^{RL} &= (z^R)^L = (z_{10u} : u \in \mathbf{2}^{n-2}), \\ f(z) &= f(z^{LL}, z^{LR}, z^R) = f(z_u : u \in \mathbf{2}^n) \\ &= \big(f^w(z_u : u \in \mathbf{2}^n) : w \in \mathbf{2}_R^L\big) = \big(f^L(z), f^R(z)\big). \end{aligned}$$

As a final example, $g\big((x_{u0} : u \in \mathbf{2}^2), (x_{0v1} : v \in \mathbf{2})\big)$ stands for

$$\begin{aligned} &\Big(g^w\big((x_{u0} : u \in \mathbf{2}^2), (x_{0v1} : v \in \mathbf{2})\big) : w \in \mathbf{2}_R^L\Big) = \\ &\quad \big(g^L(x_{000}, x_{010}, x_{100}, x_{110}, x_{001}, x_{011}), g^R(x_{000}, x_{010}, x_{100}, x_{110}, x_{001}, x_{011})\big). \end{aligned}$$

**5.2. Hypercubic compositions.** Let $f_1, f_2, \ldots$ be pairs of arbitrary quaternary terms. (Notice that $f_i^L$ and $f_i^R$ should not be confused with $f_p^L$ and $f_p^R$ for $p \in I^o(T)$, since $\mathbb{N}_0$ is disjoint from $I^o(T)$.) We are going to define the $n$-fold *hypercubic composition* $g^{(n)}_{f_1 \ldots f_n} = (g^{(n)L}_{f_1 \ldots f_n}, g^{(n)R}_{f_1 \ldots f_n})$ of these pairs of terms by induction as follows. Both components of $g^{(n)}_{f_1 \ldots f_n}$ will be $2^{n+1}$-ary terms. The variables of the $n$-fold hypercubic composition will be indexed by the elements of the (vertex set of the) hypercube $\mathbf{2}^{n+1}$. This explains the terminology "hypercubic".

The empty (0-fold) hypercubic composition is the pair $g^{(0)} = (g^{(0)L}, g^{(0)R})$ where $g^{(0)L}$ and $g^{(0)R}$ are the first and the second binary projections, respectively. That is, $g^{(0)L}(x_0, x_1) = x_0$ and $g^{(0)R}(x_0, x_1) = x_1$.

Let $n \in \mathbb{N}_0$. Assume that the $n$-fold hypercubic composition $g^{(n)}_{f_1\cdots f_n}$, which is a pair of $2^{n+1}$-ary terms, is already defined. Then let

$$g^{(n+1)}_{f_1\cdots f_{n+1}}(x_u : u \in \mathbf{2}^{n+2}) := g^{(n)}_{f_1\cdots f_n}\Big(\big(f^w_{n+1}(x_{jv} : j \in \mathbf{2}^2) : v \in \mathbf{2}^n\big) : w \in \mathbf{2}^L_R\Big), \tag{5.1}$$

that is,

$$g^{(n+1)L}_{f_1\cdots f_{n+1}}(x_u : u \in \mathbf{2}^{n+2}) := g^{(n)L}_{f_1\cdots f_n}\Big(\big(f^w_{n+1}(x_{jv} : j \in \mathbf{2}^2) : v \in \mathbf{2}^n\big) : w \in \mathbf{2}^L_R\Big),$$

and analogously for $g^{(n+1)R}_{f_1\cdots f_{n+1}}$. The particular case of (5.1) for $n = 0$ yields

$$g^{(1)}_{f_1}(x_j : j \in \mathbf{2}^2) = f_1(x_j : j \in \mathbf{2}^2), \text{ that is, } g^{(1)}_{f_1} = f_1, \tag{5.2}$$

because $\big((x_{jv} : j \in \mathbf{2}^2) : v \in \mathbf{2}^0\big)$ can obviously be replaced by $(x_j : j \in \mathbf{2}^2)$. We conclude the inductive definition of the hypercubic composition by the following example (which allows us to imagine the astronomically long formulas that would have occurred in the paper, if we had used a traditional notation):

$$\begin{aligned} g^{(2)L}_{f_1f_2}(&x_{000}, x_{001}, x_{010}, x_{011}, x_{100}, x_{101}, x_{110}, x_{111}) = \\ &f^L_1\big(f^L_2(x_{000}, x_{010}, x_{100}, x_{110}), f^L_2(x_{001}, x_{011}, x_{101}, x_{111}), \\ &\quad f^R_2(x_{000}, x_{010}, x_{100}, x_{110}), f^R_2(x_{001}, x_{011}, x_{101}, x_{111})\big). \end{aligned}$$

**Lemma 5.1.** *For $n \in \mathbb{N}_0$ and arbitrary pairs $f_1, \ldots, f_{n+1}$ of quaternary terms (of an arbitrary fixed type), the following* hypercubic identity *holds:*

$$g^{(n+1)}_{f_{n+1}f_1\cdots f_n}(x_v : v \in \mathbf{2}^{n+2}) = f_{n+1}\Big(\Big(g^{(n)w}_{f_1\cdots f_n}(x_{ui} : u \in \mathbf{2}^{n+1}) : i \in \mathbf{2}\Big) : w \in \mathbf{2}^L_R\Big). \tag{5.3}$$

*Proof.* If $n = 0$, then both sides of (5.3) coincides with $f_1(x_u : u \in \mathbf{2}^2)$ by (5.2). Assume that $n \in \mathbb{N}_0$ such that (5.3) holds. Let us compute:

$$\begin{aligned} &g^{(n+2)}_{f_{n+2}f_1\cdots f_{n+1}}(x_v : v \in \mathbf{2}^{n+3}) \\ &=^{(5.1)} g^{(n+1)}_{f_{n+2}f_1\cdots f_n}\Big(\big(f^L_{n+1}(x_{js} : j \in \mathbf{2}^2) : s \in \mathbf{2}^{n+1}\big), \\ &\qquad\qquad \big(f^R_{n+1}(x_{js} : j \in \mathbf{2}^2) : s \in \mathbf{2}^{n+1}\big)\Big). \end{aligned} \tag{5.4}$$

The induction hypothesis says that we have to consider the even part and the odd part of the vector on which $g^{(n+1)w}_{f_{n+2}f_1\cdots f_n}$ acts, and we have to do it first for

$w = L$ and then for $w = R$. Hence, by the induction hypothesis, (5.4) equals

$$f_{n+2}\Bigg(\bigg(g^{(n)w}_{f_1\dots f_n}\Big(\big(f^L_{n+1}(x_{jvi} : j \in \mathbf{2}^2) : v \in \mathbf{2}^n\big),$$

$$\big(f^R_{n+1}(x_{jvi} : j \in \mathbf{2}^2) : v \in \mathbf{2}^n\big)\Big) : i \in \mathbf{2}\bigg) : w \in \mathbf{2}^L_R\Bigg).$$

Applying (5.1) to each of the subterms $g^{(n)w}_{f_1\dots f_n}(\dots)$, this coincides with

$$f_{n+2}\bigg(\Big(g^{(n+1)w}_{f_1\dots f_{n+1}}(x_{ui} : u \in \mathbf{2}^{n+2}) : i \in \mathbf{2}\Big) : w \in \mathbf{2}^L_R\bigg). \tag{5.5}$$

We have obtained that the lefthand side of (5.4) equals (5.5). This completes the induction step. □

**5.3. The definition of $\mathcal{H}(T)$.** In the rest of Section 5, the current section, we will only consider algebras and terms of $\mathcal{V}(T)$. Remember that, for $p \in I^o(T)$, the pair $f_p = (f^L_p, f^R_p)$ has been defined in (4.2). For brevity, if $p_1, \dots, p_n \in I^o(T)$, then the following notation will apply:

$$\hat{g}^{(n)}_{p_1\dots p_n} = (\hat{g}^{(n)L}_{p_1\dots p_n}, \hat{g}^{(n)R}_{p_1\dots p_n}) := g^{(n)}_{f_{p_1}\dots f_{p_n}} = (g^{(n)L}_{f_{p_1}\dots f_{p_n}}, g^{(n)R}_{f_{p_1}\dots f_{p_n}}). \tag{5.6}$$

For $\vec{x} = (x_u : u \in \mathbf{2}^n)$ and $\vec{y} = (y_u : u \in \mathbf{2}^n)$, let us define

$$\vec{x} \approx_{p_1\dots p_n} \vec{y} \iff \hat{g}^{(n)L}_{p_1\dots,p_n}(x, y) = \hat{g}^{(n)R}_{p_1\dots,p_n}(x, y). \tag{5.7}$$

In other words, $\approx_{p_1\dots p_n}$ denotes the LR-equalizer relation of the pair $\hat{g}^{(n)}_{p_1\dots,p_n}$, where the $p_i$ are not assumed to be distinct. In particular, if $n = 0$, then

$$\approx_\emptyset \text{ is the equality relation,} \tag{5.8}$$

since $\hat{g}^{(0)}$ is the pair of projections. For $n = 1$, (4.3) and (5.2) imply that

$$\approx_p \text{ coincides with } \sim_p. \tag{5.9}$$

Consider the quasi-identities

$$\begin{gathered} \vec{x} \approx_{p_1\dots p_n} \vec{y} \Rightarrow \vec{y} \approx_{p_1\dots p_n} \vec{x} \text{ and} \\ \vec{x} \approx_{p_1\dots p_n} \vec{y} \wedge \vec{y} \approx_{p_1\dots p_n} \vec{z} \Rightarrow \vec{x} \approx_{p_1\dots p_n} \vec{z}. \end{gathered} \tag{5.10}$$

Then we define the quasivariety

$$\begin{gathered} \mathcal{H}(T) := \{A \in \mathcal{C}(T) : A \text{ satisfies the quasi-identities (5.10)} \\ \text{for all } n \in \mathbb{N}_0 \text{ and } p_1, \dots, p_n \in I^o(T)\}. \end{gathered} \tag{5.11}$$

**5.4. Some congruences on cancellative barycentric algebras.** The following trivial lemma is well-known, so we present it without proof.

