GENERALIZED CONVEXITY AND CLOSURE CONDITIONS

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ABSTRACT. Convex subsets of affine spaces over the field of real numbers are described by so-called barycentric algebras. In this paper, we discuss extensions of the geometric and algebraic definitions of a convex set to the case of more general coefficient rings. In particular, we show that principal ideal subdomains of the reals provide a good framework for such a generalization. Since the closed intervals of these subdomains play an essential role, we provide a detailed analysis of certain cases, and discuss differences from the "classical" intervals of the reals. We introduce a new concept of an algebraic closure of "geometric" convex subsets of affine spaces over the subdomains in question, and investigate their properties. We show that this closure provides a purely algebraic description of topological closures of geometric generalized convex sets. Our closure corresponds to one instance of the very general closure introduced in an earlier paper of the authors. The approach used in this paper allows to extend some results from that paper. Moreover, it provides a very simple description of the closure, with concise proofs of existence and uniqueness.

Affine spaces over a subring R of the field $\mathbb{R}$ of real numbers may be described as abstract algebras with infinitely many binary operations indexed by the elements of R and the ternary Mal'cev operation. If R is a subfield F of $\mathbb{R}$, the restriction of the set of basic operations to the operations indexed by the open unit interval of F provides an algebraic description of convex subsets of affine spaces over F. (See e.g. [24].)

Traditionally, a convex set is defined as a subset of a real affine space which contains the entire line segment joining any pair of its points. This fact was used to describe convex sets as abstract algebras. Both

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the concepts of convex sets and the corresponding algebras were then generalized to the case where, instead of the field $\mathbb{R}$, one considers any of its subfields. The algebras one obtains are idempotent and entropic, so they form a class of modes (see [19] and [24]). There are a number of further algebraic properties of convex sets providing several other equivalent definitions. These are summarized in Section 1.3.

We would like to extend the concept of a convex set to the case of convex subsets of affine spaces over some more general commutative unital rings. The natural requirements for such rings R are the following: They should be linearly ordered, and should have a non-trivial unit interval. Moreover, our generalized convex sets should be embeddable into affine spaces over the ring R, but not into an affine space over a non-trivial homomorphic image of R (that may not be an ordered ring). It is well known that linearly ordered commutative unital rings are integral domains of characteristic 0 (see [12, Ch.V.1]). Among these, principal ideal domains play a special role. Modules and affine spaces (affine modules) over such rings have a very well-known structure. Moreover, there exists a nice characterization of quasivarieties of modules over such rings provided by Belkin [1] that carries over to the corresponding quasivarieties of affine spaces [14]. In particular, faithful affine spaces over such a ring form a (minimal) quasivariety, and our generalized convex sets may be considered as embeddable into such affine spaces. Hence they also form a quasivariety. It follows that principal ideal subdomains of $\mathbb{R}$ properly containing the ring $\mathbb{Z}$ should be suitable candidates for our extensions.

If we replace the ring $\mathbb{R}$ of real numbers by a principal ideal subdomain R, not all of the equivalent properties defining convex sets over subfields of $\mathbb{R}$ carry over. However, at least some of them can be used in a possible definition of a generalized convex set.

The paper starts with introductory sections providing a short survey of basic definitions and facts concerning the algebraic description of affine spaces and convex sets, necessary for understanding the subsequent parts of the paper. The main objective of the paper is to develop an appropriate concept of a convex subset of an affine space over a principal ideal subdomain of $\mathbb{R}$, and to investigate some of its properties. The discussion of such generalized convexity is contained in Section 2. Convex subsets of affine spaces over a principal ideal subdomain R of $\mathbb{R}$ are specified as subreducts (subalgebras of reducts) of faithful affine spaces over R, defined by binary operations determined by the open unit interval of R. For a fixed R, they form a (minimal) quasivariety $\mathcal{C}v(R)$. The definition of such convex sets forms a direct generalization of the algebraic definition of convex subsets of real affine spaces. We then select a subclass of the quasivariety $\mathcal{C}v(R)$, consisting of so-called geometric convex subsets of affine spaces over R that are defined in a "geometric" fashion that stays close to the traditional geometric definition of convex sets over the field of reals. This class generates the quasivariety $\mathcal{C}v(R)$. If R is not a subfield of $\mathbb{R}$, the two classes do not coincide. The class of geometric convex sets also plays a significant role in the final sections of the paper.

The algebraization of convex subsets of real affine spaces was possible because all non-trivial line segments (bounded closed intervals) of $\mathbb{R}$, considered as algebras, are isomorphic and are generated by their endpoints. This is no longer true if the ring of reals is replaced by a subring R that is not a subfield. As the closed intervals of R play a basic role in the definition of convex subsets of affine spaces over R, it is essential to understand their structure. Such an analysis was previously carried out for the case of the ring $R = \mathbb{D} = \mathbb{Z}[1/2]$ of dyadic rational numbers. (See [15].) The second part (Section 3) of the paper is devoted to the (bounded) closed intervals of the subdomain $R = \mathbb{Z}[1/p]$ of $\mathbb{R}$, where p is a prime number. We show that, unlike the classical case, there are infinitely many isomorphism types of such intervals, and they are not necessarily generated by their endpoints. However, as we show, they are all finitely generated. This result extends a result of [15] concerning the case $p = 1/2$. The section also provides some insight into the issues involved in developing an appropriate extension of the concept of a convex set.

Finally, the third part of the paper (Sections 4 and 5) discusses the concept of algebraic closure of a generalized convex set. A very general concept of closures of subreducts of "classical" convex sets was introduced and investigated in [7]. The construction of such closures was complex, and the corresponding proofs were long and complicated. In the present paper, we are interested in the concept of a closure in relation to convex subsets of affine spaces over the subdomains R of $\mathbb{R}$ mentioned above, and in particular in relation to the geometric convex sets introduced in Section 2. We take an approach different from the approach of [7], and define our algebraic closure in a different way. The concept of algebraic closure presented in this paper is defined in a simple, direct fashion, and has a very easy and natural interpretation. Moreover, the proofs of its existence and uniqueness, quite long and complex in the general case of [7], are much simpler and more concise here. We show, however, that in the case of interest our algebraic closure coincides with a special instance of the closure introduced in [7]. If R is a field, all the concepts of closure considered in both papers coincide. If R is not a field, then this is no longer true. Moreover, one obtains new results extending some results of [7]. In particular, we show that the algebraic and topological closure of a geometric convex subset of an affine space over a principal ideal subdomain of $\mathbb{R}$ coincide.

Although this paper should be mostly self-contained, we refer the readers to the monographs [19], [24], and [22] for additional information about algebraic concepts used in the paper, especially those concerning convex sets, barycentric algebras and affine spaces; to [4] for basic geometric properties of convex subsets of $\mathbb{R}^n$, and to [7] for more information about the general closures of subreducts of convex sets. Our notation generally follows the conventions established in the first three monographs mentioned above.

1. Preliminaries

The algebras under consideration in this section, affine spaces and barycentric algebras, are all modes. That means they are idempotent and entropic algebras, as defined and investigated in [19], [20] and [24].

1.1. **Affine spaces.** Let R be a commutative unital ring. An *affine space* over R (or an *affine R-space*) may be defined as the reduct $(A, P, \underline{R})$ of an R-module $(A, +, R)$, where P is the Mal'cev operation

$$P : A^3 \to A; (x, y, z) \mapsto xyzP = x - y + z, \tag{1.1}$$

and $\underline{R}$ is the family of binary operations

$$\underline{r} : A^2 \to A; (x, y) \mapsto xy\underline{r} = x(1 - r) + yr \tag{1.2}$$

for each $r \in R$. Equivalently, it is defined as the full idempotent reduct of such a module (and is sometimes called an *affine R-module*). The class of all affine R-spaces is a variety. (See [5].) Abstractly, this variety may be defined as the class $\underline{\underline{R}}$ of Mal'cev modes $(A, P, \underline{R})$ with a ternary Mal'cev operation P and binary operations $\underline{r}$ for each $r \in R$, satisfying the identities:

$$\begin{aligned}
&xy\underline{0} = x = yx\underline{1},\\
&xy\underline{p}\; xy\underline{q}\; \underline{r} = xy\; \underline{pq\underline{r}},\\
&xy\underline{p}\; xy\underline{q}\; xy\underline{r}\; P = xy\; \underline{pqrP}
\end{aligned}$$

for all $p, q, r \in R$. (See [19] and [24, S. 6.3]). If $2 = 1+1$ is invertible in R, then $xyzP = y\, \underline{xz\underline{2^{-1}}}\; \underline{2}$ and affine R-spaces may be equivalently described as modes $(A, \underline{R})$ satisfying the identities in the first and second displayed lines above.

Note as well that the variety $\underline{\underline{R}}$ satisfies the *entropic* identities

$$xy\underline{p}\; zt\underline{p}\; \underline{q} = xz\underline{q}\; yt\underline{q}\; \underline{p} \tag{1.3}$$

for all $p, q \in R$. In other words, any two of the operations $\underline{p}$ and $\underline{q}$ commute. Also, the operations $\underline{p}$ commute with P. This shows that affine spaces are indeed modes. Moreover, since the binary operations $\underline{r}$ are interpreted by (1.2), it follows that for each invertible $p \in R$, the operation $\underline{p}$ satisfies the cancellation law

$$(xy\underline{p} = xz\underline{p}) \longrightarrow (y = z). \tag{1.4}$$

If R is a subring of $\mathbb{R}$, the cancellation laws are satisfied in any faithful[1] affine R-space for all $p \in R$ with $p \neq 0$. (Note that in general all cancellative modes embed into affine spaces [23] and [24, S. 7.7].) The structure of a ring R is determined by the free $\underline{\underline{R}}$-algebra on two generators. If we denote the free generators by 0 and 1, then the set of elements of this free algebra $\{0,1\}R$ coincides with the set $\{01\underline{r} \mid r \in R\} = R$, and the algebra is in fact isomorphic to the R-*line* $(R, \underline{R})$, the one-dimensional affine R-space. If the ring R is a field F, then the F-line $(F, \underline{F})$ is generated by any two distinct points. This is no longer true if R is not a field. Consider for example the ring $\mathbb{D} = \mathbb{Z}[2^{-1}]$ of rational dyadic numbers. Then the affine $\mathbb{D}$-space $(\mathbb{D}, \underline{\mathbb{D}})$ is freely generated by 0 and 1. However, 0 and 3 generate a $\mathbb{D}$-subspace $(A, \underline{\mathbb{D}})$ of $(\mathbb{D}, \underline{\mathbb{D}})$ that is isomorphic but not equal to $(\mathbb{D}, \underline{\mathbb{D}})$.

Note that a non-trivial faithful affine R-space contains, along with any two distinct points a and b, the *line* $\ell_R(a, b) = \{ab\underline{r} \mid r \in R\}$ generated by a and b, as a subalgebra. As an $\underline{\underline{R}}$-algebra, the line $\ell_R(a, b)$ is isomorphic to the R-line $(R, \underline{R})$.

Note also that varieties of affine spaces over fields are minimal as varieties and as quasivarieties, i.e. they do not contain non-trivial subvarieties or quasivarieties [2].

1.2. **Quasivarieties of affine spaces over principal ideal domains.** Let R be a principal ideal domain, and let Mod_R be the variety of unital R-modules. See e.g. [8] and [22] for basic facts about the structure of such modules. In particular, it is well-known that a finitely generated torsion-free R-module is free, and hence it is isomorphic to a finite power of the R-module R. There exists a nice classification of subquasivarieties of Mod_R provided by D. V. Belkin [1]. (See also [14], where this result is surveyed.) It shows that one of the minimal subquasivarieties of Mod_R is the quasivariety $\mathsf{Q}_{\mathsf{M}}(R)$ generated by the R-module R. It does not contain torsion modules, and is defined by infinitely many quasi-identities

$$(xp = 0) \rightarrow (x = 0) \tag{1.5}$$

[1] Recall that an affine R-space is faithful if any two operations $\underline{r}$ and $\underline{s}$, for distinct r and s in R, are different. See [24, S. 5.3].

for all p in the set $\mathbb{P}(R)$ of representatives of irreducible (and hence prime) elements of R modulo invertible elements. One easily observes that $\mathsf{Q_M}(R)$ consists precisely of faithful R-modules, and that finitely generated members of $\mathsf{Q_M}(R)$ are free and are (finite) powers of R. The classification of subquasivarieties of Mod_R carries over to the classification of subquasivarieties of the variety $\underline{\underline{R}}$ of affine R-spaces [14]. The quasivariety corresponding to $\mathsf{Q_M}(R)$ is the quasivariety $\mathsf{Q_A}(R)$ of affine R-spaces generated by the affine R-space R, with the following characteristic properties:

$$\mathsf{Q_A}(R) \text{ consists precisely of the faithful affine } R\text{-spaces.} \tag{1.6}$$

$$\text{Finitely generated members of } \mathsf{Q_A}(R) \text{ are free.} \tag{1.7}$$

$$\begin{gathered}\text{The free affine } R\text{-space on } n+1 \text{ generators,}\\ \text{where } n = 0, 1, \dots, \text{ is isomorphic to } R^n.\end{gathered} \tag{1.8}$$

If an affine R-space is isomorphic to R^n, where $n = 1, \dots,$ then the number n is called the *dimension* of the affine R-space.