**Lemma 5.2.** *Let $U$ and $V$ be arbitrary algebras of the same type. Let $\psi\colon U \to V$ be a surjective homomorphism, and let $\Theta \subseteq V^2$ be an arbitrary relation on $V$. Then $\Theta$ is a congruence on $V$ iff $\{(u, v) \in U^2 : \psi(u)\ \Theta\ \psi(v)\}$ is a congruence on $U$.*

**Lemma 5.3.** *Let $A \in \mathcal{C}(T)$, let $p \in I^o(T)$, and suppose that $\sim_p$ is a congruence on $A^2$. Denote the quotient mode $A^2/{\sim_p}$ by $B$. Let $p_1, \dots, p_n \in I^o(T)$ be arbitrary (not necessarily distinct) elements of $I^o(T)$. Then $\approx_{p_1 \dots p_n}$ is a congruence on $B^{2^n}$ iff $\approx_{pp_1 \dots p_n}$ is a congruence on $A^{2^{n+1}}$.*

*Proof.* For $z = (z_u : u \in \mathbf{2}^{n+1}) \in A^{n+1}$, let $\big((z_{u0}, z_{u1})^{\sim_p} : u \in \mathbf{2}^n\big) \in B^{2^n}$ be denoted by $z^{\sim_p^*}$. Clearly, each element of $B^{2^n}$ is of this form. We are going to show that, for all $x, y \in A^{2^{n+1}}$,

$$x^{\sim_p^*} \approx_{p_1 \dots p_n} y^{\sim_p^*} \iff x \approx_{pp_1 \dots p_n} y. \tag{5.12}$$

By definition, $x^{\sim_p^*} \approx_{p_1 \dots p_n} y^{\sim_p^*}$ iff

$$\hat{g}^{(n)L}_{p_1 \dots, p_n}\Big(\big((x_{u0}, x_{u1})^{\sim_p} : u \in \mathbf{2}^n\big), \big((y_{u0}, y_{u1})^{\sim_p} : u \in \mathbf{2}^n\big)\Big) =$$
$$\hat{g}^{(n)R}_{p_1 \dots, p_n}\Big(\big((x_{u0}, x_{u1})^{\sim_p} : u \in \mathbf{2}^n\big), \big((y_{u0}, y_{u1})^{\sim_p} : u \in \mathbf{2}^n\big)\Big)$$

iff

$$\hat{g}^{(n)L}_{p_1 \dots, p_n}\Big(\big((x_{u0}, x_{u1}) : u \in \mathbf{2}^n\big), \big((y_{u0}, y_{u1}) : u \in \mathbf{2}^n\big)\Big) \sim_p$$
$$\hat{g}^{(n)R}_{p_1 \dots, p_n}\Big(\big((x_{u0}, x_{u1}) : u \in \mathbf{2}^n\big), \big((y_{u0}, y_{u1}) : u \in \mathbf{2}^n\big)\Big)$$

iff

$$\Big(\hat{g}^{(n)L}_{p_1 \dots, p_n}\big((x_{ui} : u \in \mathbf{2}^n), (y_{ui} : u \in \mathbf{2}^n)\big) : i \in \mathbf{2})\Big) \sim_p$$
$$\Big(\hat{g}^{(n)R}_{p_1 \dots, p_n}\big((x_{u0} : u \in \mathbf{2}^n), (y_{u0} : u \in \mathbf{2}^n)\big) : i \in \mathbf{2}\Big).$$

By virtue of (4.3), this holds iff

$$f_p^L\bigg(\Big(\hat{g}^{(n)w}_{p_1 \dots, p_n}\big((x_{ui} : u \in \mathbf{2}^n), (y_{ui} : u \in \mathbf{2}^n)\big) : i \in \mathbf{2}\Big) : w \in \mathbf{2}^{L}_{R}\bigg) =$$
$$f_p^R\bigg(\Big(\hat{g}^{(n)w}_{p_1 \dots, p_n}\big((x_{ui} : u \in \mathbf{2}^n), (y_{ui} : u \in \mathbf{2}^n)\big) : i \in \mathbf{2}\Big) : w \in \mathbf{2}^{L}_{R}\bigg).$$

Keeping (5.6) in mind, Lemma 5.1 yields that this equation is equivalent to

$$\hat{g}^{(n+1)L}_{pp_1 \dots, p_n}\big((x_u : u \in \mathbf{2}^n), (y_u : u \in \mathbf{2}^n)\big) = \hat{g}^{(n+1)R}_{pp_1 \dots, p_n}\big((x_u : u \in \mathbf{2}^n), (y_u : u \in \mathbf{2}^n)\big),$$

which is equivalent to $x \approx_{pp_1 \dots p_n} y$ by definition. This proves (5.12).

Finally, since $\psi \colon A^{2^{n+1}} \to B^{2^n}$, $z \mapsto z^{\sim_p^*}$ is clearly a surjective homomorphism, the last sentence of Lemma 5.3 follows from (5.12) and Lemma 5.2. □

For $A \in \mathcal{C}(T)$ and a fixed sequence $\vec{p} = (p_0, p_1, p_2 \dots)$ of elements of $I^o(T)$, we define

$$A^{[0]} := A,$$
$$A^{[n+1]} := \big(A^{[n]} \times A^{[n]}\big)/{\sim_{p_n}}, \text{ provided } \sim_{p_n} \text{ is a congruence.}$$

At present, $A^{[n]}$ is not necessarily defined. However, if it is defined, then $A^{[k]}$ is also defined for every $k < n$.

**Lemma 5.4.** *Let $A \in \mathcal{C}(T)$, and let $\vec{p} = (p_0, p_1, p_2 \dots)$ be a fixed sequence of elements of $I^o(T)$. Assume that $n \in \mathbb{N}_0$ such that $A^{[n]}$ is defined. Then $\sim_{p_n}$ is a congruence on $A^{[n]} \times A^{[n]}$ iff $\approx_{p_0 \dots p_n}$ is a congruence on $A^{2^{n+1}}$.*

*Proof.* We need some auxiliary mappings. All of them will be surjective homomorphisms. Namely, we will define $\psi_k \colon A^{2^k} \to A^{[k]}$ for $k = 0, \dots, n$, $\varphi_k \colon A^{2^k} \to A^{[k-1]} \times A^{[k-1]}$ for $k = 1, \dots, n$, and $\pi_k \colon A^{[k-1]} \times A^{[k-1]} \to A^{[k]}$ for $k = 1, \dots, n$. We also define them for $k = n+1$ without claiming that $\pi_{n+1}$ and $\psi_{n+1}$ always make sense. Let $\psi_0$ be the identity mapping.

For $0 \le k \le n$, assume that the surjective homomorphism $\psi_k \colon A^{2^k} \to A^{[k]}$ is already defined. Let

$$\begin{aligned}
&\varphi_{k+1} \colon A^{2^{k+1}} \to A^{[k]} \times A^{[k]}, \quad x \mapsto \big(\psi_k(x^L), \psi_k(x^R)\big),\\
&\pi_{k+1} \colon A^{[k]} \times A^{[k]} \to (A^{[k]} \times A^{[k]})/{\sim_{p_k}} = A^{[k+1]}, \quad x \mapsto x^{\sim_{p_k}}, \text{ and}\\
&\psi_{k+1} = \pi_{k+1} \circ \varphi_{k+1} \colon A^{2^{k+1}} \to A^{[k+1]}, \quad x \mapsto \pi_{k+1}(\varphi_{k+1}(x)).
\end{aligned}$$

Since $\psi_{k+1}$ is the composite of $\psi_k \times \psi_k$ and the natural projection to a quotient mode, it is clearly a surjective homomorphism, provided $\sim_{p_k}$ is a congruence, which is surely the case when $k < n$. Hence, $\varphi_k$, $\pi_\ell$ and $\psi_\ell$ are surjective homomorphisms for $k \le n+1$ and $\ell \le n$, but $\pi_{n+1}$ and $\psi_{n+1}$ make sense only if $\sim_{p_n}$ is a congruence on $A^{[n]}$.

Next, we prove by induction on $k$ that, for all $k \le n$ and $x, y \in A^{2^{k+1}}$,

$$x \approx_{p_0 \dots p_k} y \iff \varphi_{k+1}(x) \sim_{p_k} \varphi_{k+1}(y). \tag{5.13}$$

If $k = 0$ and $z = (z_0, z_1) \in A^{2^1}$, then $\varphi_1(z) = (\psi_0(z_0), \psi_0(z_1)) = (z_0, z_1) = z$, whence (5.13) clearly follows from (5.9). So, assume that (5.13) holds for some $k < n$, let $x, y \in A^{2^{k+2}}$, and compute:

$$\begin{aligned}
x \approx_{p_0 \dots p_{k+1}} y \iff^{(5.7)} \hat{g}^{(k+2)L}_{p_0 \dots p_{k+1}}(x, y) = \hat{g}^{(k+2)R}_{p_0 \dots p_{k+1}}(x, y) \iff^{(5.1),(5.6)}&\\
\hat{g}^{(k+1)L}_{p_0 \dots p_k}\Big(\big(f^w_{p_{k+1}}(x_{0v}, x_{1v}, y_{0v}, y_{1v}) : v \in \mathbf{2}^{k+1}\big) : w \in \mathbf{2}^L_R\Big)&\\
= \hat{g}^{(k+1)R}_{p_0 \dots p_k}\Big(\big(f^w_{p_{k+1}}(x_{0v}, x_{1v}, y_{0v}, y_{1v}) : v \in \mathbf{2}^{k+1}\big) : w \in \mathbf{2}^L_R\Big)&.
\end{aligned}$$

By (5.7), this means that

$$\begin{aligned}
&\big(f^L_{p_{k+1}}(x_{0v}, x_{1v}, y_{0v}, y_{1v}) : v \in \mathbf{2}^{k+1}\big) \approx_{p_0 \dots p_k}\\
&\big(f^R_{p_{k+1}}(x_{0v}, x_{1v}, y_{0v}, y_{1v}) : v \in \mathbf{2}^{k+1}\big).
\end{aligned}$$

By the induction hypothesis, this holds iff

$$\begin{aligned}
&\varphi_{k+1}\big(f^L_{p_{k+1}}(x_{0v}, x_{1v}, y_{0v}, y_{1v}) : v \in \mathbf{2}^{k+1}\big) \sim_{p_k}\\
&\varphi_{k+1}\big(f^R_{p_{k+1}}(x_{0v}, x_{1v}, y_{0v}, y_{1v}) : v \in \mathbf{2}^{k+1}\big).
\end{aligned}$$

Since $x_{0v} = x^L_v, \dots, y_{1v} = y^R_v$, and $f^L_{p_{k+1}}$ acts componentwise on the elements $x^L$, $x^R$, $y^L$, $y^R$ of $A^{2^{k+1}}$, the above condition is equivalent to

$$\varphi_{k+1}\big(f^L_{p_{k+1}}(x^L, x^R, y^L, y^R)\big) \sim_{p_k} \varphi_{k+1}\big(f^R_{p_{k+1}}(x^L, x^R, y^L, y^R)\big).$$

Since $\varphi_{k+1}$ is a homomorphism, this holds iff

$$f^L_{p_{k+1}}\big(\varphi_{k+1}(x^L),\varphi_{k+1}(x^R),\varphi_{k+1}(y^L),\varphi_{k+1}(y^R)\big)$$
$$\sim_{p_k} f^R_{p_{k+1}}\big(\varphi_{k+1}(x^L),\varphi_{k+1}(x^R),\varphi_{k+1}(y^L),\varphi_{k+1}(y^R)\big).$$

Since congruences become equations in the corresponding quotient mode, this is equivalent to

$$f^L_{p_{k+1}}\big(\varphi_{k+1}(x^L)^{\sim_{p_k}},\varphi_{k+1}(x^R)^{\sim_{p_k}},\varphi_{k+1}(y^L)^{\sim_{p_k}},\varphi_{k+1}(y^R)^{\sim_{p_k}}\big)$$
$$= f^R_{p_{k+1}}\big(\varphi_{k+1}(x^L)^{\sim_{p_k}},\varphi_{k+1}(x^R)^{\sim_{p_k}},\varphi_{k+1}(y^L)^{\sim_{p_k}},\varphi_{k+1}(y^R)^{\sim_{p_k}}\big).$$

Using $\varphi_{k+1}(z)^{\sim_{p_k}} = \psi_{k+1}(z)$ and (4.3), the above equation is equivalent to

$$\big(\psi_{k+1}(x^L),\psi_{k+1}(x^R)\big) \sim_{p_{k+1}} \big(\psi_{k+1}(y^L),\psi_{k+1}(y^R)\big),$$

that is, to $\varphi_{k+2}(x) \sim_{p_{k+1}} \varphi_{k+2}(y)$. This completes the induction step, and we have seen that (5.13) holds for all $k \leq n$.

Finally, Lemma 5.2 applied to (5.13) with $k = n$ completes the proof. □

## 6. Basic properties of $\mathcal{H}(T)$

**Lemma 6.1.** *If $1/6 \in T$, then $\mathcal{H}(T) = \mathcal{C}(T)$.*

*Proof.* Let $n \in \mathbb{N}_0$, and let $p_0, \dots, p_n$ be an arbitrary sequence of elements of $I^o(T)$. We conclude from Lemmas 4.1 and 4.2 that for every $B \in \mathcal{C}(T)$ and $p \in I^o(T)$, $(B \times B)/{\sim_p}$ makes sense and belongs to $\mathcal{C}(T)$. Letting $B$ equal $A = A^{[0]}$, $A^{[1]}$, $A^{[2]}$, ..., we conclude that, for any $A \in \mathcal{C}(T)$, $A^{[n]}$ is defined and belongs to $\mathcal{C}(T)$. Hence, $\sim_{p_n}$ is a congruence on $A^{[n]}$ by Lemma 4.2, and Lemma 5.4 yields that $\approx_{p_0 \dots p_n}$ is a congruence on $A^{2^{n+1}}$. So, shifting the subscripts by one, we obtain that the quasi-identities (5.10) hold in $\mathcal{C}(T)$ for all $n \in \mathbb{N}$. Finally, we get from (5.8) that (5.10) trivially holds for $n = 0$. □

If $1/6$ does *not* belong to $T$, then we can prove only a weaker statement.

**Lemma 6.2.** *$\mathcal{Q}(T) \subseteq \mathcal{H}(T) \subseteq \mathcal{C}(T)$.*

*Proof.* Let $A = (T; \underline{I}^o(T))$. Since it generates $\mathcal{Q}(T)$, it suffices to show that the quasi-identities (5.10) hold in $A$. Hence, it suffices to show that, for any $p_1, \dots, p_n \in \underline{I}^o(T)$, $\approx_{p_1 \dots p_n}$ is a congruence on $A^{2^n}$. (This property of $\approx_{p_1 \dots p_n}$ is stronger than (5.10), but it will be easier to prove.)

First, by induction on $n$, we show that $\approx_{p_1 \dots p_n}$ is reflexive. For $n = 0$ and $n = 1$, it is reflexive by (5.8) and by (5.9) together with Lemma 4.1, respectively. So, assume that $n \in \mathbb{N}$ such that $\approx_{p_1 \dots p_n}$ is reflexive. We have to show that, for any $x \in A^{2^{n+1}}$, $x \approx_{p_1 \dots p_{n+1}} x$. That is, by (5.7), we have to show that $xx\,\hat{g}^{(n+1)L}_{p_1 \dots, p_{n+1}} = xx\,\hat{g}^{(n+1)R}_{p_1 \dots, p_{n+1}}$. (Note that we have returned to the Polish notation.) Taking the recursive definition (5.1) and the notational convention

(5.6) into account (and converting to Polish notation), this is equivalent to

$$(x_{0v}x_{1v}x_{0v}x_{1v}f^L_{p_{n+1}} : v \in \mathbf{2}^n)\,(x_{0v}x_{1v}x_{0v}x_{1v}f^R_{p_{n+1}} : v \in \mathbf{2}^n)\,\hat{g}^{(n)L}_{p_1\dots,p_n}$$
$$= (x_{0v}x_{1v}x_{0v}x_{1v}f^L_{p_{n+1}} : v \in \mathbf{2}^n)\,(x_{0v}x_{1v}x_{0v}x_{1v}f^R_{p_{n+1}} : v \in \mathbf{2}^n)\,\hat{g}^{(n)R}_{p_1\dots,p_n}.$$

By (5.7), this holds iff

$$(x_{0v}x_{1v}x_{0v}x_{1v}f^L_{p_{n+1}} : v \in \mathbf{2}^n) \approx_{p_1\dots p_n} (x_{0v}x_{1v}x_{0v}x_{1v}f^R_{p_{n+1}} : v \in \mathbf{2}^n). \qquad (6.1)$$

Since $\approx_{p_1\dots p_n}$ is reflexive by the induction hypothesis, (6.1) follows from (4.4). This completes the induction step. Thus, for all $n \in \mathbb{N}$ and $p_1, \dots, p_n \in \underline{I}^o(T)$, $\approx_{p_1\dots p_n}$ is a reflexive relation on $A^{2^n}$. It is compatible by (4.6).

Let $A' = (T; \underline{I}^o(T) \cup \{P\})$ where $P$ is the ternary ring term $xyzP := x-y+z$. A trivial calculation shows (and we know from [16]) that $P$ commutes with all the $\underline{p}$, $p \in I^o(T)$. That is, $A'$ is still a mode. Therefore any two term functions of $A'$ commute, and the compatibility of $\approx_{p_1\dots p_n}$, as a relation on $A'^{2^n}$ , follows again from (4.6).

Clearly, $P$ satisfies the identities $xxyP = y$ and $xyyP = x$ in $A'^{2^n}$. That is, $P$ is a Mal'cev term on $A'^{2^n}$, see Mal'cev [7]; see also Burris and Sankappanavar [1] or Smith [17]. On the other hand, if an algebra has a Mal'cev term, then all of its compatible reflexive relations are congruences, see Proposition 143 in Smith [17]. Thus, $\approx_{p_1\dots p_n}$ is a congruence on $A'^{2^n}$, whence it is also a congruence on $A^{2^n}$. □

**Lemma 6.3.** *If $A \in \mathcal{H}(T)$ and $p \in I^o(T)$, then the quotient mode $(A \times A)/{\sim_p}$ makes sense and belongs to $\mathcal{H}(T)$.*

*Proof.* Let $B = (A \times A)/{\sim_p}$. It makes sense since $\sim_p$ coincides with $\approx_p$, see (5.9), and $\approx_p$ is a congruence by the definition of $\mathcal{H}(T)$. Lemma 5.3 clearly implies that $B \in \mathcal{H}(T)$. □

## 7. One-step closure

Let $A \in \mathcal{C}(T)$ and $p \in I^o(T)$. Remember that a pair $(a, b) \in A^2$ is said to be a $\underline{p}$-eligible pair if for each $x \in (a, b\rangle$, there exists an element $y = y(x) \in A$ such that $b = xy\underline{p}$. Let

$$E_p(A) := \{(a, b) \in A^2 : (a, b) \text{ is a } \underline{p}\text{-eligible pair of } A\}.$$

**Lemma 7.1.** *For $A \in \mathcal{C}(T)$, $E_p(A)$ is a submode of $A^2$. Furthermore, for all $a \in A$, $(a, a) \in E_p(A)$.*

*Proof.* For every $a \in A$, $E_p(A)$ contains $(a, a)$ since $(a, a\rangle = \emptyset$.

Assume that $(a_1, b_1), (a_2, b_2) \in E_p(A)$ and $r \in I^o(T)$. Let

$$(a, b) := (a_1, b_1)(a_2, b_2)\underline{r} = (a_1a_2\underline{r}, b_1b_2\underline{r});$$

see Figure 3 for an illustration. We have to show that $(a, b) \in E_p(A)$. Let $y \in (a, b\rangle$. We have to find a $z \in A$ with $b = yz\underline{p}$. We can assume that $y \neq b$

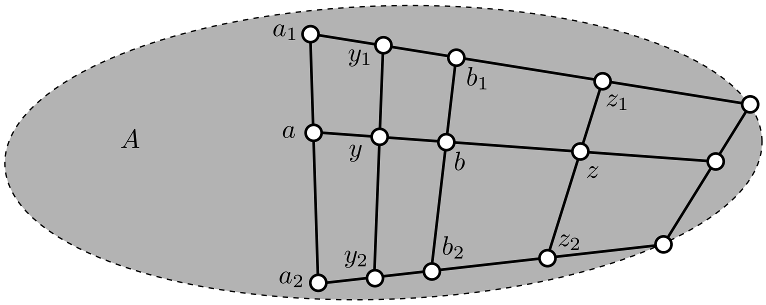


FIGURE 3. Illustration with $(p, q, r) = (1/3, 1/2, 2/5)$

since otherwise we can choose $z = b$. Then, by Lemma 2.2, $y = ab\underline{q}$ for some $q \in I^o(T)$. Let $y_i = a_i b_i \underline{q} \in \langle a_i, b_i \rangle$ for $i = 1, 2$. If $y_i \neq a_i$, then there is an element $z_i \in A$ with $b_i = y_i z_i \underline{p}$ since $(a_i, b_i) \in E_p(A)$. If $y_i = a_i$, then $a_i a_i \underline{q} =^i a_i = y_i = a_i b_i \underline{q}$ implies $a_i = b_i$ by cancellativity, whence $b_i =^i y_i b_i \underline{p}$, and we can choose $z_i := b_i$ to ensure $b_i = y_i z_i \underline{p}$. Finally, with $z := z_1 z_2 \underline{r}$,

$$
\begin{aligned}
yz\underline{p} &= ab\underline{q}z\underline{p} = a_1 a_2 \underline{r}\, b_1 b_2 \underline{r}\, \underline{q} z \underline{p} =^e a_1 b_1 \underline{q}\, a_2 b_2 \underline{q}\, \underline{r} z \underline{p} \\
&= y_1 y_2 \underline{r} z \underline{p} = y_1 y_2 \underline{r}\, z_1 z_2 \underline{r}\, \underline{p} =^e y_1 z_1 \underline{p}\, y_2 z_2 \underline{p}\, \underline{r} = b_1 b_2 \underline{r} = b. \qquad \square
\end{aligned}
$$

Assume that $A \in \mathcal{H}(T)$. Then $\sim_p$ is a congruence and $(A \times A)/{\sim_p} \in \mathcal{H}(T)$ by Lemma 6.3. Therefore, its restriction to the subalgebra $E_p(A)$, which will also be denoted by $\sim_p$, is a congruence on $E_p(A)$. So, we can define

$$
K_p^{(1)}(A) := E_p(A)/{\sim_p}\,, \tag{7.1}
$$

which we call the *one-step $p$-closure* of $A$. Notice that we write $K_p^{(1)}(A; \underline{I}^o(T))$ instead of $K_p^{(1)}((A; \underline{I}^o(T)))$, and the same convention applies for similar constructs. Since $K_p^{(1)}(A)$ is clearly a submode of $(A \times A)/{\sim_p} \in \mathcal{H}(T)$, Lemma 6.3 implies the following statement.

**Lemma 7.2.** *If $A \in \mathcal{H}(T)$ and $p \in I^o(T)$, then $K_p^{(1)}(A) \in \mathcal{H}(T)$.*

For $A, B \in \mathcal{C}(T)$ such that $B$ is an extension of $A$ (that is, $A$ is a submode of $B$), we say the extension $B$ *closes the $\underline{p}$-eligible pairs of* $A$ if for each $(a, b) \in E_p(A)$ there exists a $c \in B$ such that $b = ac\underline{p}$. If, in addition, for each $c \in B$ there exists an $(a, b) \in E_p(A)$ with $b = ac\underline{p}$, then we say that $B$ *accurately closes the $\underline{p}$-eligible pairs* of $A$.