Most of the rings considered in this paper are principal ideal subdomains of the ring $\mathbb{R}$ of real numbers. Note however that not all unital subrings of the ring $\mathbb{R}$ are principal ideal domains. An example is provided by the ring $\mathbb{Z}[\pi]$. We will be especially interested in affine spaces over the rings $\mathbb{Z}[p^{-1}] = \{m/p^n \mid m, n \in \mathbb{Z}\}$, where p is a prime number. The ring $\mathbb{Z}[p^{-1}]$ is the localization of the ring $\mathbb{Z}$ at the monoid $(M, \cdot, 1)$, where $M = \{p^n \mid n \in \mathbb{N}\}$. And since the localization of a principal ideal domain is again a principal ideal domain [11, Ch. 2], it follows that the ring $\mathbb{Z}[p^{-1}]$ is a principal ideal subdomain of the ring $\mathbb{R}$. The set $\mathbb{P}(\mathbb{Z}[p^{-1}])$ consists of prime numbers different from p. The minimal quasivariety $\mathsf{Q_A}(\mathbb{Z}[p^{-1}])$ of affine $\mathbb{Z}[p^{-1}]$-spaces, generated by the affine $\mathbb{Z}[p^{-1}]$-space $\mathbb{Z}[p^{-1}]$, consists of faithful affine $\mathbb{Z}[p^{-1}]$-spaces (cf.1.6). Note also that principal ideal subdomains of $\mathbb{R}$ are not necessarily subrings of the ring of rational numbers. The ring $\mathbb{Z}[\sqrt{2}]$ is a principal ideal ring, however the ring $\mathbb{Z}[\sqrt{10}]$ is not. (See e.g. [18].)

1.3. **Convex sets and barycentric algebras.** Recall that reducts and subreducts (subalgebras of reducts) of modes are modes again. In this section we are interested in certain subreducts of affine F-spaces, where F is a subfield of the field $\mathbb{R}$ of real numbers. Note that the subreducts of a given type of algebras in a given quasivariety also form a quasivariety. (See [13, Section 11.1].) This concerns, in particular, the subreducts of a given type of affine R-spaces.

Let $I(F) = [0, 1] \subset F$ be the closed unit interval $\{x \in F \mid 0 \leq x \leq 1\}$ of F. Let $I^o(F) =]0, 1[$ be the corresponding open unit interval $\{x \in F \mid 0 < x < 1\}$. More generally, for $a, b \in F$ with $a \neq b$, the closed interval $[a, b]_F$ is the set $\{x \in F \mid a \leq x \leq b\}$, and the open interval $]a, b[_F$ is the set $\{x \in F \mid a < x < b\}$. If a subfield F of $\mathbb{R}$ is replaced by a (unital) subring R of $\mathbb{R}$, then $I(R)$, $I^o(R)$, $[a, b]_R$ and $]a, b[_R$ are defined in a similar way. It is well known that the convex subsets of affine F-spaces may be described as $\underline{I}^o(F)$-subreducts of affine F-spaces $(A, \underline{F})$ (see [19] and [24]), which means as algebras $(B, \underline{I}^o(F))$ of type $\underline{I}^o(F) \times \{2\}$, equipped with a binary operation

$$\underline{p} : B \times B \to B; (x, y) \mapsto xy\,\underline{p}$$

for each p in $I^o(F)$. In particular all closed (and open) intervals of F form such algebras.

To be more specific, let us first recall that traditionally, a convex set is defined as a subset C of the space $A = \mathbb{R}^n$, where $n = 1, 2, \ldots,$ containing together with any two different points a and b, all points of the segment $[a, b]_{\ell_{\mathbb{R}}(a,b)} := \{ab\underline{p} \mid p \in I = I(\mathbb{R})\}$ of the (real) line $\ell_{\mathbb{R}}(a, b)$ joining them. It is easy to see that C is a subalgebra of the algebra $(A, \underline{I}^o = \underline{I}^o(\mathbb{R}))$, i.e. it is an $\underline{I}^o$-subreduct of the affine $\mathbb{R}$-space $(A, \underline{\mathbb{R}})$. Note that, as an $\underline{I}^o$-algebra, $[a, b]_{\ell_{\mathbb{R}}(a,b)}$ is isomorphic to the closed unit interval $I = I(\mathbb{R})$. More generally, *convex sets over* $\mathbb{R}$ (or $\mathbb{R}$-*convex sets*) are defined as $\underline{I}^o$-subreduct of the affine $\mathbb{R}$-spaces. (See [19] and [24].) The class $\mathcal{C}v(\mathbb{R})$ of convex sets described in this way generates the variety $\mathcal{B}(\mathbb{R})$ of *barycentric algebras over* $\mathbb{R}$ (or $\mathbb{R}$-*barycentric algebras*). The variety $\mathcal{B}(\mathbb{R})$ is defined by the identities

$$xx\,\underline{p} = x \tag{1.9}$$

of *idempotence* for each p in I^o, the identities

$$xy\,\underline{p} = yx\,\underline{1-p} \tag{1.10}$$

of *skew-commutativity* for each p in I^o, and the identities

$$xy\,\underline{p}\;z\,\underline{q} = x\;yz\,\underline{q/(p \circ q)}\;\;\underline{p \circ q} \tag{1.11}$$

of *skew-associativity* for each p, q in I^o. Here $p \circ q = p + q - pq$. (See [16], [19], [20] and [24].)

Among $\mathbb{R}$-barycentric algebras, $\mathbb{R}$-convex sets are characterized by cancellativity, i.e. they form the subquasivariety $\mathcal{C}(\mathbb{R})$ of $\mathcal{B}(\mathbb{R})$ defined by the cancellation laws (1.4) that hold for all $p \in I^o$, whence $\mathcal{C}v(\mathbb{R}) = \mathcal{C}(\mathbb{R})$. (See [16] and [24, S. 5.8]. Note however that, by [16], each of these cancellative laws implies the remaining ones.)

The definition of the variety $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$-barycentric algebras extends to the definition of the variety $\mathcal{B}(F)$ of *F-barycentric algebras*, barycentric algebras over any subfield F of the field $\mathbb{R}$, with the same axiomatization as above for all $p, q \in I^o(F)$ [24, S. 5.8]. Convex sets over F (or *F-convex sets*) are defined in a similar way as in the case $F = \mathbb{R}$, as $\underline{I}^o(F)$-subreducts of affine F-spaces. The class $\mathcal{C}v(F)$ of F-convex sets forms a quasivariety that coincides with the subquasivariety $\mathcal{C}(F)$ of the cancellative members of the variety $\mathcal{B}(F)$, that means

$$\mathcal{C}v(F) = \mathcal{C}(F). \tag{1.12}$$

(See [24, Ch. 7].) Note however that in [24] these algebras are called simply convex sets and barycentric algebras. The quasivariety $\mathcal{C}(F)$ contains algebras like subalgebras of $(F^n, \underline{I}^o(F))$ for $n = 0, 1, 2, \ldots$, but also $\underline{I}^o(F)$-reducts of G-convex sets for any subfield G of $\mathbb{R}$ containing F. (At the same time they may be considered as $\underline{I}^o(F)$-subreducts of $\mathbb{R}$-convex sets.) However, as geometric objects, G-convex sets are more naturally considered as $\underline{I}^o(G)$-algebras.

Note also that one of the consequences of a more general result in [17] is the following proposition. (Recall again that the class of subreducts of a given type of algebras in a given quasivariety is again a quasivariety.)

Proposition 1.1. [17] *Let R be a (unital) subring of $\mathbb{R}$. Then the free algebra over X in the quasivariety of subreducts of a given type τ of affine R-spaces is isomorphic to the τ-subreduct, generated by X, of the free affine R-space over X.*

Corollary 1.2. [17] *Let R be a (unital) subring of $\mathbb{R}$. Then the free algebra over X in the quasivariety of $\underline{I}^o(R)$-subreducts of affine R-spaces is isomorphic to the $\underline{I}^o(R)$-subreduct, generated by X, of the free affine R-space over X.*

The set of elements of the free $\underline{I}^o(R)$-algebra over $X = \{x_0, \ldots, x_n\}$ coincides with the set

$$\{x_0 a_0 + \cdots + x_n a_n \mid a_i \in I(R), \sum_{i=1}^{n} a_i = 1\}. \tag{1.13}$$

This set will be called an n-dimensional *simplex over R* and will be denoted by $S_n(R)$.

In particular, the free $\mathcal{C}(F)$-algebra on $n+1$ generators is just the n-dimensional simplex over the field F. Since each F-polytope (finitely generated F-convex set) contains as a subalgebra an n-dimensional simplex for some maximal natural number n, it also generates an affine F-space $(F^n, \underline{F})$. The affine F-space $(F^n, \underline{F})$ will be considered as a subreduct of the corresponding real space $(\mathbb{R}^n, \underline{\mathbb{R}})$, and may be equipped with the usual coordinate axes.

Some essential properties of F-barycentric algebras and F-convex sets are summarized in the following propositions. (Recall that F is a subfield of $\mathbb{R}$.)

Proposition 1.3. [24, S. 5.8] [17] *The following conditions are equivalent for any non-trivial subalgebra $(A, \underline{I}^o(F))$ of $(F, \underline{I}^o(F))$:*

(a) *$(A, \underline{I}^o(F))$ is a line segment of $(F, \underline{I}^o(F))$;*
(b) *$(A, \underline{I}^o(F))$ is isomorphic to $(I(F), \underline{I}^o(F))$;*
(c) *$(A, \underline{I}^o(F))$ is generated by two (different) elements;*
(d) *$(A, \underline{I}^o(F))$ is a free algebra on two free generators in the quasivariety $\mathcal{C}(F)$ and in the variety $\mathcal{B}(F)$.*

In particular, any two segments of the line F are isomorphic as F-barycentric algebras, and they are generated by their endpoints. Moreover, the algebra $(I(F), \underline{I}^o(F))$ embeds into each non-trivial F-convex set.

Proposition 1.4. [9] [24, S. 7.6] *The quasivariety $\mathcal{C}(F) = \mathcal{C}v(F)$ of F-convex sets is a minimal subquasivariety of the variety $\mathcal{B}(F)$.*

In particular, $\mathcal{C}v(F)$ coincides with the subquasivariety $\mathsf{Q}_{\mathcal{B}}(F)$ generated by $(F, \underline{I}^o(F))$, and with the subquasivariety $\mathsf{Q}_{\mathcal{B}}(I(F))$ generated by $(I(F), \underline{I}^o(F))$.

The variety $\mathcal{B}(F)$ is equivalently described as the class of homomorphic images of F-convex sets [24, Ch. 7].

2. Convex subsets of affine spaces over principal ideal subdomains of $\mathbb{R}$

Our aim is to extend the algebraic definitions of convex sets and barycentric algebras over subfields of the field $\mathbb{R}$ of reals to the case of principal ideal subdomains R of $\mathbb{R}$. However such a generalization is not so obvious as one could expect. We would like to keep most of the characteristic properties of traditional convex sets. In particular, it would be natural to define convex sets, as subsets of affine R-spaces containing with any two points all points lying between them on each one-dimensional subspace containing them. However, as we will see, not all of the (equivalent) basic properties of convex sets described in Propositions 1.3 and 1.4 of Section 1.3 will then be preserved, and our definition would require some further clarifications.

Recall that, for a subfield F of $\mathbb{R}$, all line segments (bounded closed non-trivial intervals) of the line F, considered as barycentric algebras, are isomorphic to the closed unit interval $(I(F), \underline{I}^o(F))$, and are generated by their endpoints. This is no longer true for subrings R of $\mathbb{R}$ which are not fields. First note that the closed unit interval of $\mathbb{Z}$ consists only of 0 and 1. So in what follows we will always assume that $R \neq \mathbb{Z}$. In [15], a subset of $\mathbb{D}^k$, for $k = 1, 2, \ldots$, is called a *dyadic convex set* if it is the intersection of a convex subset C of $\mathbb{R}^k$ with its subspace $\mathbb{D}^k$. Such sets may also be called *convex relative to* $\mathbb{D}$. (Compare [3], for instance.) In fact, dyadic convex sets were considered in [15] not as algebras $(C, \underline{I}^o(\mathbb{D}))$ but as (term) equivalent algebras $(C, \underline{1/2})$. It was shown there that the closed intervals $[a, b]_{\mathbb{D}} = \{x \in \mathbb{D} \mid a \leq x \leq b\}$ of $\mathbb{D}$ are not necessarily generated by their endpoints, and are not necessarily pairwise isomorphic. There are infinitely many isomorphism types of them. Moreover, a subalgebra of $(\mathbb{D}, \underline{I}^o(\mathbb{D}))$ generated by two different elements may not be a closed interval of the line $\mathbb{D}$. A simple example was provided by a subalgebra of $(\mathbb{D}, \underline{I}^o(\mathbb{D}))$ generated by 0 and 3, since it does not contain, say, 1 and 2. However, such a two generated subalgebra of $(\mathbb{D}, \underline{I}^o(\mathbb{D}))$ is always isomorphic to the interval $(I(\mathbb{D}), \underline{I}^o(\mathbb{D}))$, and this interval is a free algebra on two free generators 0 and 1 in the (quasi)variety it generates. (See Proposition 1.1 and [17].)