**Lemma 7.3.** *For $p \in I^o(T)$ and $A \in \mathcal{H}(T)$, the mapping $\psi\colon A \to K_p^{(1)}(A)$, $a \mapsto (a, a)^{\sim_p}$ is an embedding. Moreover, $K_p^{(1)}(A)$ is an extension of $\psi(A)$ that accurately closes the $\underline{p}$-eligible pairs of $\psi(A)$.*

We usually identify $A$ with $\psi(A)$, so Lemma 7.3 simply says that $K_p^{(1)}(A)$ is an extension of $A$ that closes the $\underline{p}$-eligible pairs of $A$.

*Proof of Lemma 7.3.* The second sentence of Lemma 7.1 shows that $\psi$ is indeed a well defined $A \to K_p^{(1)}(A)$ mapping. Let $\psi(a_1) = \psi(a_2)$. Then $(a_1, a_1){\sim_p}(a_2, a_2)$, whence

$$a_1a_2\underline{p}\,a_1a_2\underline{p}\,\underline{p} =^i a_1a_2\underline{p} = a_1a_2\underline{p}\,a_1\underline{p} =^i a_1a_2\underline{p}\,a_1a_1\underline{p}\,\underline{p}.$$

Applying cancellativity twice, we get $a_2 = a_1$. This shows that $\psi$ is injective. Since $\psi$ is the composite of the homomorphism $A \to E_p(A)$, $x \mapsto (x, x)$, and the natural homomorphism $E_p(A) \to E_p(A)/{\sim_p} = K_p^{(1)}(A)$, $y \mapsto y^{\sim_p}$, $\psi$ is a homomorphism. Thus, $\psi$ is an embedding.

Since $b\,ab\underline{p}\,\underline{p} =^i bb\underline{p}ab\underline{p}\,\underline{p} =^e ba\underline{p}bb\underline{p}\,\underline{p} =^i ba\underline{p}\,b\underline{p}$, we conclude that

$$(b, b) \sim_p (a, ab\underline{p}) =^i (aa\underline{p}, ab\underline{p}) = (a, a)(a, b)\underline{p},$$

for all $a, b \in A$. Hence, $\psi(b) = \psi(a)(a, b)^{\sim_p}\,\underline{p}$ for $(a, b) \in E_p(A)$, proving that $K_p^{(1)}(A)$ closes and accurately closes the $\underline{p}$-eligible pairs of $\psi(A)$. □

We conjecture that $K_p^{(1)}(A)$ is not $\underline{p}$-closed in general because of some possible *new* $\underline{p}$-eligible pairs. The next statement is about uniqueness, but we need a stronger statement.

**Proposition 7.4.** *Assume that $A, B \in \mathcal{H}(T)$, and $B$ is an extension of $A$ such that $B$ closes the $\underline{p}$-eligible pairs of $A$. Then the following statements hold.*

(i) *There is a* unique *submode $C$ of $B$ such that $C$ accurately $\underline{p}$-closes the $\underline{p}$-eligible pairs of $A$.*
(ii) *There is exactly one embedding $\alpha_1\colon K_p^{(1)}(A) \to B$ whose restriction to $A$ is the identical mapping.*
(iii) *$C = \alpha_1(K_p^{(1)}(A))$, whence $C$ is isomorphic to $K_p^{(1)}(A)$ over $A$.*
(iv) *If $B$* accurately *closes the $\underline{p}$-eligible pairs of $A$, then $B$ is isomorphic to $K_p^{(1)}(A)$ over $A$.*

*Proof.* Consider the mapping

$$\alpha_0\colon E_p(A) \to B, \quad (a, b) \mapsto c \iff b = ac\underline{p}.$$

By cancellativity, $c$ above is unique, whence $\alpha_0$ is well-defined. It is a homomorphism since in case of $(a_1, b_1) \mapsto c_1$, $(a_2, b_2) \mapsto c_2$ and $q \in I^o(T)$ we have $b_1b_2\underline{q} = a_1c_1\underline{p}\,a_2c_2\underline{p}\,\underline{q} =^e a_1a_2\underline{q}\,c_1c_2\underline{q}\,\underline{p}$, that is, $(a_1, b_1)(a_2, b_2)\underline{q} = (a_1a_2\underline{q}, b_1b_2\underline{q}) \mapsto c_1c_2\underline{q}$.

Next, we show that $\operatorname{Ker}\alpha_0$, the congruence kernel of $\alpha_0$, coincides with $\sim_p$. If $(a_1, b_1) \mapsto c_1$, $(a_2, b_2) \mapsto c_2$, and $(a_1, b_1) \sim_p (a_2, b_2)$, then

$$\begin{aligned} a_1\,a_2c_2\underline{p}\,\underline{p} &= a_1b_2\underline{p} = a_1a_2\underline{p}\,b_1\underline{p} = a_1a_2\underline{p}\,a_1c_1\underline{p}\,\underline{p} \\ &=^e a_1a_1\underline{p}\,a_2c_1\underline{p}\,\underline{p} =^i a_1\,a_2c_1\underline{p}\,\underline{p}. \end{aligned}$$

Hence, applying cancellativity twice we conclude that $c_2 = c_1$. This means that $\sim_p$ is included in $\operatorname{Ker}\alpha_0$. Secondly, we assume that $(a_i, b_i) \mapsto c$, that is,

$b_1 = a_1 c \underline{p}$ and $b_2 = a_2 c \underline{p}$. Then

$$a_1 b_2 \underline{p} = a_1\, a_2 c \underline{p}\, \underline{p} =^i a_1 a_1 \underline{p}\, a_2 c \underline{p}\, \underline{p} =^e a_1 a_2 \underline{p}\, a_1 c \underline{p}\, \underline{p} = a_1 a_2 \underline{p}\, b_1\, \underline{p}$$

gives that $(a_1, b_1) \sim_p (a_2, b_2)$. Hence, $\operatorname{Ker} \alpha_0$ is included in $\sim_p$, so these two congruences are equal. Therefore, since $K_p^{(1)}(A) = E_p(A)/{\sim_p}$, the mapping

$$\alpha_1 \colon K_p^{(1)}(A) \to B, \quad (a,b)^{\sim_p} \mapsto \alpha_0((a,b))$$

is an embedding. Since $a =^i a a \underline{p}$, $\alpha_1$ acts identically on $A$.

To show the uniqueness stated in the second part of Proposition 7.4, assume that $\beta \colon K_p^{(1)}(A) \to B$ is an embedding over $A$. Let $y \in K_p^{(1)}(A) \setminus A$. Then $b = a y \underline{p}$ for some $(a,b) \in E_p(A)$ since $K_p^{(1)}(A)$ accurately closes the $\underline{p}$-eligible pairs of $A$ by Lemma 7.3. Hence, $b = \beta(b) = \beta(a y \underline{p}) = \beta(a)\beta(y)\underline{p} = a\beta(y)\underline{p}$, and the cancellativity of $\underline{p}$ yields the uniqueness of $\beta$. This together with the previously constructed $\alpha_1$ proves the second part of the proposition, and also shows the existence of a $C$ according to the first part.

To prove the uniqueness of $C$, assume that $D$ is another submode of $B$ that accurately closes the $\underline{p}$-eligible pairs of $A$. Take an arbitrary $c \in C \setminus A$, and choose an $(a,b) \in E_p(A)$ with $b = a c \underline{p}$. Since $D$ also closes $(a,b)$, there is a $d \in D$ with $b = a d \underline{p}$. Then the cancellativity of $\underline{p}$ implies $c = d$, whence $c \in D$. This shows $C \subseteq D$, and $D \subseteq C$ follows the same way.

Finally, the last two parts of the proposition clearly follow from the first two. □

## 8. Multi-step closure and the rest of the main proof

In this section, let $\nu = \nu(T)$ denote the *smallest* ordinal number whose cardinality, denoted by $|\nu|$, is larger than that of our fixed ring $T$, that is, $|\nu| > |T|$. Let $\emptyset \neq \Gamma \subseteq I^o(T)$. A transfinite sequence $\vec{p} = (p_\iota : \iota < \nu)$ will be called a *strong* $\Gamma$*-sequence* if the following properties hold:

(i) $\Gamma = \{p_\iota : \iota < \nu\}$;

(ii) each $p \in \Gamma$ is cofinal in the sequence $\vec{p}$, that is, for arbitrary $p \in \Gamma$ and $\iota < \nu$, there exists an ordinal $\lambda$ such that $\iota < \lambda < \nu$ and $p = p_\lambda$.

**Lemma 8.1.** *There exists a strong* $\Gamma$*-sequence.*

*Proof.* Let $U = (U; <)$ and $V = (V; <)$ be well-ordered sets such that $V$ is of order type $\nu$ and $|U| = |\Gamma|$. Take a bijection $\tau \colon U \to \Gamma$, $u \mapsto p_u$. Consider the anti-lexicographic ordering $<$ on $U \times V$. That is, for $(u_1, v_1), (u_2, v_2) \in U \times V$, let $(u_1, v_1) < (u_2, v_2)$ mean that either $v_1 < v_2$, or $v_1 = v_2$ and $u_1 < u_2$. Then $(U \times V; <)$ is a well-ordered set. Let $\eta$ denote its order type. Since $\Gamma$ is clearly cofinal in $\vec{p} := \big(p_u : (u,v) \in U \times V\big)$, it suffices to show that $\eta = \nu$.

By the definition of $\nu$, $|\{y : y < v\}| < |\nu|$. That is $|\{y : y < v\}| \leq |T|$ holds for every $v \in V$. Hence, each initial segment $\{(u', v') : (u', v') < (u, v)\}$ of $U \times V$ is of cardinality at most $|\Gamma| \cdot |T| = |T|$. This implies that $\eta \leq \nu$. On the other hand, $|U \times V| \geq |V| = |\nu|$, whence $\nu \leq \eta$. □

**Definition 8.2.** Let $\vec{p} = (p_\iota : \iota < \nu)$ be a fixed strong $\Gamma$-sequence, and let $A \in \mathcal{H}(T)$. We are going to define a *directed system* of modes $\tilde{K}_{\vec{p}}^{(\lambda)}(A)$, $\lambda \le \nu$, with embeddings $\psi_{\xi,\lambda}\colon \tilde{K}_{\vec{p}}^{(\xi)}(A) \to \tilde{K}_{\vec{p}}^{(\lambda)}(A)$, $\xi \le \lambda \le \nu$. As usual, "directed system" means that $\psi_{\xi,\xi}$ will be the identity mapping and $\xi \le \zeta \le \lambda \le \nu$ will imply $\psi_{\xi,\lambda} = \psi_{\zeta,\lambda} \circ \psi_{\xi,\zeta}$.

Let $\tilde{K}_{\vec{p}}^{(0)}(A) := A$. If $\lambda = \mu + 1$ is a successive ordinal, then let

$$\begin{aligned}
&\tilde{K}_{\vec{p}}^{(\lambda)}(A) := K_{p_\mu}^{(1)}\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big)\\
&\qquad\qquad \text{or , equivalently, } \tilde{K}_{\vec{p}}^{(\mu+1)}(A) := K_{p_\mu}^{(1)}\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big),\\
&\psi_{\mu,\lambda} : \tilde{K}_{\vec{p}}^{(\mu)}(A) \to \tilde{K}_{\vec{p}}^{(\lambda)}(A), \quad a \mapsto (a,a)^{\sim_{p_\mu}}, \text{ and}\\
&\psi_{\xi,\lambda} : \tilde{K}_{\vec{p}}^{(\xi)}(A) \to \tilde{K}_{\vec{p}}^{(\lambda)}(A) \text{ is the composite } \psi_{\mu,\lambda} \circ \psi_{\xi,\mu} \text{ for } \xi < \mu.
\end{aligned} \tag{8.1}$$

Note that $\psi_{\mu,\lambda}$ is the embedding $\psi$ from Lemma 7.3 and $\tilde{K}_{\vec{p}}^{(1)}(A) = K_{p_0}^{(1)}(A)$.

If $\lambda$ is a limit ordinal and $\lambda \le \nu$, then let

$$\tilde{K}_{\vec{p}}^{(\lambda)}(A) := \bigcup_{\mu<\lambda} \tilde{K}_{\vec{p}}^{(\mu)}(A) \text{ (directed union), and}$$

$$\psi_{\mu,\lambda} := \bigcup_{\mu\le\iota<\lambda} \psi_{\mu,\iota}\,, \text{ if } \mu < \lambda.$$

Notice that a category theorists would say that, for a limit ordinal $\lambda \le \nu$, $\tilde{K}_{\vec{p}}^{(\lambda)}(A)$ is the *directed colimit* of the functor $F$ from the small category $\{\mu : \mu < \lambda\}$ to $\mathcal{V}(T)$ such that $F(\iota) = \tilde{K}_{\vec{p}}^{(\iota)}(A)$ and $F(\iota \to \mu) = \psi_{\iota,\mu}$ for $\iota < \mu < \lambda$, see Mac Lane [6, pages 67–68]. Finally, define

$$K_\Gamma^\infty(A) := \tilde{K}_{\vec{p}}^{(\nu)}(A) \text{ and } K_{\text{all}}^\infty(A) := K_{\underline{I^o(T)}}^\infty(A).$$

It follows from the forthcoming Lemmas 8.3 and 8.4 that, up to isomorphism over $A$, $K_\Gamma^\infty(A)$ does not depend on the actual choice of the strong $\Gamma$-sequence $\vec{p}$. The notion of $\underline{\Gamma}$-closures has been defined right before Theorem 2.3.

**Lemma 8.3.** *If $A \in \mathcal{H}(T)$ and $\emptyset \ne \Gamma \subseteq I^o(T)$, then the definition of $K_\Gamma^\infty(A)$ makes sense, $K_\Gamma^\infty(A)$ belongs to $\mathcal{H}(T)$, and it is a $\underline{\Gamma}$-closure of $A$.*

*Proof.* For modes $X$ and $Y$, $X \le Y$ will denote that $X$ is a submode of $Y$. Clearly, $\tilde{K}_{\vec{p}}^{(0)}(A) = A \in \mathcal{H}(T)$. If $\tilde{K}_{\vec{p}}^{(\mu)}(A) \in \mathcal{H}(T)$, then Lemma 7.2 implies that $\tilde{K}_{\vec{p}}^{(\mu+1)}(A) =^{(8.1)} K_{p_\mu}^{(1)}\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big)$ belongs to $\mathcal{H}(T)$. Quasivarieties are closed with respect to directed colimits since this construction clearly preserves the quasi-identities. Hence, $K_\Gamma^\infty(A)$ belongs to $\mathcal{H}(T)$.

Next, let $p \in \Gamma$ and let $(a,b) \in E_p\big(K_\Gamma^\infty(A)\big)$. We want to show that $(a,b)$ is $\underline{p}$-closed in $K_\Gamma^\infty(A)$. We know that $\nu$ is a limit ordinal, so there is a least $\iota < \nu$ such that $a, b \in \tilde{K}_{\vec{p}}^{(\iota)}(A)$. Clearly, $(a,b\rangle = \{ab\underline{p} : p \in I^o(T)\} \cup \{b\}$ (both in $\tilde{K}_{\vec{p}}^{(\iota)}(A)$ and in $K_\Gamma^\infty(A)$). Let $x_p := ab\underline{p}$. Then $\{x_p : p \in I^o(T)\} = (a,b\rangle \setminus \{b\}$, but $b$ is not interesting from our perspective here. Since $(a,b)$ is $\underline{p}$-closed in $K_\Gamma^\infty(A)$, for each $p \in I^o(T)$ there is a $y_p \in K_\Gamma^\infty(A)$ such that $b = x_p y_p \underline{p}$.

Hence, there is a least ordinal $\lambda_p$ such that $\iota \leq \lambda_p < \nu$ and $y_p \in \tilde{K}_{\vec{p}}^{(\lambda_p)}(A)$. Let $\mu_{ab} := \sum_{p \in I^o(T)} \lambda_p$. Then $|\mu_{ab}| = \sum_{p \in I^o(T)} |\lambda_p| \leq |T| \cdot |T| = |T|$, so $\mu_{ab} < \nu$. Clearly, $(a,b)$ is $\underline{p}$-eligible in $\tilde{K}_{\vec{p}}^{(\eta)}(A)$ for every $\eta$ such that $\mu_{ab} \leq \eta < \nu$. Since $p$ is cofinal in the $\Gamma$-sequence, we can choose this $\eta$ such that $p_\eta = p$. Then, since $(a,b)$ is $\underline{p} = \underline{p_\eta}$-eligible in $\tilde{K}_{\vec{p}}^{(\eta)}(A)$, the construction (see (8.1) and Lemma 7.3) yields a $c \in \tilde{K}_{\vec{p}}^{(\eta+1)}(A) \subseteq K_\Gamma^\infty(A)$ such that $b = ac\underline{p_\eta} = ab\underline{p}$. Thus, $K_\Gamma^\infty(A)$ is $\underline{\Gamma}$-closed.

Finally, suppose that $A \leq B \leq K_\Gamma^\infty(A)$ such that $B$ is a $\underline{\Gamma}$-closed extension of $A$. We have to prove that $B = K_\Gamma^\infty(A)$. It suffices to show, by induction on $\mu$, that $\tilde{K}_{\vec{p}}^{(\mu)}(A) \subseteq B$ for all $\mu < \nu$. For $\mu = 0$, this is trivial since $\tilde{K}_{\vec{p}}^{(0)}(A) = A$. Assume that $\tilde{K}_{\vec{p}}^{(\mu)}(A) \subseteq B$. By the second part of Proposition 7.4, there is a (unique) embedding $\beta$ of $\tilde{K}_{\vec{p}}^{(\mu+1)}(A) =^{(8.1)} K_{p_\mu}^{(1)}\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big)$ into $B$ such that $\beta$ acts identically on $\tilde{K}_{\vec{p}}^{(\mu)}(A)$. This $\beta$ is also a $\tilde{K}_{\vec{p}}^{(\mu+1)}(A) \to K_\Gamma^\infty(A)$ embedding. On the other hand, $\alpha\colon \tilde{K}_{\vec{p}}^{(\mu+1)}(A) \to K_\Gamma^\infty(A)$, $x \mapsto x$, is also an embedding that acts identically on $\tilde{K}_{\vec{p}}^{(\mu)}(A)$. Hence, the second part of Proposition 7.4, applied to $\tilde{K}_{\vec{p}}^{(\mu)}(A)$ and $K_\Gamma^\infty(A)$, yields that $\alpha$ and $\beta$ are the same mappings. Hence,

$$\tilde{K}_{\vec{p}}^{(\mu+1)}(A) = \alpha\big(\tilde{K}_{\vec{p}}^{(\mu+1)}(A)\big) = \beta\big(\tilde{K}_{\vec{p}}^{(\mu+1)}(A)\big) \subseteq B,$$

indeed. The induction step for limit ordinals is trivial. □

**Lemma 8.4.** *Let $A \in \mathcal{H}(T)$, and let $\emptyset \neq \Gamma \subseteq I^o(T)$. Then, in $\mathcal{H}(T)$, the $\underline{\Gamma}$-closure of $A$ is unique up to isomorphism over $A$.*

*Proof.* We know from Lemma 8.3 that $K_\Gamma^\infty(A)$ is a $\underline{\Gamma}$-closure of $A$. Let $B \in \mathcal{H}(T)$ be another $\Gamma$-closure of $A$. We are going to define a sequence of embeddings $\beta_\iota\colon \tilde{K}_{\vec{p}}^{(\mu)}(A) \to B$, $\mu \leq \nu$, by induction on $\mu$ such that $\beta_\iota$ is a restriction of $\beta_\lambda$ for all $\iota < \lambda \leq \nu$. (Here $\tilde{K}_{\vec{p}}^{(\nu)}(A)$ is understood as $K_\Gamma^\infty(A)$.) Let $\beta_0$ be the identical mapping.

Assume that $\beta_\mu$ is already defined. Then $\beta_\mu\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big)$ is a submode of $B$. Since $B$ is $\underline{p_\mu}$-closed, the second part of Proposition 7.4 gives a unique embedding

$$\alpha_1\colon K_{p_\mu}^{(1)}\big(\beta_\mu\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big)\big) \to B.$$

Clearly, we can extend the isomorphism $\beta_\mu\colon \tilde{K}_{\vec{p}}^{(\mu)}(A) \to \beta_\mu\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big)$ to a unique isomorphism $\hat{\beta}_\mu\colon K_{p_\mu}^{(1)}\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big) \to K_{p_\mu}^{(1)}\big(\beta_\mu\big(\tilde{K}_{\vec{p}}^{(\mu)}(A)\big)\big)$ in the natural way. Since the domain of $\hat{\beta}_\mu$ is just $\tilde{K}_{\vec{p}}^{(\mu+1)}(A)$, the composite mapping $\beta_{\mu+1} = \alpha_1 \circ \hat{\beta}_\mu$ is an embedding that extends $\beta_\mu$.

If $\lambda \leq \nu$ is a limit ordinal, then let $\beta_\lambda = \bigcup_{\mu<\lambda} \beta_\mu$.

Since $\beta_\nu$ is a $K_\Gamma^\infty(A) \to B$ embedding, $\beta_\nu\big(K_\Gamma^\infty(A)\big)$ is a $\underline{\Gamma}$-closed submode of $B$. Hence, since $B$ is a $\underline{\Gamma}$-closure of $A$, we conclude that $\beta_\nu\big(K_\Gamma^\infty(A)\big) = B$. Consequently, $\beta_\nu$ is surjective, whence it is an isomorphism. □

*The proof of Theorem 2.3.* Lemmas 6.1, 6.2, 8.3 and 8.4. □

## 9. When $T$ is more or less a field

This section deals with the question whether $\underline{p}$-closed members of $\mathcal{C}(T)$ are necessarily $\underline{q}$-closed, for given $p, q \in I^o(T)$. Since $p$ and $q$ can be algebraically independent over $\mathbb{Q}$, we conjecture that the answer is negative in general. However, if $T$ contains certain quotients of its elements, then we can give a satisfactory answer.

**Lemma 9.1.** *Let $p, q \in I^o(T)$ with $p < q$. Suppose that*

(1) $\hat{r}(p,q) := (q-p)/((1-p)q) \in T$, *and*
(2) $\hat{u}(p,q) := p/(p+q) \in T$.

*Then every $\underline{p}$-closed member of $\mathcal{C}(T)$ is $\underline{q}$-closed.*

Note that (1) and (2), on which the proof relies, do not hold automatically. For example, let us consider $\mathbb{D} = \mathbb{Z}[1/2] = \{a/2^k : a \in \mathbb{Z},\ k \in \mathbb{N}\}$, the *ring of dyadic rational numbers.* Then $1/4$ and $1/2$ are in $I^o(\mathbb{D})$, but $\hat{r}(1/4, 1/2) = 2/3 \notin \mathbb{D}$ and $\hat{u}(1/4, 1/2) = 1/3 \notin \mathbb{D}$.