Yet another difference between the case of affine F-spaces and affine $\mathbb{D}$-spaces is that the dyadic unit interval does not embed into each $\underline{I}^o(\mathbb{D})$-subreduct of an affine $\mathbb{D}$-space. Since the residue class rings $\mathbb{Z}_{2n+1}$ are homomorphic images of the ring $\mathbb{D}$, some affine $\mathbb{D}$-spaces are in fact equivalent to affine $\mathbb{Z}_{2n+1}$-spaces. Moreover, $\underline{I}^o(\mathbb{D})$-subreducts of such affine $\mathbb{D}$-spaces are equivalent to affine $\mathbb{Z}_{2n+1}$-spaces. (See [14].) So these affine spaces should be excluded from our considerations. Let us note as well that each dyadic convex set belongs to the quasivariety of $\underline{I}^o(\mathbb{D})$-subreducts of faithful affine $\mathbb{D}$-spaces. Moreover the (closed) dyadic unit interval embeds into each (non-trivial) dyadic convex set.

The considerations above together with results of Section 1.3 suggest the following definition.

Definition 2.1. Let R be a principal ideal subdomain of the ring $\mathbb{R}$ such that $\mathbb{Z} \subset R$. Then a subset C of an affine R-space $(A, P, \underline{R})$ is called an *R-convex set* if the affine space is faithful and C is an $\underline{I}^o(R)$-subreduct of $(A, P, \underline{R})$.

In other words, R-convex subsets of a faithful affine R-space $(A, P, \underline{R})$ are subsets of A closed under $\underline{I}^o(R)$-operations.

Let us note that the above mentioned subalgebra of $(\mathbb{D}, \underline{I}^o(\mathbb{D}))$ generated by 0 and 3 is a $\mathbb{D}$-convex set but it does not contain all points of the $\mathbb{D}$-line $(\mathbb{D}, \underline{\mathbb{D}})$ lying between 0 and 3. In particular, it is not a dyadic convex set as defined above. This example shows that two distinct points of an R-convex subset of an affine R-space $(A, P, \underline{R})$ may belong to more than one of its one-dimensional subspaces.

As the class of faithful affine R-spaces forms a quasivariety, the class of $\underline{I}^o(R)$-subreducts of its members is also a quasivariety. Similarly as before, this quasivariety will be denoted by $\mathcal{C}v(R)$.

Recall Proposition 1.1 and Corollary 1.2, which hold also if R is a principal ideal subdomain of $\mathbb{R}$. Then the free $\underline{I}^o(R)$-algebra over $X = \{x_0, \dots, x_n\}$ is the n-dimensional simplex $S_n(R)$ over R.

If R is a principal ideal subdomain of $\mathbb{R}$, then an n-dimensional affine R-space, where $n = 1, 2, \dots$, is isomorphic to the affine R-space R^n, it may be considered as a subreduct of the corresponding affine $\mathbb{R}$-space $\mathbb{R}^n$, and may be equipped with the usual coordinate axes. As the set of free generators of the affine $\mathbb{R}$-space $\mathbb{R}^n$, one can take the standard affine basis consisting of $e_0 = (0, \dots, 0), e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$. This set of generators also generates the free $\underline{I}^o(R)$-algebra, and the affine R-space R^n.

If C is an R-convex subsets of a finite dimensional affine R-space A, then the *affine R-hull* $\texttt{aff}_R(C)$ of C is the intersection of all affine subspaces of A containing C. If $\texttt{aff}_R(C)$ is of (finite) dimension n, then we say that C is *finite dimensional*, and that its dimension $\dim C$ equals n.

For an R-convex subset C of an affine R-space R^n, the convex $\mathbb{R}$-hull $\texttt{conv}_{\mathbb{R}}(C)$ of C in the space $\mathbb{R}^n$ can be considered as the subalgebra of $(\mathbb{R}^n, \underline{I}^o)$ generated by the set C. Then the *convex R-hull* of C in R^n is defined to be

$$\texttt{conv}_R(C) := \texttt{conv}_{\mathbb{R}}(C) \cap R^n.$$

The *relative interior* of an n-dimensional R-convex set C is the intersection of the relative interior of $\texttt{conv}_{\mathbb{R}}(C)$ in its affine $\mathbb{R}$-hull and R^n.

Note the following lemma.

Lemma 2.2. *Let C be an n-dimensional R-convex set, where $n = 1, 2, \dots$. Then C contains a subalgebra isomorphic to the n-dimensional simplex $S_n(R)$.*

Proof. Let A be the affine R-hull $\texttt{aff}_R(C)$ of C. Since R is a principal ideal subdomain of $\mathbb{R}$, the affine R-space A can be identified with R^n, and considered as a subreduct of the affine $\mathbb{R}$-space $\mathbb{R}^n$. Let $e_0 = (0, \dots, 0), e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$ be the standard affine basis of $\mathbb{R}^n$, the set of free generators of the affine $\mathbb{R}$-space $\mathbb{R}^n$, of the affine R-space R^n and of the free $\underline{I}^o(R)$-algebra $S_n(R)$.

Now let $a = (a_0 = 1 - \sum_{i=1}^n a_i, a_1, \dots, a_n)$ be an element of the relative interior of C different from e_0. Then there is $b \in R$ small enough so that the elements $a'_0 = a$, $a'_1 = (a_0 - b, a_1 + b, a_2, \dots, a_n), \dots, a'_n = (a_0 - b, a_1, \dots, a_{n-1}, a_n + b)$ are also in C. These elements form an affinely independent set both in R^n and in $\mathbb{R}^n$ (i.e. no a'_i is in the subspace generated by the remaining elements). The mapping $e_0 \mapsto a'_0, e_1 \mapsto a'_1, \dots, e_n \mapsto a'_n$ extends to an isomorphism between $S_n(R)$ and its image contained in C. $\square$

In particular, if C is an n-dimensional R-convex subset of the affine R-space R^n, then C generates this affine R-space and also the corresponding affine $\mathbb{R}$-space $\mathbb{R}^n$.

We will show that similarly as in the case of subfields of $\mathbb{R}$, the quasivariety $\mathcal{C}v(R)$ is a minimal quasivariety, i.e. it does not contain any non-trivial subquasivariety.

Proposition 2.3. *The quasivariety $\mathcal{C}v(R)$ is generated by either of the algebras $(R, \underline{I}^o(R))$ and $(I(R), \underline{I}^o(R))$. Hence, it does not contain any non-trivial subquasivariety.*

Proof. Let $\mathsf{Q}(R)$ be the subquasivariety of $\mathcal{C}v(R)$ generated by $(R, \underline{I}^o(R))$, and let $\mathsf{Q}(I(R))$ be the subquasivariety generated by $(I(R), \underline{I}^o(R))$. As $(I(R), \underline{I}^o(R))$ is a subalgebra of $(R, \underline{I}^o(R))$, and $(R, \underline{I}^o(R))$ is an $\underline{I}^o(R)$-reduct of $(R, \underline{R})$, it follows that $\mathsf{Q}(I(R)) \leq \mathsf{Q}(R) \leq \mathcal{C}v(R)$.

To show that $\mathsf{Q}(R) \leq \mathsf{Q}(I(R))$, let us first note that the mode $(R, \underline{I}^o(R))$ is a directed co-limit of its finitely generated subalgebras. If $(A, \underline{I}^o(R))$ is a submode of $(R, \underline{I}^o(R))$, finitely generated by say $g_1, \dots, g_k$, where $g_1 < \cdots < g_k$, then A is a subalgebra of the interval $[g_1, g_k] := [g_1, g_k]_R$. (In what follows all intervals of the line R are denoted in a similar way.) This interval is isomorphic to the interval $[0, a = g_k - g_1]$, and the last one embeds isomorphically into the interval $(I(R), \underline{I}^o(R))$. Indeed, first note that for any element $r \in R$ there is $\varepsilon_r \in I^o(R)$ small enough so that $r\varepsilon_r < 1$. Then consider the mapping $h : [0, a] \to [0, a\varepsilon_a]; x \mapsto x\varepsilon_a$. It is easy to check that h is an $\underline{I}^o(R)$-monomorphism. Since each quasivariety is closed under directed co-limits, it follows that $\mathsf{Q}(I(R)) \geq \mathsf{Q}(R)$. Consequently, both quasivarieties coincide, i.e.

$$\mathsf{Q}(I(R)) = \mathsf{Q}(R).$$

To show that $\mathcal{C}v(R) \leq \mathsf{Q}(R)$, it is sufficient to prove that each finitely generated member of $\mathcal{C}v(R)$ belongs to $\mathsf{Q}(R)$, and then use the fact that each algebra is a directed co-limit of its finitely generated subalgebras, and the fact that quasivarieties are closed under directed co-limits. So let $(C, \underline{I}^o(R))$ be a finitely generated algebra in $\mathcal{C}v(R)$. This means that $(C, \underline{I}^o(R))$ is an $\underline{I}^o(R)$-subreduct of a faithful affine R-space. In fact, it is a subreduct of its affine R-hull $A = \mathtt{aff}_R(C)$. Since C is finitely generated, the affine space $(A, P, \underline{R})$ has a finite dimension, say k, and is isomorphic to $(R^k, P, \underline{R})$. Hence $(A, \underline{I}^o(R)) \cong (R^k, \underline{I}^o(R)) \cong (R, \underline{I}^o(R))^k$ belongs to $\mathsf{Q}(R)$. Since $(C, \underline{I}^o(R))$ is a subalgebra of $(A, \underline{I}^o(R))$, it follows that $(C, \underline{I}^o(R))$ is also a member of $\mathsf{Q}(R)$. Consequently, $\mathcal{C}v(R) \subseteq \mathsf{Q}(R)$, and finally one obtains the required equalities:

$$\mathsf{Q}(I(R)) = \mathsf{Q}(R) = \mathcal{C}v(R). \tag{2.1}$$

As each non-trivial $\underline{I}^o(R)$-subreduct of a faithful affine R-space contains a subalgebra isomorphic to $(I(R), \underline{I}^o(R))$, it follows that the quasivariety $\mathcal{C}v(R)$ is minimal. □

Note that by [13, Sect. 14.3], a non-trivial quasivariety is minimal precisely when it is generated by any of its non-trivial members.

We will select a class of R-convex sets that can be defined similarly as traditional real convex sets. To do so, we will need a precise definition of the concept of a line segment joining two given points of an R-convex set.

Definition 2.4. Let $a \neq b$ be elements of an affine R-space $(A, P, \underline{R})$. As before, let $\ell_R(a, b) = \{ab\underline{r} \mid r \in R\}$ be the line generated by a and b. Let us agree that, for $c = ab\underline{r}$ and $d = ab\underline{s}$ belonging to $\ell_R(a, b)$, we write $c < d$ if $r < s$. This defines a *linear order on the line* $\ell_R(a, b)$. In particular, $a < b$. Note that $\ell_R(a, b)$ and $\ell_R(b, a)$ are the same lines but with opposite orders.

For $c, d \in \ell_R(a, b)$, with $c < d$, the *segment* of $\ell_R(a, b)$ joining c and d is defined to be the set

$$[c, d]_{\ell_R(a,b)} := \{x \in \ell_R(a, b) \mid c \leq x \leq d\}.$$

For $u, v \in A$, the *geometric segment* of A joining u and v is defined as follows:

$$[u, v]_{geom} := \bigcup \{[u, v]_{\ell_R(a,b)} \mid a, b \in A \,\text{and}\, u, v \in \ell_R(a, b) \,\text{with}\, u < v\}.$$

Finally, an R-convex subset C of a faithful affine R-space $(A, P, \underline{R})$ is called *geometric*, if $u, v \in C$ implies $[u, v]_{geom} \subseteq C$.

Note that C is a geometric R-convex subset of A if $c, d \in C \cap \ell_R(a, b)$ and $c < d$ imply $[c, d]_{\ell_R(a,b)} \subseteq C$ for all $a, b \in A$ with $a \neq b$.

If R is a subfield F of the field $\mathbb{R}$, all F-convex subsets are geometric. If R is not a field, then roughly speaking, an R-convex subset C of a finite dimensional affine R-space A is geometric, if it contains together with any two of its different points c and d, all points lying between them on each one-dimensional subspace of $(A, P, \underline{R})$ containing c and d.