*Proof of Lemma 9.1.* Let $A \in \mathcal{C}(T)$ be $\underline{p}$-closed. Let $r = \hat{r}(p,q)$ and $u = \hat{u}(p,q)$. They belong to $T$ by (1) and (2), whence $0 < p < q < 1$ implies that $r, u \in I^o(T)$. We will use the fact that, under the assumptions $p < q$, $r = \hat{r}(p,q)$, and $u = \hat{u}(p,q)$, the identities (9.1) and (9.2) below hold in $\mathcal{V}(T)$. Hence, they hold in $A$ as well.

Let $(a,b)$ be a $\underline{q}$-eligible pair. We have to find a $c \in A$ such that $b = ac\underline{q}$. Let $a_1 = ab\underline{r}$. We claim that $(a_1, b)$ is $\underline{p}$-eligible. Suppose that $x \in (a_1, b\rangle$ and, without loss of generality, $x \neq b$ and $a \neq b$. By Lemma 2.2, $x = a_1 b\underline{s}$ for some $s \in I^o(T)$. Since $ab\underline{s} \in (a,b\rangle$ and $(a,b)$ is $\underline{q}$-eligible, there exists a $y \in A$ with $ab\underline{s}\, y\underline{q} = b$. Hence, the identity

$$(b)\,(ab\underline{s}\, y\underline{q}\,)\,\underline{u} = (ab\underline{r}\, b\underline{s}\, y\underline{p}\,)\,(b)\,\underline{u} \tag{9.1}$$

together with the cancellativity of $\underline{u}$ yields the first equation of the following formula:

$$b = ab\underline{r}\, b\underline{s}\, y\underline{p} = a_1 b\underline{s}\, y\underline{p} = xy\underline{p}.$$

Hence, $(a_1, b)$ is $\underline{p}$-eligible. Since $A$ is $\underline{p}$-closed, there exists a $c$ with $a_1 c\underline{p} = b$, that is, $ab\underline{r}\, c\underline{p} = b$. Hence, the identity

$$(ab\underline{r}\, c\underline{p})\,(b)\,\underline{u} = (b)\,(ac\underline{q})\,\underline{u} \tag{9.2}$$

and the cancellativity of $\underline{u}$ imply $ac\underline{q} = b$. □

**Proposition 9.2.** *Let $T$ be subfield of $\mathbb{R}$, let $p \in I^o(T)$, and let $A \in \mathcal{C}(T)$. Then $A$ is $\underline{p}$-closed if and only if $A$ is closed.*

*Proof.* Assume that $A$ is $\underline{p}$-closed. We have to show that it is $\underline{q}$-closed for all $q \in I^o(T)$. In virtue of Lemma 9.1, we know that $X := \{q \in I^o(T) : A$ is $\underline{q}$-closed$\}$ is an order filter (in other words, up-set) in $(I^o(T); \leq)$. Hence, it suffices to show that whenever $p \in X$ is greater than $1/2$, then $1-p$ also belongs to $X$. So, we assume that $1/2 < p \in I^o$, $A \in \mathcal{C}(T)$ is $\underline{p}$-closed and $q := 1-p$. We have to show that $A$ is $\underline{q}$-closed.

Let $(a,b)$ be a $\underline{q}$-eligible pair, that is, $(a,b) \in E_q(A)$. We have to show that $(a,b)$ is $\underline{q}$-closed. That is, we have to find a $c \in A$ with $b = ac\underline{q}$. The case $a = b$ is evident, so we assume that $a \neq b$. Define

$$r := q/p \in I^o(T) \text{ and } d := ab\underline{r}. \tag{9.3}$$

Since $(a,b)$ is $\underline{q}$-eligible and $A$ is cancellative, there is a unique $e \in A$ with

$$b = de\underline{q}. \tag{9.4}$$

With the notation $h = 1/2 \in I^o(T)$ (coming from "$\underline{\text{h}}$alf") and $w = 1/(2p+1) \in I^o(T)$, and with the previous meaning of $p, q, r$, consider the following identity:

$$(ab\underline{r})\,(b)\underline{h}\,(de\underline{q})\,\underline{w} = (d)\,(ae\underline{r})\,\underline{h}\,(b)\,\underline{w}. \tag{9.5}$$

This identity holds in $\mathcal{V}(T)$. In the present situation, the first and the third parenthesized subterms on both sides give the same elements by (9.3) and (9.4). Hence, the cancellativity of $\underline{w}$ and $\underline{h}$ yields that

$$b = ae\underline{r}. \tag{9.6}$$

We want to show that $(a,e)$ is a $\underline{p}$-eligible pair. Let $x \in (a,e\rangle$, we can assume that $x \neq e$. Then

$$x = ae\underline{s} \tag{9.7}$$

for some $s \in I^o(T)$. Define

$$z := a\,ax\underline{r}\,\underline{r}. \tag{9.8}$$

Since

$$\begin{aligned} z &=^{(9.8)} a\,ax\underline{r}\,\underline{r} =^{(9.7)} a\,a\,ae\underline{s}\,\underline{r}\,\underline{r} =^{i} a\,(aa\underline{s}\,ae\underline{s}\,\underline{r})\,\underline{r} \\ &=^{e} a\,aa\underline{r}\,(ae\underline{r})\,\underline{s}\,\underline{r} =^{(9.6)} a\,aa\underline{r}\,b\underline{s}\,\underline{r} =^{i} a\,ab\underline{s}\,\underline{r}, \end{aligned}$$

we get that $z \in \langle a,b\rangle$. In fact, $z$ belongs to $(a,b\rangle$, because otherwise $a\,aa\underline{r}\,\underline{r} =^{i} a = z =^{(9.8)} a\,ax\underline{r}\,\underline{r}$, and the cancellativity of $\underline{r}$ would imply $x = a$, contradicting $x \in (a,e\rangle$. Therefore the $\underline{q}$-eligibility of $(a,b)$ yields an element $y \in A$ such that

$$b = zy\underline{q}. \tag{9.9}$$

If $h = 1/2$, $q = 1-p$ and $r = q/p$, as before, then the identity

$$(ab\underline{r}\,e\underline{q})\,(b)(xy\underline{p})\,\underline{q}\,\underline{h} = (b)\,(a\,ax\underline{r}\,\underline{r}\,y\underline{q})(e)\,\underline{q}\,\underline{h} \tag{9.10}$$

holds in $\mathcal{V}(T)$. In the present situation, the first parenthesized elements (on the left-hand and right-hand sides of (9.10)) are equal by (9.3) and (9.4). The second parenthesized elements are equal by (9.8) and (9.9). Therefore, by the cancellativity of $\underline{h}$ and $\underline{q}$, the third parenthesized elements are also equal, that is, $x y \underline{p} = e$. This shows that $(a, e) \in E_p(A)$. Hence, there exists an element $c \in A$ such that

$$e = ac\underline{p}. \tag{9.11}$$

Finally, for $q = 1 - p$ and $r = q/p$ as before, the identity

$$a\, ac\underline{p}\, \underline{r} = ac\underline{q} \tag{9.12}$$

holds in $\mathcal{V}(T)$. Hence, we conclude that $ac\underline{q} = a\, ac\underline{p}\, \underline{r} =^{(9.11)} ae\underline{r} =^{(9.6)} b$. This means that $(a, b)$ is $\underline{q}$-closed. □

**Proposition 9.3.** *If $T$ is a subfield of $\mathbb{R}$, $A \in \mathcal{C}(T)$, and $p \in I^o(T)$, then $K^\infty_{\text{all}}(A) = K^\infty_{\{p\}}(A)$.*

*Proof.* Apply Theorem 2.3 and Proposition 9.2. □

## 10. Relating our closure to the topological one

*Proof of Proposition 2.4.* We will consider $\mathbb{R}^n$ as an *affine space* ${}_{\mathbb{R}}\mathbb{R}^n$, which is the full idempotent reduct of the vector space (denoted the same way). The *affine subspaces* are exactly the cosets of the subspaces of the vector space. The elements of the affine space ${}_{\mathbb{R}}\mathbb{R}^n$ will be called *affine points* while the corresponding elements of the vector space ${}_{\mathbb{R}}\mathbb{R}^n$ are called *vectors*. Sometimes we consider $\mathbb{R}^n$ even as a Euclidean space. Let $a_0, \dots, a_k \in {}_{\mathbb{R}}\mathbb{R}^n$ be affine points. We say that these points are *independent*, if they are pairwise distinct and none of them belongs to the affine subspace spanned (generated) by the rest. Alternatively, they are independent iff $\{a_0, \dots, a_k\}$ is a free generating set of the affine space ${}_{\mathbb{R}}\mathbb{R}^n$. Yet another definition is that $a_0, \dots, a_k$ are independent iff they are the vertices of a $k$-dimensional simplex. Clearly, the independence of $a_0, \dots, a_k \in {}_{\mathbb{R}}\mathbb{R}^n$ implies $k \le n$.

Let $k$ denote the largest integer such that $H$ contains $k+1$ independent affine points. Fix $k+1$ independent affine points, say, $v_0, \dots, v_k$ in $H$, and let $V$ denote the affine subspace they span. We claim that $H \subseteq V$. Indeed, otherwise we could take an arbitrary affine point $v_{k+1}$ in $H \setminus V$, and, as an easy consequence of the Exchange Axiom, $v_0, \dots, v_k, v_{k+1}$ would be an independent set of more than $k+1$ elements.

The topological notion of an open set will be used within $V$, which we consider a Euclidean space of dimension $k$ with the topology induced by the usual metric. To emphasize this, we will say that a subset of $V$ is "*open in $V$*". Let $H'$ denote the *interior* of $H$, understood again in $V$. If $k = 0$, then $H$ is a singleton set, whence the statement of the proposition is trivial. Hence, we can assume that $k \ge 1$. Then $H'$ is a non-empty open set in $V$ since the interior of the simplex determined by $v_0, \dots, v_k$ is a subset of $H'$. Clearly, $H' \subseteq H \subseteq H'^{tc}$, whence $H'^{tc} = H^{tc}$.

Since $V$ is still subset of $\mathbb{R}^n$, we can form $H^{tc} \cap S^n$, which is a subset of $V$. For short, we will denote the barycentric algebras $\big(H^{tc} \cap S^n; \underline{I}^o(T)\big)$, $\big(H \cap S^n, \underline{I}^o(T)\big)$, and $\big(H' \cap S^n, \underline{I}^o(T)\big)$ by $H^{tc}_S = H'^{tc}_S$, $H_S$, and $H'_S$, respectively.

All of them belong to $\mathcal{Q}(T)$ by Lemma 3.1, and therefore also to $\mathcal{H}(T)$ by Lemma 6.2 (or by Theorem 2.3).

For $p \in I^o(T)$, let

$$\underline{p}^{(\mathrm{i})} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \text{ where } (x, y) \mapsto -\frac{1-p}{p} \cdot x + \frac{1}{p} \cdot y.$$

This operation is called the *right inverse* of $\underline{p}$. Indeed, for any $x, y, z \in \mathbb{R}^n$,

$$xy\underline{p} = z \iff y = xz\,\underline{p}^{(\mathrm{i})}\,.$$

Note that $H$ is not closed with respect to $\underline{p}^{(\mathrm{i})}$ in general. Note also that formally $\underline{p}^{(\mathrm{i})}$ is $\underline{1/p}$. However, we will not use the notation $\underline{1/p}$, since $1/p \notin I^o(T)$ for $p \in I^o(T)$.