In the last case, geometric convex sets may be described in a more direct way. As a k-dimensional affine R-space A over a principal ideal subdomain R of $\mathbb{R}$ is isomorphic to the affine R-space R^k, we will consider now only the affine R-spaces R^k. As noted before, the affine R-space R^k is a subreduct of the affine $\mathbb{R}$-space $\mathbb{R}^k$. The space $\mathbb{R}^k$ will be equipped with the standard basis, and with the standard coordinatization with $\bar{0} = (0, \dots, 0)$ as the origin. R-convex subsets of R^k are $\underline{I}^o(R)$-subreducts of the affine R-space R^k and of the affine $\mathbb{R}$-space $\mathbb{R}^k$. In particular, the affine R-space R is a subreduct of the affine $\mathbb{R}$-space $\mathbb{R}$, both generated by 0 and 1. In the space $\mathbb{R}^k$, we will also consider lines $\ell_{\mathbb{R}}(a, b)$ for $a, b \in R^k$ and their affine R-subspaces $\ell_{\mathbb{R}}(a, b) \cap R^k$. Since translations of affine spaces are their automorphisms, such lines are isomorphic to translated lines with one of the points a, b being the origin $\bar{0}$ of $\mathbb{R}^k$. In many situations, this allows us to consider only the translate $\ell_{\mathbb{R}}(\bar{0}, c)$, where $\bar{0} \neq c = b - a \in R^k$, of $\ell_{\mathbb{R}}(a, b)$ instead of $\ell_{\mathbb{R}}(a, b)$, and its affine R-subspaces $\ell_R(\bar{0}, c)$.

Lemma 2.5. *Let a and b be different points of R^k. Then there is an affine $\mathbb{R}$-space isomorphism $\iota_{\mathbb{R}} : \ell_{\mathbb{R}}(a, b) \to \mathbb{R}$ which restricts to the $\underline{R}$-embedding $\iota_R : \ell_{\mathbb{R}}(a, b) \cap R^k \to R$.*

Proof. Without loss of generality, we will consider the translate $\ell_{\mathbb{R}}(\bar{0}, c)$ of $\ell_{\mathbb{R}}(a, b)$.

Each axis of $\mathbb{R}^k$, considered as an affine $\mathbb{R}$-space, is isomorphic to the affine $\mathbb{R}$-space $\mathbb{R}$. If c belongs to one of the axes, then $\ell_{\mathbb{R}}(\bar{0}, c)$ is isomorphic to the axis, and for any different $a, b \in \mathbb{R}$, we have

$$\ell_{\mathbb{R}}(\bar{0}, c) = \ell_{\mathbb{R}}(a, b) \cong \mathbb{R}.$$

Obviously, $\mathbb{R} \cap R = R = \ell_R(0, 1)$. Then $\iota_{\mathbb{R}} : \ell_{\mathbb{R}}(0, c) \to \mathbb{R}$ is just the identity mapping, and ι_R is its restriction to R.

Let $k \geq 2$. Assume that c does not belong to any of the axes. Then the isomorphism $\iota_{\mathbb{R}}$ is given by the projection onto any of the axis, e.g. Ox_i-axis, where $i = 1, \dots, n$,

$$\iota_{\mathbb{R}} : \ell_{\mathbb{R}}(\bar{0}, c) \to \mathbb{R} : x = (x_1, \dots, x_k) \mapsto x_i. \tag{2.2}$$

Obviously, its restriction to $\ell_{\mathbb{R}}(\bar{0}, c) \cap R^k$,

$$\iota_R : \ell_{\mathbb{R}}(\bar{0}, c) \cap R^k \to R : x = (x_1, \dots, x_k) \mapsto x_i, \tag{2.3}$$

is an R-homomorphism. It is injective, but not necessarily surjective. $\square$

Lemma 2.6. *Let c and d be different points of R^k. Let $x \in \ell_{\mathbb{R}}(d,c) \cap R^k$ and assume that on the line $\ell_{\mathbb{R}}(d,c)$, $d < x < c$. Then there are a and b in R^k such that the three elements d, x and c belong to $\ell_R(a,b)$.*

Proof. As in the proof of Lemma 2.5, we will consider only lines of the form $\ell_{\mathbb{R}}(\bar{0},c)$ for $c \neq \bar{0}$, and in the case $k \geq 2$, with c not belonging to any of the axes. Let $c = (c_1, \dots, c_k)$ and $x = (x_1, \dots, x_k)$. Consider the mappings $\iota_{\mathbb{R}}$ and ι_R of (2.2) and (2.3). By Lemma 2.5, $\ell_{\mathbb{R}}(\bar{0},c) \cap R^k$ is isomorphic to the affine R-subspace $I = \iota_R(\ell_{\mathbb{R}}(\bar{0},c) \cap R^k)$ of R. Since R is a principal ideal domain, the elements c_1 and x_1 have a greatest common divisor g_1. There are $c', x' \in R$ such that $c_1 = g_1 c'$ and $x_1 = g_1 x'$. Moreover, there are $s, t \in R$ such that $g_1 = sc_1 + tx_1$. Then $c_1 = 0(1-c') + g_1 c' = 0g_1\underline{c'}$ and $x_1 = 0(1-x') + g_1 x' = 0g_1\underline{x'}$. Hence the elements c_1, x_1 of I belong to $\ell_R(0, g_1)$. Now note that $g := sc + tx$ belongs to $\ell_{\mathbb{R}}(\bar{0},c) \cap R^k$ since $s, t \in R$ and $c, x \in \ell_{\mathbb{R}}(\bar{0},c) \cap R^k$. Moreover $\iota_R(g) = g_1$. Hence the elements $0, c, x$ belong to $\ell_R(\bar{0}, \iota_R^{-1}(g))$. $\square$

Let C be an R-convex subset of the affine R-space R^k. Recall that the convex R-hull of C in R^k was defined by

$$\mathtt{conv}_R(C) := \mathtt{conv}_{\mathbb{R}}(C) \cap R^k.$$

Proposition 2.7. *Let $(C, \underline{I}^o(R))$ be an R-convex subset of an affine space $(R^k, P, \underline{R})$, where $k = 1, 2, \dots$. Then the following two conditions are equivalent.*

(a) *C is a geometric convex subset of R^k;*

(b) $C = \mathtt{conv}_R(C)$.

Proof. ($\Leftarrow$) Let $c, d \in C$. Let $a, b \in R^k$ with $a \neq b$ such that $c, d \in \ell_R(a,b)$ and let $x = ab\underline{r}$ with $r \in R$ such that x is between c and d on the R-line $\ell_R(a,b)$. Obviously, $x \in \mathtt{conv}_{\mathbb{R}}(C) \cap R^k$. Hence $x \in C$.

($\Rightarrow$) As in Lemmas 2.5 and 2.6, instead of lines $\ell_{\mathbb{R}}(c,d)$, for $c, d \in R^k$, we will consider only lines $\ell_{\mathbb{R}}(\bar{0},c)$ for $\bar{0} \neq c \in R^k$. Assume that $\bar{0}, c \in C$. Let $x \in \ell_{\mathbb{R}}(\bar{0},c) \cap R^k$ be strictly between $\bar{0}$ and c. By Lemma 2.6, there are $a, b \in R^k$ such that $x \in \ell_R(a,b)$. Since C is geometric, it follows that $x \in C$. $\square$

In particular, dyadic convex subsets of affine $\mathbb{D}$-spaces $\mathbb{D}^k$, as defined above and in [15], are precisely geometric $\mathbb{D}$-convex subsets of finite dimensional affine $\mathbb{D}$-spaces.

Let us note that by Proposition 2.3 and the comment following it, the quasivariety $\mathcal{C}v(R)$ of R-convex sets is generated by any non-trivial geometric R-convex subset of a faithful affine R-space.

The subspace $3\mathbb{D} = \{3m/(2^n) \mid m, n \in \mathbb{Z}\}$ of $\mathbb{D}$, considered as an $\underline{I}^o(\mathbb{D})$-convex subset of the affine $\mathbb{D}$-space $(\mathbb{D}, \underline{\mathbb{D}})$, is surely not geometric. However, it is isomorphic to the geometric convex set $(\mathbb{D}, \underline{I}^o(\mathbb{D}))$, and it is a geometric subset of the affine $\mathbb{D}$-subspace $(3\mathbb{D}, \underline{\mathbb{D}})$. Below is a more complicated example.

Example 2.8. Consider the $\mathbb{D}$-convex subset C of the affine $\mathbb{D}$-space $\mathbb{D}^2$, consisting of all points of the half plane $\{(x, y) \in \mathbb{D}^2 \mid y > 0\}$ together with the subspace $\{(x, 0) \mid x \in 3\mathbb{D}\}$. Proposition 2.7 shows that C is not geometric in $\mathbb{D}^2$. And since the smallest affine $\mathbb{D}$-space containing C is $\mathbb{D}^2$, it cannot be geometric in any proper subspace of $\mathbb{D}^2$ containing C.

3. Intervals of the lines $\mathbb{Z}[1/p]$

In this section we will classify, up to isomorphism, the closed intervals $[a, b] := [a, b]_{\mathbb{Z}[1/p]} = \{x \in \mathbb{Z}[1/p] \mid a \leq x \leq b\}$ of the lines $\mathbb{Z}[1/p]$, where $a, b \in \mathbb{Z}[1/p]$ and $a < b$. We will call them simply intervals, and consider them as $\underline{I}^o(\mathbb{Z}[1/p])$-subreducts of affine $\mathbb{Z}[1/p]$-spaces. The classification allows one to understand the issues involved in developing an appropriate extension of the concept of convex sets. On the other hand, it provides a nice generalization of the classification of dyadic intervals [15] and a good starting point for further investigation of the corresponding generalizations of polygons and polytopes in affine $\mathbb{Z}[1/p]$-spaces.

First recall that $\mathbb{Z}[1/2] = \mathbb{D}$ and that the operation $x \cdot y = (x + y)/2$ endows each $\mathbb{D}$-convex set with the algebraic structure of a commutative binary mode ($\mathcal{CB}$-mode or CBM-groupoid for short). Recall that such groupoids have a well developed algebraic theory (see [10] and [24, Ch. 5, 6, 7]). It can be easily deduced from [24, S. 5.5 and 7.5] that each $\mathbb{D}$-convex set is (term) equivalent to its $\underline{1/2}$-reduct. Consequently, all $\mathbb{D}$-convex sets can be considered as $\mathcal{CB}$-modes. In [15], it was shown that each (closed) interval of the line $\mathbb{D}$ (considered as a $\mathcal{CB}$-mode or as an $\underline{I}^o(\mathbb{D})$-algebra) is minimally generated by two or three generators. $\mathbb{D}$-convex sets, considered as $\mathcal{CB}$-modes, form the (minimal) quasivariety consisting of subalgebras of $\underline{1/2}$-reducts of faithful affine $\mathbb{D}$-spaces. It is equivalent to the quasivariety $\mathcal{C}v(\mathbb{D})$. The variety generated by this quasivariety is the variety of commutative binary modes, and may be considered as a dyadic counterpart of the variety $\mathcal{B}(\mathbb{R})$ of real barycentric algebras. (Cf. [14], [15], [21] and [24].)

Now $\mathbb{Z}[1/p]$-convex sets form an obvious generalization of $\mathbb{D}$-convex sets. We will show that similarly as in the dyadic case, also for $p > 2$, the intervals of the lines $\mathbb{Z}[1/p]$ are not necessarily pairwise isomorphic and are not necessarily generated by their endpoints. They are, however, all finitely generated.

The classification of intervals of $\mathbb{Z}[1/p]$ is based on the following general observation.

Proposition 3.1. *Let R be a principal ideal subdomain of the ring $\mathbb{R}$. Let $(C, \underline{I}^o(R))$ and $(C', \underline{I}^o(R))$ be finite dimensional R-convex sets. Then $(C, \underline{I}^o(R))$ and $(C', \underline{I}^o(R))$ are isomorphic if and only if there is an isomorphism $h : (A, P, \underline{R}) \to (A', P, \underline{R})$ from the affine R-hull $A = \mathtt{aff}_R(C)$ onto the affine R-hull $A' = \mathtt{aff}_R(C')$ such that $h(C) = C'$.*

Proof. It is evident that the restriction of the isomorphism h provides an $\underline{I}^o(R)$-isomorphism between C and its image $C' = h(C)$.

Now assume that $g : (C, \underline{I}^o(R)) \to (C', \underline{I}^o(R))$ is an isomorphism, and that $\mathtt{dim}C = \mathtt{dim}C' = k$. By Lemma 2.2, C contains a subalgebra $(S, \underline{I}^o(R))$ isomorphic to the k-dimensional simplex $S_k(R)$, the free $\underline{I}^o(R)$-algebra with vertices as free generators. Its image is an isomorphic simplex contained in C'. The generators of $(S, \underline{I}^o(R))$ also generate the affine R-hull A of C, isomorphic to the free affine space $(R^k, P, \underline{R})$. (Cf. Corollary 1.2). Similarly the images of those generators generate the affine R-hull A' of C'. Hence g extends to an isomorphism from the affine R-space A to the affine R-space A'. □

Corollary 3.2. *Let $(C, \underline{I}^o(R))$ and $(C', \underline{I}^o(R))$ be geometric R-convex subsets of a finite dimensional affine R-space B. Then $(C, \underline{I}^o(R))$ and $(C', \underline{I}^o(R))$ are isomorphic if and only if there is an automorphism h of the affine space B such that $h(C) = C'$.*

Proof. It is evident that the restriction of the automorphism h provides an $\underline{I}^o(R)$-isomorphism between C and its image $C' = h(C)$.

Assume that C and C' are isomorphic under an $\underline{I}^o(R)$-isomorphism g, and that they are k-dimensional. By Proposition 3.1, their affine R-hulls A and A' are isomorphic under the isomorphism $\bar{g}$ extending g. Note that A and A' are maximal affine subspaces of B of dimension k. If $k = n = \mathtt{dim}B$, then $\bar{g}$ is an automorphism of B. Now let $k < n$ and let $\{a_0, a_1, \ldots, a_k\}$ be a set of free generators of $\mathtt{aff}_R C$ contained in C. Then $\{g(a_0), g(a_1), \ldots, g(a_k)\}$ is the set of free generators of $\mathtt{aff}_R(C')$ contained in C'. Both these sets may be extended to sets of free generators: $\{a_0, a_1, \ldots, a_k, b_1, \ldots, b_{n-k}\}$ and $\{g(a_0), g(a_1), \ldots, g(a_k), c_1, \ldots, c_{n-k}\}$ of B. Then the affine R-space homomorphism extending the mapping $a_i \mapsto g(a_i)$ for $i = 0, 1, \ldots, k$ and $b_i \mapsto c_i$ for $i = 1, \ldots, n-k$, is an automorphism of B. □

Recall that, for any positive integer n, the automorphisms of an affine $\mathbb{Z}[1/p]$-space $\mathbb{Z}[1/p]^n$ form the n-dimensional affine group $GA(n, \mathbb{Z}[1/p])$, the group generated by the linear group $GL(n, \mathbb{Z}[1/p])$ and the group of translations of the affine space $\mathbb{Z}[1/p]^n$. (Cp. e.g. [22, Ex. III.2.4.6].) Each element of the affine group $GA(n, \mathbb{Z}[1/p])$ is also an automorphism of any reduct of the affine space $\mathbb{Z}[1/p]^n$. In particular if $n = 1$, then an element of $GA(1, \mathbb{Z}[1/p])$ transforms any interval of $\mathbb{Z}[1/p]$ into an isomorphic interval. On the other hand, since each interval of $\mathbb{Z}[1/p]$ is a geometric convex subset of $\mathbb{Z}[1/p]$, Corollary 3.2 shows that an isomorphism between two intervals of $\mathbb{Z}[1/p]$ extends to an automorphism of the affine space $\mathbb{Z}[1/p]$.

We will use this observation to consider isomorphisms of intervals of $\mathbb{Z}[1/p]$ as restrictions of automorphisms of the affine space $\mathbb{Z}[1/p]$. Now the affine automorphisms of the affine space $\mathbb{Z}[1/p]$ are given by

$$\kappa(a, b) : \mathbb{Z}[1/p] \to \mathbb{Z}[1/p]; x \mapsto ax + b, \tag{3.1}$$

where $a = \pm p^r$ for some integer r, and $b \in \mathbb{Z}[1/p]$. The linear group $GL(1, \mathbb{Z}[1/p])$ is isomorphic to the group $\mathbb{Z}[1/p]^*$ of units, consisting of the elements $a = \pm p^r$ for some integer r.

The following lemma generalizes Lemma 3.1 in [15].

Lemma 3.3. *The following hold for intervals in $\mathbb{Z}[1/p]$:*

1. *For each positive integer k and each integer r, the intervals $[0, k]$ and $[0, kp^r]$ are isomorphic.*
2. *An interval is generated by its endpoints precisely when it is isomorphic to the interval $I(\mathbb{Z}[1/p])$.*
3. *Two intervals $[0, k]$ and $[0, l]$, where k and l are positive integers not divisible by p, are isomorphic precisely if $k = l$.*

Proof. First observe that each interval $[a, b]$ in $\mathbb{Z}[1/p]$ translates isomorphically to an interval of the form $[0, k]$, where k is a positive integer. Indeed, $[a, b]$ is surely isomorphic to $[0, b-a]$. Without loss of generality assume that $b - a = kp^r$ for some positive integer k and an integer r. Then it is very easy to check that the mapping

$$[0, kp^r] \to [0, k]; x \mapsto xp^{-r}$$

is an $\underline{I}^o(\mathbb{Z}[1/p])$-isomorphism, whence (1) holds. In particular, the intervals $[0, 1]$ and $[0, p^r]$ are isomorphic for each integer r.

Note that obviously, each interval $[0, p^r]$ is generated by its endpoints. Now assume that $k > 1$ is a positive integer and the interval $[0, k]$ is generated by its endpoints. Note that 0 and k generate the set

$$A = \{km/p^r \mid r \in \mathbb{N},\, m = 0, 1, \ldots, p^r\}\,. \tag{3.2}$$

If for each positive integer r, one has $k \neq p^r$, then none of the numbers $1, 2, \ldots, k-1$ belongs to A. This contradicts the assumption that $[0, k]$ is generated by 0 and k. It follows that k is a power of p, whence by the first part of the proof, the interval $[0, k]$ is isomorphic to the interval $[0, 1]$.

Now assume that k and l are positive integers bigger than 1 and not divisible by p. Suppose that the intervals $[0, k]$ and $[0, l]$ are isomorphic. Then by Corollary 3.2, and by (3.1), one is obtained from the other by an action of the group $\mathbb{Z}[1/p]^*$. In particular, this means that l belongs to the orbit of k, whence $l = kp^r$ for some integer r. Since l is an integer not divisible by p, it follows that $k = l$. $\square$

Note that $\mathbb{Z}[1/p]$ is the disjoint union of the $GL(1, \mathbb{Z}[1/p])$-orbits: $\{0\} = 0\mathbb{Z}[1/p]^*$ and $k\mathbb{Z}[1/p]^*$ for positive integers k not divisible by p. The isomorphism classes of intervals in $\mathbb{Z}[1/p]$ are determined by such non-zero orbits. For a positive integer k not divisible by p, the class containing the interval $[0, k]$ is the set of intervals

$$\{[a, a + kp^r] \mid a \in \mathbb{Z}[1/p],\, r \in \mathbb{Z}\}\,.$$

As a corollary one obtains the following theorem.

Theorem 3.4. *Each interval in $\mathbb{Z}[1/p]$ is isomorphic to some interval $[0, k] := [0, k]_{\mathbb{Z}[1/p]}$, where k is a positive integer not divisible by p. Two such intervals are isomorphic precisely when their right-hand ends are equal.*

Let us note that the natural numbers contained in the interval $[0, k]$ provide a finite set of its generators. We will describe another, smaller set.

Each positive integer k has a unique representation

$$k = k_1 p^{n_1} + k_2 p^{n_2} + \cdots + k_j p^{n_j}, \tag{3.3}$$

where all n_i, for $i = 1, \ldots, j$, are pairwise different non-negative integers and $k_i \in \{0, 1, \ldots, p-1\}$.

Lemma 3.5. *Let $k > 1$ be a positive integer with the representation* (3.3)*. Then*

$$\begin{aligned} G = \{&0, p^{n_1}, 2p^{n_1}, \ldots, k_1 p^{n_1}, k_1 p^{n_1} + p^{n_2}, \ldots, \\ &k_1 p^{n_1} + k_2 p^{n_2}, \ldots, \sum_{i=1}^{j} k_i p^{n_i} = k\} \end{aligned}$$

forms a set of generators of the interval $[0, k]$.

We omit the obvious proof. The set G provides a quite convenient set of generators of the interval $[0, k]$. However, it is not necessarily a minimal one.

Corollary 3.6. *Each interval of* $\mathbb{Z}[1/p]$ *is finitely generated.*

4. Algebraic closures of geometric R-convex sets

Consider a ring R such that $\mathbb{Z} \subset R \subseteq \mathbb{R}$. In [7], a concept of a closure was introduced for cancellative members of the variety generated by the mode $(R, \underline{I}^o(R))$. At this level of generality, it was long and complicated to prove that this closure exists and is unique. In this section, we take an approach different from that in [7], and define an algebraic closure in a different, much simpler and more direct way, however only in the case of R-convex subsets of affine R-spaces over a principal ideal subdomain R of the ring $\mathbb{R}$. We show that in the case of finite dimensional geometric R-convex subsets of affine R-spaces our algebraic closure has indeed all properties attributed to a closure.

We start with some basic definitions.

Definition 4.1. [7] Let R be a ring such that $\mathbb{Z} \subset R \subseteq \mathbb{R}$, and consider a mode $(C, \underline{I}^o(R))$. Let $(a, b) \in C \times C$. Denote by $\langle a, b\rangle$ the subalgebra generated by a and b, and by $\langle a, b\rangle^o$ its subalgebra $\langle a, b\rangle \setminus \{a, b\}$.

Let $s \in I^o(R)$. The pair (a, b) is called *s-eligible*, if for each $x \in \langle a, b\rangle^o$ there is a $y \in C$ with $b = xy\underline{s}$. The symbol $E_s(C)$ denotes the set of s-eligible pairs of $(C, \underline{I}^o(R))$.

Recall that a groupoid $(A, \cdot)$ is a *left quasigroup* if the equation $a \cdot x = b$ has a unique solution for each pair $(a, b) \in A^2$. (See e.g. [22, Ch.1, S.4.3].)

Lemma 4.2. *Let R be a ring such that $\mathbb{Z} \subset R \subseteq \mathbb{R}$, containing an invertible element $s \in I^o(R)$. Let $(A, \underline{I}^o(R))$ be the $\underline{I}^o(R)$-reduct of an affine R-space $(A, P, \underline{R})$. Then $(A, \underline{s})$ is a left quasigroup and*

$$E_s(A) = A \times A.$$

Proof. First note that A is in fact closed under each operation $\underline{r}$ for $r \in R$. Hence for $a, b \in A$, one has $ab\,\underline{1/s} \in A$. Let $c = ab\,\underline{1/s}$. Note that

$$a\,ab\,\underline{1/s}\,\underline{s} = a\,ab\underline{s}\,\underline{1/s} = b. \tag{4.1}$$

Indeed, the first equality follows by the idempotent and entropic laws, and the second by the following:

$$\begin{aligned} ac\underline{s} &= a\, ab\, \underline{1/s}\, \underline{s} \\ &= a(1-s) + [a(s-1)/s) + b(1/s)]s \\ &= a(1-s) + a(s-1) + b = b. \end{aligned}$$

By cancellativity, c is the unique element of A such that $ac\underline{s} = b$. This shows that $(A, \underline{s})$ is a left quasigroup, and also that each pair of elements of A is s-eligible. It follows that $E_s(A) = A \times A$. $\square$

In what follows we will assume that the ring R is a principal ideal subdomain of $\mathbb{R}$, such that $\mathbb{Z} \subset R$, containing an invertible element $s \in I^o(R)$.

Lemma 4.3. *Let $(C, \underline{I}^o(R))$ be an R-convex set. Then the set $E_s(C)$ forms a subalgebra of the algebra $(C \times C, \underline{I}^o(R))$, whence it is a member of the quasivariety $\mathcal{C}v(R)$.*

Proof. The first part is a direct consequence of [7, Lemma 7.1]. The second follows by the fact that R is a principal ideal subdomain of $\mathbb{R}$, whence the R-convex set $(C, \underline{I}^o(R))$ is a member of the quasivariety $\mathcal{C}v(R)$, the class of $\underline{I}^o(R)$-subreducts of faithful affine R-spaces. (Cp. the comments following Definition 2.1.) $\square$

Note that if $a \in C$, then $\langle a, a \rangle^o = \varnothing$, whence $(a, a) \in E_s(C)$. It follows that $E_s(C)$ is in fact a subdirect square of C.

Lemma 4.4. *Let $(C, \underline{I}^o(R))$ be an R-convex subset of an affine R-space $(A, P, \underline{R})$. Let $(a, b) \in C \times C$. Then (a, b) is an s-eligible pair of $(C, \underline{I}^o(R))$ if and only if $xb\underline{1/s} \in C$ for each $x \in \langle a, b \rangle^o$.*

Proof. By (4.1), for any elements $x, b \in A$ the element $y = xb\underline{1/s}$ is the unique element $y \in A$ such that $xy\underline{s} = b$. $\square$

Recall that R is a principal ideal subdomain of the ring $\mathbb{R}$, such that $\mathbb{Z} \subset R$, containing an invertible element $s \in I^o(R)$.