First we show that, for every $p \in I^o(T)$,

$$H_S^{tc} \text{ is } \underline{p}\text{-closed.} \tag{10.1}$$

Let $(a, b) \in E_p(H_S^{tc})$ such that $a \neq b$. Then there is unique $c \in \mathbb{R}^n$ such that $ac\underline{p} = b$, that is, $c = ab\,\underline{p}^{(\mathrm{i})}$. Assume for a contradiction that $c$ is not in $H_S^{tc}$. Then there is a small neighborhood $U$ of $c$ in $V$ such that $U \cap H_S^{tc} \subseteq U \cap H^{tc} = \emptyset$, because $V \setminus H^{tc}$ is open in $V$. (By a neighborhood of $c$ we mean a superset of $\{c\}$ that is open in $V$.) Since $x \to xb\,\underline{p}^{(\mathrm{i})}$ is a continuous $V \to V$ mapping, $U' := \{x \in V : xb\,\underline{p}^{(\mathrm{i})} \in U\}$ is an open set in $V$. We get from $ab\,\underline{p}^{(\mathrm{i})} = c \in U$ that $a \in U'$. Since $T$ is distinct from $\mathbb{Z}$, $I^o(T)$ is everywhere dense in the closed interval $I(\mathbb{R}) = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Hence, we can choose a sufficiently small $q \in I^o(T)$ such that $x_0 := ab\underline{q} \in (a, b\rangle \cap U'$. Let $y_0 = x_0 b\,\underline{p}^{(\mathrm{i})}$, then $x_0 y_0 \underline{p} = b$. It follows from the definition of $U'$ that $y_0 \in U$. On the other hand, $y_0 \in H_S^{tc}$ since $(a, b) \in E_p(H_S^{tc})$. This yields that $y_0 \in H^{tc} \cap U = \emptyset$, a contradiction proving (10.1).

Consider the mapping

$$\alpha \colon E_p(H_S) \to H_S^{tc}, \quad (a, b) \mapsto ab\,\underline{p}^{(\mathrm{i})}\,.$$

First of all, we have to check that $(a, b) \in E_p(H_S)$ implies $ab\,\underline{p}^{(\mathrm{i})} \in H_S^{tc}$. Since $S$ is a field, $ab\,\underline{p}^{(\mathrm{i})} \in S^n$. The $\underline{p}$-eligibility of $(a, b)$ yields that $xb\,\underline{p}^{(\mathrm{i})} \in H_S$ for all $x \in (a, b\rangle$. Since $\underline{p}^{(\mathrm{i})}$ is continuous and $H^{tc}$ is closed with respect to limits,

$$\alpha(a, b) = ab\,\underline{p}^{(\mathrm{i})} = (\lim_{x \to a,\ x \in (a,b\rangle} x) b\,\underline{p}^{(\mathrm{i})} = \lim_{x \to a,\ x \in (a,b\rangle} (xb\,\underline{p}^{(\mathrm{i})}) \in H^{tc}.$$

Hence, $\alpha(a, b) \in H_S^{tc}$, indeed.

To show that $\alpha$ is a homomorphism, assume that $(a_1, b_1), (a_2, b_2) \in E_p(H_S)$, $q \in I^o(T)$, and $\alpha(a_i, b_i) = a_i b_i\,\underline{p}^{(\mathrm{i})} = c_i \in S^n$ for $i = 1, 2$. Then $b_i = a_i c_i \underline{p}$ for $i = 1, 2$. Since $b_1 b_2 \underline{q} = a_1 c_1 \underline{p}\, a_2 c_2 \underline{p}\, \underline{q} =^e a_1 a_2 \underline{q}\, c_1 c_2 \underline{q}\, \underline{p}$, we obtain that $c_1 c_2 \underline{q} = a_1 a_2 \underline{q}\, b_1 b_2 \underline{q}\, \underline{p}^{(\mathrm{i})}$. Hence, $\alpha((a_1, b_1)(a_2, b_2)\,\underline{q}) = \alpha(a_1 a_2 \underline{q}, b_1 b_2 \underline{q}) = c_1 c_2 \underline{q} = \alpha(a_1, b_1)\alpha(a_2, b_2)\underline{q}$, showing that $\alpha$ is a homomorphism.

Next, we show that $\alpha$ is surjective. For $c \in H'_S$, $c$ is clearly the $\alpha$-image of $(c, c)$, which belongs to $E_p(H_S)$ by Lemma 7.1. So, assume that $c \in H_S^{tc} \setminus H'_S$, and fix an element $a \in H'_S$. Then $b := ac\underline{p} \in H_S^{tc}$. The convexity of $H^{tc}$ easily implies that $(a,b) \in E_p(H_S^{tc})$. Take the hyperplane (that is, an affine subspace of dimension $k-1$) $D$ of $V$ through $a$ that is orthogonal to the line $\ell$ through $a$ and $c$. Since $H'$ is an open set in $V$, there is a small positive $\varrho$ such that the $(k-1)$-dimensional closed sphere $G$ in $D$ with center $a$ and radius $\varrho$ is a subset of $H'$. By the convexity of $H^{tc}$, all points of the (bounded) cone determined by $c$ and $G$ (in $V$) belong to $H^{tc}$. Since all points of $\ell$ strictly between $a$ and $c$ belong to the interior of this cone, and therefore to the interior of $H^{tc}$, it follows in a straightforward way that these points belong to $H$. This fact together with $(a,b) \in E_p(H_S^{tc})$ yields that $(a,b) \in E_p(H_S)$. So, $(a,b)$ is in the domain of $\alpha$, and $\alpha(a,b) = ab\underline{p}^{(\mathrm{i})} = c$ proves the surjectivity of $\alpha$.

Next, we show that the congruence kernel $\operatorname{Ker}\alpha$ of $\alpha$ coincides with the congruence $\sim_p$ on $E_p(H_S)$. Let $(a,b),(a',b') \in E_p(H_S)$, and denote $\alpha(a,b)$ and $\alpha(a',b')$ by $c$ and $c'$, respectively. Then $b = ac\underline{p}$ and $b' = a'c'\underline{p}$. First, assume that $(a,b) \sim_p (a',b')$. Then

$$\begin{aligned} aa'\underline{p}\,ac\underline{p}\,\underline{p} = aa'\underline{p}\,b\underline{p} &=^{(4.1)} ab'\underline{p} = a\,a'c'\underline{p}\,\underline{p} \\ &=^{i} aa\underline{p}\,a'c'\underline{p}\,\underline{p} =^{e} aa'\underline{p}\,ac'\underline{p}\,\underline{p}. \end{aligned}$$

Hence, by applying the cancellativity of $\underline{p}$ twice, we obtain $c = c'$. This means that $\sim_p$ is included in $\operatorname{Ker}\alpha$. Conversely, assume that $c = c'$, that is, $\operatorname{Ker}\alpha$ collapses $(a,b)$ and $(a',b')$. Then

$$aa'\underline{p}\,b\underline{p} = aa'\underline{p}\,ac\underline{p}\,\underline{p} =^{e} aa\underline{p}\,a'c\underline{p}\,\underline{p} =^{i} a\,a'c\underline{p}\,\underline{p} = a\,a'c'\,\underline{p}\,\underline{p} = ab'\underline{p}$$

means that $(a,b) \sim_p (a',b')$. Thus, $\operatorname{Ker}\alpha$ is $\sim_p$.

Finally, taking the canonical embedding given by Lemma 7.3 into account, the homomorphism theorem yields that $K_p^{(1)}(H_S) =^{(7.1)} E_p(H_S)/{\sim_p}$ is isomorphic to $H_S^{tc}$ over $H_S$. This together with (10.1), where $p$ was an arbitrary element of $I^o(T)$, imply that $K_p^{(1)}(H_S)$ is closed. In particular, $K_p^{(1)}(H_S)$ closes its own $\underline{p}$-eligible pairs. Either from Lemma 7.3 combined with $H_S \subseteq K_p^{(1)}(H_S)$, or from Lemma 7.1 applied for $a = b := c$, we obtain that for each $c \in K_p^{(1)}(H_S)$, there is a pair $(a,b) \in E_p(K_p^{(1)}(H_S))$ such that $b = ac\underline{p}$. Hence, $K_p^{(1)}(H_S)$ accurately closes its own $\underline{p}$-eligible pairs. So we infer from Proposition 7.4 that $\tilde{K}_{\vec{p}}^{(2)}(H_S) = K_p^{(1)}(K_p^{(1)}(H_S)) = K_p^{(1)}(H_S)$. This implies $\tilde{K}_{\vec{p}}^{(\lambda)}(H_S) = K_p^{(1)}(H_S)$ for all $\lambda$. Therefore $K_{\mathrm{all}}^{\infty}(H_S) = K_p^{(1)}(H_S) \cong H_S^{tc}$, over $H_S$. □

*Proof of Corollary 2.5.* We are going to use Proposition 2.4, applied for $T = S = \mathbb{R}$. Since $K_{\mathrm{all}}^{\infty}\big(H; \underline{I}^o(\mathbb{R})\big)$ is isomorphic to $\big(H^{tc}; \underline{I}^o(\mathbb{R})\big)$ over $H$, we conclude that $\big(H; \underline{I}^o(\mathbb{R})\big)$ is closed in our sense iff $\big(H; \underline{I}^o(\mathbb{R})\big)$ equals $K_{\mathrm{all}}^{\infty}\big(H; \underline{I}^o(\mathbb{R})\big)$ iff $\big(H; \underline{I}^o(\mathbb{R})\big)$ equals $\big(H^{tc}; \underline{I}^o(\mathbb{R})\big)$ iff $H$ is topologically closed.

Let $p \in I^o(\mathbb{R})$. We obtain from Proposition 2.4 again that $\big(H; \underline{I}^o(\mathbb{R})\big)$ is $\underline{p}$-closed iff $\big(H; \underline{I}^o(\mathbb{R})\big)$ equals $K_p^{(1)}\big(H; \underline{I}^o(\mathbb{R})\big)$, and the statement follows from $K_p^{(1)}\big(H; \underline{I}^o(\mathbb{R})\big) = K_{\mathrm{all}}^{\infty}\big(H; \underline{I}^o(\mathbb{R})\big)$ and the first part of the proof. □

Let $A \in \mathcal{V}(T)$ and $a \in A$. Then $a$ is called a *wall element* if

$$xy\underline{p} = a \text{ implies } x = y = a, \text{ for all } p \in I^o(T), \tag{10.2}$$

that is, if $\{a\}$ is a wall according to [16].

*Proof of Lemma 2.7.* Let $F$ denote $\langle T\rangle_{\text{field}}$. Consider $\mathbb{R}^2$ and $S^2$ as vector spaces ${}_F\mathbb{R}^2$ and ${}_FS^2$ over $F$, respectively. Let $\varrho$ and $\sigma$ denote the smallest ordinal numbers whose cardinalities are the dimension of ${}_F\mathbb{R}^2$ and that of ${}_FS^2$, respectively. Then $\omega \le \sigma$ by the assumption, and $\sigma \le \varrho$, evidently.

Take a basis $(e_\lambda : \lambda < \sigma)$ of the vector space ${}_FS^2$. We can assume that each $e_\lambda$ belongs to $C$. Indeed, otherwise $e_\lambda$ can be replaced by $e_\lambda/m$ for a sufficiently large $m = m(\lambda) \in \mathbb{N} \subseteq F$.