Definition 4.5. An R-convex subset $(C, \underline{I}^o(R))$ of an affine R-space $(A, P, \underline{R})$ is called *algebraically s-closed*, if for each s-eligible pair $(a, b) \in C \times C$, there is a $c \in C$ such that $b = ac\underline{s}$.

Note that in [7], such a set C would be called $\underline{s}$-closed.

Proposition 4.6. *An R-convex subset $(C, \underline{I}^o(R))$ of an affine R-space $(A, P, \underline{R})$ is algebraically s-closed if and only if $ab\underline{1/s} \in C$ for each s-eligible pair $(a, b) \in C \times C$.*

Proof. The proof follows by the fact that $c = ab\underline{1/s}$ is the unique element of A such that $b = ac\underline{s}$. (Cf. (4.1).) $\square$

Now for an R-convex subset $(C, \underline{I}^o(R))$ of a (necessarily faithful) affine R-space $(A, P, \underline{R})$ over a principal ideal subdomain R of $\mathbb{R}$ and an invertible element $s \in I^o(R)$, let

$$\mathtt{cl}_{R,s}(C) := \{ab\underline{1/s} \mid (a,b) \in E_s(C)\}. \tag{4.2}$$

We will call the set $\mathtt{cl}_{R,s}(C)$ the *s-closure* of $(C, \underline{I}^o(R))$. Obviously, if C is algebraically s-closed, then $\mathtt{cl}_{R,s}(C) = C$. We will show in this section that the operator $\mathtt{cl}_{R,s}$, when applied to geometric R-convex sets, has indeed all the properties usually attributed to a closure operator.

Lemma 4.7. *The s-closure $\mathtt{cl}_{R,s}(C)$ of an R-convex subset $(C, \underline{I}^o(R))$ of an affine R-space $(A, P, \underline{R})$ is a subalgebra of $(A, \underline{I}^o(R))$.*

Proof. Let $(a,b), (a',b') \in E_s(C)$. To prove that $\mathtt{cl}_{R,s}(C)$ is a subalgebra $(A, \underline{I}^o(R))$, it is sufficient to show that for each $p \in I^o(R)$, one has

$$ab\underline{1/s}\ a'b'\underline{1/s}\ \underline{p} \in \mathtt{cl}_{R,s}(C).$$

Now, the entropicity implies that

$$ab\underline{1/s}\ a'b'\underline{1/s}\ \underline{p} = aa'\underline{p}\ bb'\underline{p}\ \underline{1/s},$$

and by Lemma 4.3,

$$(aa'\underline{p}, bb'\underline{p}) = (a,b)(a',b')\,\underline{p} \in E_s(C).$$

It follows that the element $ab\underline{1/s}\ a'b'\underline{1/s}\ \underline{p}$ has the required form, whence indeed, it belongs to $\mathtt{cl}_{R,s}(C)$. $\square$

Note that the s-closure $\mathtt{cl}_{R,s}(C)$ is also a member of the quasivariety $\mathcal{C}v(R)$.

Lemma 4.8. *The following hold for the s-closures $\mathtt{cl}_{R,s}(B)$ and $\mathtt{cl}_{R,s}(C)$ of R-convex subsets $(C, \underline{I}^o(R))$ and $(B, \underline{I}^o(R))$ of an affine R-space $(A, P, \underline{R})$.*

(a) $C \le \mathtt{cl}_{R,s}(C)$;
(b) *If* $(B, \underline{I}^o(R)) \le (C, \underline{I}^o(R))$, *then*
$(\mathtt{cl}_{R,s}(B), \underline{I}^o(R)) \le (\mathtt{cl}_{R,s}(C), \underline{I}^o(R))$.

Proof. Let $c \in C$. To show that $c \in \mathtt{cl}_{R,s}(C)$, one should find $(a,b) \in E_s(C)$ such that $c = ab\,\underline{1/s}$. And indeed, $(c,c) \in E_s(C)$ and $c = cc\,\underline{1/s}$. Hence $c \in \mathtt{cl}_{R,s}(C)$. Obviously C is an $\underline{I}^o(R)$-subalgebra of $\mathtt{cl}_{R,s}(C)$.

To show (b), note that any pair $(a,b) \in E_s(B)$ is also a member of $E_s(C)$. By (4.1), $c = ab\,\underline{1/s}$ is the unique element of A such that

$ac\,\underline{s} = b$. It follows that if $ab\,\underline{1/s}$ is in $\mathtt{cl}_{R,s}(B)$, then it is also a member of $\mathtt{cl}_{R,s}(C)$. Clearly, $\mathtt{cl}_{R,s}(B)$ is a subalgebra of $\mathtt{cl}_{R,s}(C)$. □

From now on all R-convex sets we consider in this Section will be geometric convex subsets of a finite dimensional affine R-spaces $A = R^k$ over a principal ideal subdomain of $\mathbb{R}$ different from $\mathbb{Z}$. The affine R-space R^k will be considered as a subreduct of the affine $\mathbb{R}$-space $\mathbb{R}^k$, and may be equipped with the standard basis. (See the comments following Definition 2.1.) As before, geometric R-convex subsets of R^k will be considered as $\underline{I}^o(R)$-subreducts of the affine R-space R^k and of the affine $\mathbb{R}$-space $\mathbb{R}^k$. In particular, the affine R-space R will be considered as a subreduct of the affine $\mathbb{R}$-space $\mathbb{R}$, both generated by 0 and 1.

For distinct $a, b \in R^k$, consider the line $\ell_R(a, b)$ and the geometric segment $[a, b]_{geom}$. Define

$$[a, b[_{geom} := [a, b]_{geom} \setminus \{b\}.$$

Lemma 4.9. *Let C be a geometric R-convex subset of an affine R-space $(R^k, P, \underline{R})$ and let $(a, b) \in E_s(C)$. Then*

$$[a, ab\,\underline{1/s}[_{geom} \subseteq C.$$

Proof. Recall that $(a, b) \in E_s(C)$ means that $xb\,\underline{1/s} \in C$ for each $x \in \langle a, b\rangle^o$. If $a < x < y < b$ on the line $\ell_R(a, b)$, then obviously $b < yb\,\underline{1/s} < xb\,\underline{1/s} < ab\,\underline{1/s}$. Since C is geometric, it follows by Proposition 2.7 that

$$[a, ab\,\underline{1/s}[_{geom} = \bigcup_{x \in \langle a, b\rangle^o} [a, xb\,\underline{1/s}]_{geom} \subseteq C.$$

□

Corollary 4.10. *A geometric R-convex subset C of an affine R-space $(R^k, P, \underline{R})$ is s-closed if an only if for all $(a, b) \in E_s(C)$*

$$[a, ab\,\underline{1/s}]_{geom} \subseteq C.$$

Proof. This is a direct consequence of Proposition 4.6 and Lemma 4.9. □

Example 4.11. Let C' be the interior of the (closed) square C in the dyadic plane $\mathbb{D}^2$ with the vertices $(0, 0), (0, 1), (1, 0), (1, 1)$, together with the (closed) side joining the points $(0, 0)$ and $(1, 0)$. Obviously, both C and C' are geometric convex subsets of $\mathbb{D}^2$. The convex set C' is not algebraically 1/2-closed. (E.g. C' does not contain the point $(1/2, 1/2)(1/2, 3/4)\underline{2} = (1/2, 1)$ for the eligible pair $((1/2, 1/2), (1/2, 3/4))$.) On the other hand, it is easy to see that the square C contains all points $ab\underline{2}$ for all $1/2$-eligible pairs (a,b) of elements of C'. In particular, for any c on the boundary of C and $a \in C'$, there is a $b \in C'$ such that $ab\underline{2} = c$, namely $b = ac\underline{1/2}$. It follows that C is the algebraic $1/2$-closure of C'.

Lemma 4.12. *Let s and t be two invertible elements of the ring R, belonging to $I^o(R)$. Let $(C, \underline{I}^o(R))$ be a geometric R-convex subset of an affine R-space $(R^k, P, \underline{R})$. Then its s-closure $\mathtt{cl}_{R,s}(C)$ and t-closure $\mathtt{cl}_{R,t}(C)$ coincide.*

Proof. We can assume that $s \neq t$. Let $c \in \mathtt{cl}_{R,s}(C)$. Then (by definition) $c = ab\underline{1/s}$ for some $(a,b) \in E_s(C)$. We will show that there is $b' \in C$ such that $(a,b') \in E_t(C)$ and $ab\,\underline{1/s} = ab'\,\underline{1/t}$. Let $b' = ab\,\underline{t/s}$. Obviously, $b' \in L = \ell_R(a,b)$. Since $0 < t/s < 1/s$, it follows that $a < b' < c$, whence by Lemma 4.9, $b' \in C$. Then

$$\begin{aligned} ac\,\underline{t} &= a\,(ab\underline{1/s})\,\underline{t} \\ &= a(1-t) + a(t - t/s) + b(t/s) \\ &= a(1-t/s) + b(t/s) = ab\,\underline{t/s} = b'. \end{aligned}$$

Hence, by (4.1),

$$ab'\,\underline{1/t} = a\,(ac\underline{t})\,\underline{1/t} = c = ab\,\underline{1/s}.$$

To prove that $(a,b') \in E_t(C)$, we have to show that $xb'\,\underline{1/t} \in C$ for each $x \in \langle a, b'\rangle^o$. Indeed, by Lemma 4.9 , the segment $[a, c[_{geom}$ is contained in C, moreover all $xb'\,\underline{1/t}$ belong to $[a, c[_{geom}$ and hence to C, since $c = ab'\,\underline{1/t} > xb'\,\underline{1/t}$.

By Lemma 4.4, it follows that $c \in \mathtt{cl}_{R,t}(C)$, whence $\mathtt{cl}_{R,s}(C) \subseteq \mathtt{cl}_{R,t}(C)$. Since t and s have been chosen arbitrarily, one obtains that $\mathtt{cl}_{R,s}(C) = \mathtt{cl}_{R,t}(C)$. □

Definition 4.13. A geometric R-convex subset $(C, \underline{I}^o(R))$ of an affine R-space $(R^k, P, \underline{R})$ over a principal ideal subdomain R of $\mathbb{R}$ is called *algebraically closed*, if it is s-closed for some invertible element $s \in I^o(R)$. Then, in view of Lemma 4.12, the s-closure $\mathtt{cl}_{R,s}(C)$ of C will be called the *algebraic closure* or simply the *closure* of C. It will sometimes be denoted by $\mathtt{cl}_R(C)$.

Let us note that this definition concerns only invertible elements $s \in I^o(R)$, while the definition of a closure in [7] considers all $s \in I^o(R)$.

Recall that the convex $\mathbb{R}$-hull $\mathtt{conv}_{\mathbb{R}}(C)$ of an R-convex subset C of R^k in the affine space $\mathbb{R}^k$ is the subalgebra of $(\mathbb{R}^k, \underline{I}^o)$ generated by the set C. Then the convex R-hull $\mathtt{conv}_R(C)$ of C in R^k is $\mathtt{conv}_{\mathbb{R}}(C) \cap R^k$.

Lemma 4.14. *Let C be a k-dimensional geometric convex subset of the affine R-space $(R^k, P, \underline{R})$, where $k = 1, 2, \dots$. Let s be an invertible element of $I^o(R)$. Then*

$$\mathtt{cl}_{\mathbb{R},s}(\mathtt{conv}_{\mathbb{R}}(C)) \cap R^k = \mathtt{conv}_{\mathbb{R}}(\mathtt{cl}_{R,s}(C)) \cap R^k. \tag{4.3}$$

Proof. First note that Proposition 2.7 implies

$$C = \mathtt{conv}_R(C) = \mathtt{conv}_{\mathbb{R}}(C) \cap R^k. \tag{4.4}$$

It follows that

$$\mathtt{cl}_{R,s}(C) = \mathtt{cl}_{R,s}(\mathtt{conv}_R(C)) = \mathtt{cl}_{R,s}(\mathtt{conv}_{\mathbb{R}}(C) \cap R^k). \tag{4.5}$$

On the other hand

$$\mathtt{conv}_{\mathbb{R}}(C) \subseteq \mathtt{conv}_{\mathbb{R}}(\mathtt{cl}_{R,s}(C)) \subseteq \mathtt{cl}_{\mathbb{R},s}(\mathtt{conv}_{\mathbb{R}}(C)). \tag{4.6}$$

Indeed, the first inclusion follows by Lemma 4.8(a). To show the second one, note that all s-eligible pairs of elements in C are also s-eligible in $\mathtt{conv}_{\mathbb{R}}(C)$. It follows by (4.2) that

$$\mathtt{cl}_{R,s}(C) \subseteq \mathtt{cl}_{\mathbb{R},s}(\mathtt{conv}_{\mathbb{R}}(C)). \tag{4.7}$$

Note that by Lemma 4.7 applied to $\mathbb{R}$, $\mathtt{cl}_{\mathbb{R},s}(\mathtt{conv}_{\mathbb{R}}(C))$ is an $\mathbb{R}$-convex set, i.e. it is a convex set in the usual traditional sense. This fact, together with (4.7), implies the second inclusion in (4.6).