Our plan is to construct a submode $B$ of $\big(S^2; \underline{I}^o(T)\big)$ that is everywhere dense in $\mathbb{R}^2$ in the usual topological sense. A square in $\mathbb{R}^2$ will be called a *rational square* if both coordinates of each of the four vertices belong to $\mathbb{Q}$. There are countably many rational squares. Let $(U_i : i < \omega)$ be an enumeration of them. (Here $U_i$ is understood as a closed convex subset of $\mathbb{R}^2$.) We claim that the vector space

$${}_FS^2 \text{ has a basis } (f_\iota : \iota < \sigma) \text{ such that } f_i \in U_i \text{ for all } i < \omega. \tag{10.3}$$

Let $f_0 \in U_0 \setminus \{0\}$ be arbitrary. Next, assume that $i < \omega$, and $f_0, \dots, f_i$ are defined and they are linearly independent. Let $W_i$ denote the subspace of ${}_FS^2$ spanned by $\{f_0, \dots, f_i\}$. Assume for a contradiction that $U_{i+1} \cap {}_FS^2 \subseteq W_i$. Then since $\mathbb{Q} \subseteq S$, the center point $c_{i+1}$ of $U_{i+1}$ belongs to ${}_FS^2$ and also to $W_i$. For each $v \in {}_FS^2$, there is a sufficiently large $n \in \mathbb{N}$ such that $w := c_{i+1} + \frac{1}{n} \cdot v \in U_{i+1}$. But $w \in {}_FS^2$, whence $w \in W_i$. So, $v = n \cdot (w - c_{i+1}) \in W_i$. We have obtained that ${}_FS^2 \subseteq W_i$. This is a contradiction, because the $(i+1)$-dimensional vector space ${}_FW_i$ cannot include the $\sigma$-dimensional ${}_FS^2$. Having seen that $U_{i+1} \cap {}_FS^2 \not\subseteq W_i$, we can select an $f_{i+1} \in (U_{i+1} \cap {}_FS^2) \setminus W_i$. Clearly, the system $f_0, \dots, f_i, f_{i+1}$ is still linearly independent. This way we have defined a linearly independent system $(f_\iota : \iota < \sigma)$. Since this system can be extended to a basis of ${}_FS^2$, (10.3) follows.

Armed with the two bases, the bijection $\{e_\iota : \iota < \sigma\} \to \{f_\iota : \iota < \sigma\}$, $e_\iota \mapsto f_\iota$, can be extended to an automorphism $\pi$ of the vector space ${}_FS^2$. Since $\big(S^2; \underline{I}^o(T)\big)$ is a reduct of this vector space, $\pi$ is an automorphism of $\big(S^2; \underline{I}^o(T)\big)$ as well. Define $B := \pi(C)$, then $\big(B; \underline{I}^o(T)\big) \cong \big(C; \underline{I}^o(T)\big)$. Clearly, $C^{tc} \cap S^2$ has many wall elements, for example, $(1,0)$, $(0,-1)$ and, say, $(3/5, -4/5)$. Since $C^{tc} \cap S^2 \cong K^\infty_{\text{all}}(C)$ by Proposition 2.4, we get that $K^\infty_{\text{all}}(C)$ also has wall elements. Therefore $K^\infty_{\text{all}}(B)$ also has wall elements, because $B \cong C$ and the uniqueness part of Theorem 2.3 yield that $K^\infty_{\text{all}}(B) \cong K^\infty_{\text{all}}(C)$.

On the other hand, $\{f_i : i < \omega\} = \{\pi(e_i) : i < \omega\}$ is a subset of $B$ and it is everywhere dense in $\mathbb{R}^2$. Hence, $B^{tc} = \mathbb{R}^2$, so $B^{tc} \cap S^2 = S^2$. Since $S^2 = \big(S^2; \underline{I}^o(T)\big)$ has no wall elements, $B^{tc} \cap S^2$ is not isomorphic to $K^\infty_{\text{all}}(B)$. ☐

*Solution of Exercise 2.6.* No, they are *not* isomorphic. Let $C^{tc}_{\mathbb{Q}} = \{(x,y) \in \mathbb{Q}^2 : x^2+y^2 \le 1\}$ and $D^{tc}_{\mathbb{Q}} = \{(x,y) \in \mathbb{Q}^2 : 0 \le x,\ 0 \le y,\ x+y \le 1\}$. Assume for a contradiction that the two groupoids in question are isomorphic. It is well-known, see Theorem 6.6.1 in [16], that the modes $(\mathbb{R}^2; \underline{h})$ and $(\mathbb{R}^2; \underline{I}^o(\mathbb{D}))$ are term equivalent, that is, they have the same term functions. Therefore, the same holds for their submodes. Hence, $(C; \underline{h})$ is term equivalent with $(C; \underline{I}^o(\mathbb{D}))$, and $(D; \underline{h})$ is term equivalent with $(D; \underline{I}^o(\mathbb{D}))$. Thus, $(C; \underline{I}^o(\mathbb{D}))$ is isomorphic to $(D; \underline{I}^o(\mathbb{D}))$, and Theorem 2.3 yields that $K^{\infty}_{\text{all}}(C; \underline{I}^o(\mathbb{D})) \cong K^{\infty}_{\text{all}}(D; \underline{I}^o(\mathbb{D}))$. Hence, applying Proposition 2.4, we get that

$$\big(C^{tc}_{\mathbb{Q}}; \underline{I}^o(T)\big) \cong \big(D^{tc}_{\mathbb{Q}}; \underline{I}^o(T)\big). \tag{10.4}$$

Clearly, $\big(D^{tc}_{\mathbb{Q}}; \underline{I}^o(T)\big)$ has only three wall elements, see (10.2), the vertices $(0,0)$, $(1,0)$ and $(0,1)$ of the triangle. On the other hand, $\big(D^{tc}_{\mathbb{Q}}; \underline{I}^o(T)\big)$ has at least four wall elements, namely, $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$. (In fact, it has infinitely many wall elements obtained from Pythagorean triples, for example, $(3/5, 4/5)$.) This contradicts (10.4). ☐

## 11. Relating Theorem 2.3 to some results of [16]

Now, we are going to compare our results with some results of the reference book on modes.

For a subfield $F$ of $\mathbb{R}$ and a mode $A \in \mathcal{V}(F)$, [16, Prop. 5.8.7] states that $A$ is cancellative iff at least one of the operations $\underline{p} \in \underline{I}^o(F)$ is cancellative. (For $F = \mathbb{R}$, this was originally proved by Neumann [10].) This has motivated (but seems not to imply directly) our Proposition 9.2.

The reader has surely observed that the case $1/6 \in T$ is essentially simpler than the case $1/6 \notin T$. The situation simplifies further when $T = F$ is a subfield of $\mathbb{R}$. Then, by [16, Thm. 5.8.6], each (cancellative barycentric) algebra $A$ of $\mathcal{C}(F)$ is a *convex set* over $F$, that is, a convex subset of some affine space over $F$ in the obvious sense. Since affine spaces are closed members of $\mathcal{C}(F)$, $A$ clearly possesses a closed extension in $\mathcal{C}(F)$, and the *existence* of a closure (which is a *minimal* closed extension) follows trivially. Furthermore, [16, Lemma 7.6.3] together with [16, Thm. 5.8.6] yield that $\underline{\underline{B}}_1 := \mathcal{C}(F)$ is a minimal subquasivariety of $\mathcal{V}(F)$, provided $F$ is a subfield of $\mathbb{R}$. This yields $\mathcal{Q}(F) = \mathcal{H}(F) = \mathcal{C}(F)$, which is much more than the corresponding part of Theorem 2.3 for *fields*.

On the other hand, our construction and the aiming congruences are not only for proving the existence of a closure. They are heavily used in proving the *uniqueness part* of Theorem 2.3. Note that [16] does not have a proper connection with this uniqueness (not even for subfields of $\mathbb{R}$). In fact, no concept similar to our closure is discussed there.

**Acknowledgment.** The first author is indebted to Ferenc Fodor for helpful conversations on geometry.

## References

[1] Burris, S., Sankappanavar, H.P.: A Course in Universal Algebra. Graduate Texts in Mathematics, vol 78. Springer-Verlag, New York–Berlin (1981). The Millennium Edition, http://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html
[2] Herstein, I.N.: Topics in Algebra, 2nd edn. Xerox College Publishing, Lexington, Mass.-Toronto, Ont. (1975)
[3] Ignatov, V.V.: Quasivarieties of convexors. Izv. Vyssh. Uchebn. Zaved. Mar. **29**, 12–14 (1985) (Russian)
[4] Ježek, J., Kepka, T.: Semigroup representations of commutative idempotent abelian groupoids. Comment. Math. Univ. Carolinae **16**, 487–500 (1975)
[5] Ježek, J., Kepka, T.: Medial Groupoids. Academia, Praha (1983)
[6] Mac Lane. S.: Categories for the Working Mathematician, Springer-Verlag, New York (1971)
[7] Mal'cev, A.I.: Algebraic Systems. Springer-Verlag, Berlin (1973)
[8] Matczak, K., Romanowska, A.: Quasivarieties of cancellative commutative binary modes. Studia Logica **78**, 321–335 (2004)
[9] Matczak, K., Romanowska, A.B., Smith, J.D.H.: Dyadic polygons. International Journal of Algebra and Computation **21**, 387–408 (2011)
[10] Neumann, W.D.: On the quasivariety of convex subsets of affine spaces. Arch. Math. (Basel) **21**, 11–16 (1970)
[11] Pszczoła, K., Romanowska, A., Smith, J.D.H.: Duality for some free modes. Discuss. Math. General Algebra and Appl. **23**, 45–62 (2003)
[12] Romanowska, A.B., Smith, J.D.H.: Modal Theory. Heldermann, Berlin (1985)
[13] Romanowska, A.B., Smith, J.D.H.: On the structure of barycentric algebras. Houston J. Math. **16**, 431–448 (1990)
[14] Romanowska, A.B., Smith, J.D.H.: On the structure of semilattice sums. Czechoslovak Math. J. **41**, 24–43 (1991)
[15] Romanowska, A.B., Smith, J.D.H.: Embedding sums of cancellative modes into functorial sums of affine spaces. In: Abe, J.M., Tanaka, S. (eds.) Unsolved Problems on Mathematics for the 21st Century. A Tribute to Kiyoshi Iseki's 80th Birthday, pp. 127–139. OS Press, Amsterdam (2001)
[16] Romanowska, A.B., Smith, J.D.H.: Modes. World Scientific, Singapore (2002)
[17] Smith, J.D.H.: Mal'cev Varieties. Springer, Berlin-Heidelberg-New York (1976)
[18] van Lint, H.H.: Introduction to Coding Theory. Springer, New York-Berlin-Heidelberg (1982)

Gábor Czédli

Bolyai Institute, University of Szeged, Szeged, Aradi vértanúk tere 1, HUNGARY 6720
*e-mail*: czedli@math.u-szeged.hu
*URL*: http://www.math.u-szeged.hu/~czedli/

A. B. Romanowska

Faculty of Mathematics and Information Sciences, Warsaw University of Technology, 00-661 Warsaw, Poland
*e-mail*: aroman@mini.pw.edu.pl
*URL*: http://www.mini.pw.edu.pl/~aroman/