Finally we prove the following equality

$$\mathtt{cl}_{R,s}(\mathtt{conv}_{\mathbb{R}}(C) \cap R^k) = \mathtt{cl}_{R,s}(C) = \mathtt{cl}_{\mathbb{R},s}(\mathtt{conv}_{\mathbb{R}}(C)) \cap R^k. \tag{4.8}$$

The first equality of (4.8) follows by (4.5). By (4.7), we have

$$\mathtt{cl}_{R,s}(C) \subseteq \mathtt{cl}_{\mathbb{R},s}(\mathtt{conv}_{\mathbb{R}}(C)) \cap R^k.$$

We need to show the reverse inclusion. Recall that by (4.2)

$$\mathtt{cl}_{R,s}(C) = \{ab\underline{1/s} \mid (a, b) \in E_s(C)\}. \tag{4.9}$$

Note that

$$A := \mathtt{cl}_{\mathbb{R},s}(\mathtt{conv}_{\mathbb{R}}(C)) = \{ab\underline{1/s} \mid (a, b) \in E_s(\mathtt{conv}_{\mathbb{R}}(C))\}. \tag{4.10}$$

Let $c = ab\underline{1/s} \in A \cap R^k$. Let $d \in C$ belong to the relative interior of C. (See the definition right before Lemma 2.2.) Without loss of generality assume that on the line $\ell_{\mathbb{R}}(d, c)$, one has $d < c$. Consider the (geometric) segment $[d, c]_{\mathbb{R}}$ of $\ell_{\mathbb{R}}(d, c)$.

First note that $[d, c]_{\mathbb{R}}$ is contained in A. Indeed, by Corollary 2.5 of [7], the algebraic s-closure of an $\mathbb{R}$-convex subset of $\mathbb{R}^k$ coincides with its topological closure, and obviously, the topological closure of an $\mathbb{R}$-convex subset of $\mathbb{R}^k$ is again $\mathbb{R}$-convex. Since A is an $\mathbb{R}$-convex subset of $\mathbb{R}^k$ and $d \in C \subseteq A$, it follows that $[d, c]_{\mathbb{R}} \subseteq A$.

Next we show that $[d, c[_{\mathbb{R}}$ is contained in $\texttt{conv}_{\mathbb{R}}(C)$. Recall that, by Theorem 3.3 of [4], if B is a convex subset of $\mathbb{R}^k$ in the usual sense (i.e. it is $\mathbb{R}$-convex), x belongs to the relative interior of B and y belongs to the topological closure of B with $x \neq y$, then $[x, y[\subset B$. We apply this fact to $A = \texttt{conv}_{\mathbb{R}}(C)$ and $x = d$, and to $y = c$, to obtain that $[d, c[_{\mathbb{R}} \subseteq \texttt{conv}_{\mathbb{R}}(C)$.

Now, since $dc\underline{s} \in [d, c[_{\mathbb{R}}$ and C is geometric, we conclude by (4.4) that

$$dc\underline{s} \in [d, c[_{\mathbb{R}} \cap R^k \subseteq \texttt{conv}_{\mathbb{R}}(C) \cap R^k = \texttt{conv}_R(C) = C.$$

By (4.1), we know that $c = d(dc\underline{s})\underline{1/s}$. Since $(d, dc\underline{s}) \in E_s(A)$, both d and $dc\underline{s}$ belong to C and c belongs to R^k, it follows that $(d, dc\underline{s}) \in E_s(C)$, and finally that $c \in \texttt{cl}_{R,s}(C)$. This completes the proof of (4.8).

Now we are ready to prove the equality (4.3). The inclusion $\supseteq$ follows by (4.6). By (4.8), the left-hand side of (4.3) equals $\texttt{cl}_{R,s}(C)$. Hence the inclusion $\subseteq$ of (4.3) follows by the obvious fact that $\texttt{cl}_{R,s}(C) \subseteq \texttt{conv}_{\mathbb{R}}(\texttt{cl}_{R,s}(C))$. □

Proposition 4.15. *Let C be a k-dimensional geometric convex subset of the affine R-space R^k, where $k = 1, 2, \ldots$. Let s be an invertible element of $I^o(R)$. Then the closure $\texttt{cl}_{R,s}(C)$ is also a geometric convex subset of R^k. In particular,*

$$\texttt{cl}_{R,s}(C) = \texttt{conv}_R(\texttt{cl}_{R,s}(C)). \tag{4.11}$$

Proof. First note that, by definition,

$$\texttt{conv}_R(\texttt{cl}_{R,s}(C)) = \texttt{conv}_{\mathbb{R}}(\texttt{cl}_{R,s}(C)) \cap R^k. \tag{4.12}$$

By (4.5), $\texttt{cl}_{R,s}(C) = \texttt{cl}_{R,s}(\texttt{conv}_{\mathbb{R}}(C) \cap R^k)$. This and (4.12) imply that (4.11) can be written as

$$\texttt{cl}_{R,s}(\texttt{conv}_{\mathbb{R}}(C) \cap R^k) = \texttt{conv}_{\mathbb{R}}(\texttt{cl}_{R,s}(C)) \cap R^k, \tag{4.13}$$

or by (4.8), as

$$\texttt{cl}_{\mathbb{R},s}(\texttt{conv}_{\mathbb{R}}(C)) \cap R^k = \texttt{conv}_{\mathbb{R}}(\texttt{cl}_{R,s}(C)) \cap R^k, \tag{4.14}$$

which holds by Lemma 4.14. □

The following proposition and Lemma 4.8 justify the name of a closure for the closure $\texttt{cl}_R(C)$ of a geometric convex set C.

Proposition 4.16. *Let C be a k-dimensional geometric convex subset of the affine R-space $(R^k, P, \underline{R})$, where $k = 1, 2, \ldots$. Let s be an invertible element of $I^o(R)$. Then*

$$\texttt{cl}_{R,s}(\texttt{cl}_{R,s}(C)) = \texttt{cl}_{R,s}(C). \tag{4.15}$$

Proof. By Proposition 4.15, the closure $\mathtt{cl}_{R,s}(C)$ is also a geometric convex subset of R^k. The equality (4.15) is clear if C is the $\underline{I}^o(R)$-reduct of the affine space R^k, and in the case when $\mathtt{cl}_{R,s}(C) = C$. So assume now that C is a proper subreduct of R^k different from $\mathtt{cl}_{R,s}(C)$. If $k = 1$, then C is an interval of R, and $\mathtt{cl}_{R,s}(C)$ consists of all elements of C together with those end-points of $\mathtt{conv}_{\mathbb{R}}(C)$ in $\mathbb{R}$ that belong to R. In this case, the proposition obviously holds. In what follows we assume that k is at least 2.

Consider the geometric segment $[c_1, c_2]_{geom}$ of R^k such that $(c_1, c_2) \in E_s(\mathtt{cl}_{R,s}(C))$. In view of Lemma 4.9 and Corollary 4.10, it is sufficient to show that $c := c_1c_2\,\underline{1/s}$ also belongs to $\mathtt{cl}_{R,s}(C)$.

First note that if both c_1 and c_2 are in C, then obviously $c_1c_2\,\underline{1/s} \in \mathtt{cl}_{R,s}(C)$. Now assume that at least one of c_1 and c_2 is not in C. Let, say, $c_1 \notin C$. Since $\mathtt{cl}_{R,s}(C)$ is geometric, it follows that $[c_1, c_2]_{geom} \subseteq \mathtt{cl}_{R,s}(C)$. Moreover, by Lemma 4.9, $[c_1, c[_{geom} \subseteq \mathtt{cl}_{R,s}(C)$. Since $\dim C > 1$, we may pick a point $a_1 \in C$ such that a_1, c_1 and c are affinely independent. These three points generate the R-plane, the real plane, and the closed triangle Δ contained in the real plane. Let Δ' be the triangle Δ without the side joining the vertices a_1 and c. Since $\mathtt{cl}_{R,s}(C)$ is geometric, it follows by Proposition 2.7 that $\Delta'_R = \Delta' \cap R^k \subseteq \mathtt{cl}_{R,s}(C)$. Let a be in the interior of Δ'_R. Similarly as in the proof of Lemma 4.14, one shows that $b = ac\underline{s} \in C$ and $(a, b) \in E_s(C)$. Hence $c = a\,ac\underline{s}\underline{1/s} = ab\underline{1/s} \in \mathtt{cl}_{R,s}(C)$. □

5. The algebraic and other closures

In this section we discuss relations between the algebraic closure introduced in Section 4 and the closures of [7]. We show that in the case of interest our algebraic closure coincides with the one-step p-closure of [7]. Nevertheless, the existence and uniqueness of our algebraic closure follow in a much simpler and more concise way. Moreover, we show that in the case of finite dimensional geometric convex subsets of affine R-spaces over principal ideal subdomains of $\mathbb{R}$, their algebraic and topological closures coincide.

Let R be a commutative unital ring and let $s \in R$. Consider an affine R-space $(A, P, \underline{R})$. Define the following relation $\sim_s$ on the set $A \times A$:

$$(a_1, b_1) \sim_s (a_2, b_2) \text{ if and only if } a_1b_2\underline{s} = a_1a_2\underline{s}\,b_1\underline{s}.$$

Lemma 5.1. *For any commutative unital ring R and $s \in R$, the relation $\sim_s$ is a congruence relation of the affine space $(A \times A, P, \underline{R})$.*

Proof. First note that the variety $\underline{\underline{R}}$ is a Mal'cev variety. Hence a subalgebra of the affine R-space A^4 is a congruence on $A \times A$ if and only if it is reflexive. (Cf. [25].) The reflexivity of $\sim_s$ follows easily by idempotence:

$$a_1 b_1 \underline{s} = a_1 a_1 \underline{s}\, b_1 \underline{s}.$$

To show that the operations $\underline{r}$, for $r \in R$, are compatible with $\sim_s$ assume that for $i = 1, 2$, one has $(a_i, b_i) \sim_s (c_i, d_i)$, whence $a_i d_i \underline{s} = a_i c_i \underline{s}\, b_i \underline{s}$. By entropicity and the definition of $\sim_s$

$$\begin{aligned} a_1 a_2 \underline{r}\ d_1 d_2 \underline{r}\ \underline{s} &= a_1 d_1 \underline{s}\ a_2 d_2 \underline{s}\ \underline{r} \\ &= (a_1 c_1 \underline{s}\, b_1 \underline{s})\,(a_2 c_2 \underline{s}\, b_2 \underline{s})\ \underline{r} \\ &= (a_1 c_1 \underline{s}\ \ a_2 c_2 \underline{s}\ \underline{r})\ (b_1 b_2 \underline{r})\ \underline{s} \\ &= (a_1 a_2 \underline{r}\ \ c_1 c_2 \underline{r}\ \underline{s})\ (b_1 b_2 \underline{r})\ \underline{s}. \end{aligned}$$

It follows that $(a_1 a_2 \underline{r}, b_1 b_2 \underline{r}) \sim_s (c_1 c_2 \underline{r}, d_1 d_2 \underline{r})$. Similarly, one proves that P is compatible with $\sim_s$. It follows that $\sim_s$ is a congruence on $(A \times A, P, \underline{R})$. □

Lemma 5.2. *Let R be a commutative unital ring and let $s \in R$. Let a and b be elements of an affine space $(A, P, \underline{R})$. If the operation $\underline{s}$ is cancellative, then the following hold:*

(1) $(b, b) \sim_s (a, ab\underline{s})$*;*
(2) *the mapping*

$$\varphi : A \to (A \times A)/ \sim_s\ ;\ a \mapsto (a, a)^{\sim_s}$$

is a monomorphism of affine R-spaces.

Proof. Idempotence and entropicity imply that $b\, ab\underline{s}\, \underline{s} = bb\underline{s}\, ab\underline{s}\, \underline{s} = ba\underline{s} b\, \underline{s}$. Hence the first condition holds.

To show that φ is injective, assume that $(a, a) \sim_s (b, b)$, whence $ab\underline{s} = ab\underline{s}\, a\, \underline{s} = ab\underline{s}\, aa\underline{s}\, \underline{s} = aa\underline{s}\, ba\underline{s}\, \underline{s} = a\, ba\underline{s}\, \underline{s}$. Since the operation $\underline{s}$ is cancellative, it follows that $b = ba\underline{s}$. Idempotence and cancellativity again imply that $a = b$. This shows that φ is injective. As φ is the composition of the homomorphism $A \to A \times A; a \mapsto (a, a)$ and the natural homomorphism $A \times A \to (A \times A)/ \sim_s; (a, a) \mapsto (a, a)^{\sim_s}$, it follows that φ is a homomorphism. □

Note that the relation $\sim_s$ restricts to subreducts of an affine space $(A \times A, P, \underline{R})$. In particular, if R is a subring of $\mathbb{R}$ and $s \in I^o(R)$, then it is a congruence relation of its $\underline{I}^o(R)$-subreducts.

The relation $\sim_s$ was introduced in [7] under the name of an *aiming congruence* for all cancellative members of the variety generated by $(R, \underline{I}^o(R)))$ and any $s \in I^o(R)$, where $\mathbb{Z} \subset R \subseteq \mathbb{R}$. However, in such a general case, although $\sim_s$ is reflexive and compatible, it is not necessarily a congruence relation. Some necessary conditions for this relation to be a congruence relation were discussed in [7]. In the present paper we are mainly interested in the case where the ring R is a principal ideal subdomain of $\mathbb{R}$ and the corresponding $\underline{I}^o(R)$-subreducts of affine R-spaces are R-convex sets. In such a case, for any $s \in I^o(R)$, the relation $\sim_s$ is a congruence relation of $(C \times C, \underline{I}^o(R))$ for each R-convex set $(C, \underline{I}^o(R))$. In particular, $\sim_s$ is a congruence relation of the algebra $(E_s(A), \underline{I}^o(R))$ for any faithful affine R-space A. (Compare Lemma 4.2).

Lemma 5.3. *Let R be a ring such that $\mathbb{Z} \subset R \subseteq \mathbb{R}$, containing an invertible element $s \in I^o(R)$. Let $(A, \underline{I}^o(R))$ be the $\underline{I}^o(R)$-reduct of an affine R-space $(A, P, \underline{R})$. Then $(E_s(A), \underline{I}^o(R))/{\sim_s}$ is isomorphic to $(A, \underline{I}^o(R))$.*

Proof. Let us consider the quotient $E_s(A)/{\sim_s} = (A \times A)/{\sim_s}$, and the mapping φ of Lemma 5.2 (2). The mapping is obviously a monomorphism of $\underline{I}^o(R)$-reducts. We will show that it is surjective. Let $(a, b)^{\sim_s} \in E_s(A)/{\sim_s}$. Let c be the (uniquely defined) element such that $ac\underline{s} = b$. By Lemma 5.2 (1), $(a, b) = (a, ac\underline{s}) \sim_s (c, c)$. Hence $\varphi(c) = (a, b)^{\sim_s}$, and the mapping φ is onto. □

If $R = F$, where F is a subfield of the field $\mathbb{R}$, the congruence $\sim_s$ has a nice geometric interpretation. Assume that $(C \times C, \underline{I}^o(F))$ is a subreduct of $(F^n \times F^n, \underline{F})$. Consider two pairs (a_1, b_1) and (a_2, b_2) of elements of C lying on distinct lines of F^n and intersecting at the point c, and such that $\{a_1, b_1\} \neq \{a_2, b_2\}$. Without loss of generality assume that the points c, a_1 and a_2 form the vertices of a (non-trivial) triangle. Then $(a_1, b_1) \sim_s (a_2, b_2)$ (i.e. $a_1 b_2 \underline{s} = a_1 a_2 \underline{s}\, b_1 \underline{s}$) means that

$$b_1 = a_1 c\underline{s} \text{ and } b_2 = a_2 c\underline{s}.$$

(Compare Figure 1 in [7].) Moreover the lines through a_2 and c, and through b_1 and $a_1 a_2 \underline{s}$ are parallel. We may say that the pairs (a_1, b_1) and (a_2, b_2), both "aim" at the same point c with respect to $\underline{s}$. Note that the point c is not necessarily a member of the set C.

Now let us consider the quotient $(E_s(C), \underline{I}^o(R))/{\sim_s}$ of $(E_s(C), \underline{I}^o(R))$ by the aiming congruence $\sim_s$. This quotient coincides with the one-step s-closure $K_s^{(1)}(C)$ introduced in [7, Sect. 7] for a broader class of modes. The next proposition shows that if the $I^o(R)$ contains an invertible element s, then in the case of R-convex sets, $K_s^{(1)}(C)$ coincides with the algebraic s-closure $\mathtt{cl}_{R,s}(C)$ introduced in the previous section.

Proposition 5.4. *Let $(C, \underline{I}^o(R))$ be an R-convex subset of an affine R-space $(A, P, \underline{R})$. If $I^o(R)$ contains an invertible element s, then* *the algebraic s-closure* $(\mathtt{cl}_{R,s}(C), \underline{I}^o(R))$ *of* $(C, \underline{I}^o(R))$ *is isomorphic to* $(E_s(C), \underline{I}^o(R))/{\sim_s}$ *, i.e.*

$$(\mathtt{cl}_{R,s}(C), \underline{I}^o(R)) \cong (E_s(C), \underline{I}^o(R))/{\sim_s}.$$

Proof. Let us fix an invertible $s \in I^o(R)$ and define the following mapping

$$h : E_s(C) \longrightarrow \mathtt{cl}_{R,s}(C);\ (a, b) \mapsto ab\underline{1/s}.$$

We will show that h is an $\underline{I}^o(R)$-homomorphism. Indeed, for each $p \in I^o(R)$:

$$\begin{aligned} &h((a, b)\,(c, d)\,\underline{p}) \\ &= h((ac\underline{p}, bd\underline{p})) \\ &= ac\underline{p}\ bd\underline{p}\ \ \underline{1/s} \\ &= ab\underline{1/s}\ cd\underline{1/s}\ \ \underline{p} \\ &= h((a, b))\ h((c, d))\ \ \underline{p}. \end{aligned}$$

This shows that h is indeed a homomorphism.

Now it remains to show that $ker h = {\sim_s}$. Recall that $(a_1, a_2) \sim_s (b_1, b_2)$ means that

$$a_1 b_2 \underline{s} = a_1 b_1 \underline{s}\, a_2\, \underline{s},$$

or

$$a_1(1 - s) + b_2 s = a_1(1 - s)^2 + b_1 s(1 - s) + a_2 s,$$

or equivalently

$$b_1(s^2 - s) + b_2 s = a_1(s^2 - s) + a_2 s. \tag{5.1}$$

The last equality means that

$$b_1 b_2 \underline{1/s} = b_1(1 - 1/s) + b_2(1/s) = a_1(1 - 1/s) + a_2(1/s) = a_1 a_2 \underline{1/s}.$$

□

As an immediate corollary of Proposition 5.4 we obtain the following.

Corollary 5.5. *Let R be a principal ideal subdomain of the ring $\mathbb{R}$ properly containing $\mathbb{Z}$ and containing an invertible element $s \in I^o(R)$. The following conditions are equivalent for a k-dimensional geometric R-convex subset C of the affine R-space R^k, where $k = 1, 2, \ldots$.*

(a) $(C, \underline{I}^o(R))$ *is algebraically closed, i.e.* $(C, \underline{I}^o(R)) = (\mathtt{cl}_{R,s}(C), \underline{I}^o(R))$,

(b) $(C, \underline{I}^o(R)) \cong (E_s(C), \underline{I}^o(R))/{\sim_s}$.

Recall that the quotient $(E_s(C), \underline{I}^o(R))/{\sim_s}$ coincides with the one-step s-closure $K_s^{(1)}(C)$ of C introduced in [7, S.7] for a broader class of modes. This class is a certain subquasivariety $\mathcal{H}(T)$ of cancellative members of the variety generated by the $(T, \underline{I}^o(T))$, where T is a subring of $\mathbb{R}$ properly containing $\mathbb{Z}$, and containing the quasivariety $\mathsf{Q}(T)$ generated by $(T, \underline{I}^o(T))$. The one-step s-closure (defined for any $s \in I^o(T)$) is then used to define a much more general type of a closure denoted $K_\Gamma^\infty(C)$, where $\Gamma \subseteq I^o(T)$. By the main result of [7], all members C of $\mathcal{H}(T)$ have a uniquely defined closure $K_\Gamma^\infty(C)$, which is also contained in $\mathcal{H}(T)$. This closure is constructed as a directed co-limit of certain intermediate closures (the first being a one-step closure). In particular, if T is a subring of a subfield F of $\mathbb{R}$ and C is an F-convex subset of the affine F-space F^n, then the closure $K_{I^o(T)}^\infty(C)$ of the reduct $(C, \underline{I}^o(T))$ of $(C, \underline{I}^o(F))$ is isomorphic to its one-step closure $K_p^{(1)}(C)$ for any $p \in I^o(T)$, and is also isomorphic to its topological closure. Moreover, $K_p^{(1)}(C)$ belongs to the quasivariety $\mathsf{Q}(T)$. (See [7, Prop. 2.4].) Note however that this algebra is generally not an R-convex subset of the affine R-space R^n for $n = 1, 2, \ldots$.

Proposition 5.4 shows that in the case of finite dimensional geometric convex subsets of (faithful) affine spaces over principal ideal subdomains of $\mathbb{R}$, the concept of the one-step s-closure $K_s^{(1)}(C)$ coincides with the concept of an algebraic closure introduced in this paper. And also in this case, the (algebraic) closure belongs to the quasivariety $\mathsf{Q}(R)$. Moreover, Proposition 5.4 provides a simple description of $K_s^{(1)}(C)$.

It is clear that if $R = F = \mathbb{R}$, then our algebraic closure, the closure defined in [7] and the topological closure of any $(C, \underline{I}^o(\mathbb{R}))$ coincide. (Compare [7, Cor. 2.5].) We will show that our algebraic closure and the topological closure coincide also in the case of finite dimensional geometric R-convex subsets discussed above. We consider the usual Euclidean topology on $\mathbb{R}^k$. Then R^k is a topological subspace of $\mathbb{R}^k$. Its closed (open) sets are simply closed (open) subsets of $\mathbb{R}^k$ intersected with R^k.

For a geometric convex subset C of R^k, let $\mathtt{cl}_R^{top}(C)$ be its topological closure in R^k, and $\mathtt{cl}_{\mathbb{R}}^{top}(C)$ its topological closure in $\mathbb{R}^k$. Observe that

$$\mathtt{cl}_R^{top}(C) = \mathtt{cl}_{\mathbb{R}}^{top}(C) \cap R^k \tag{5.2}$$

and since R is dense in $\mathbb{R}$,

$$\mathtt{cl}_{\mathbb{R}}^{top}(C) = \mathtt{cl}_{\mathbb{R}}^{top}(\mathtt{conv}_{\mathbb{R}}(C)). \tag{5.3}$$

Theorem 5.6. *Let R be a principal ideal subdomain of the ring $\mathbb{R}$ properly containing $\mathbb{Z}$. Let $(C, \underline{I}^o(R))$ be a k-dimensional geometric* *convex subset of an affine R-space $(R^k, P, \underline{R})$. Then the algebraic closure $\mathtt{cl}_R(C)$ of C and the topological closure $\mathtt{cl}_R^{top}(C)$ of C in R^k coincide:*

$$\mathtt{cl}_R(C) = \mathtt{cl}_R^{top}(C).$$

Proof. First note that if C is the $I^o(R)$-reduct of the affine space R^k, then C is obviously both algebraically and topologically closed. So in what follows we assume that C is a proper $\underline{I}^o(R)$-subreduct of R^k.

Recall that by Proposition 2.7, an R-convex subset C of R^k is geometric precisely if $C = \mathtt{conv}_R(C)$. In particular,

$$C = \mathtt{conv}_R(C) = \mathtt{conv}_{\mathbb{R}}(C) \cap R^k.$$

Note also that by (4.3)

$$\mathtt{cl}_{\mathbb{R}}(\mathtt{conv}_{\mathbb{R}}(C)) \cap R^k = \mathtt{conv}_{\mathbb{R}}(\mathtt{cl}_R(C)) \cap R^k. \tag{5.4}$$

Now since algebraic and topological closures of convex subsets in affine $\mathbb{R}$-spaces coincide, it follows by (5.2) , (5.3), (5.4), (4.12) and (4.11) that

$$\begin{aligned}
\mathtt{cl}_R^{top}(C) &= \mathtt{cl}_{\mathbb{R}}^{top}(C) \cap R^k = \mathtt{cl}_{\mathbb{R}}^{top}(\mathtt{conv}_{\mathbb{R}}(C)) \cap R^k \\
&= \mathtt{cl}_{\mathbb{R}}(\mathtt{conv}_{\mathbb{R}}(C)) \cap R^k = \mathtt{conv}_{\mathbb{R}}(\mathtt{cl}_R(C)) \cap R^k \\
&= \mathtt{conv}_R(\mathtt{cl}_R(C)) = \mathtt{cl}_R(C).
\end{aligned}$$

□

Let us note that Theorem 5.6 and Proposition 4.16 hold also without the assumption that C is k-dimensional. Indeed, if $\mathtt{dim}C = n < k$, then C is n-dimensional in its affine R-hull.

Note also that Theorem 5.6 and Proposition 5.4 complement and extend Proposition 2.4 of [7].

Example 5.7. In particular, Theorem 5.6 holds for geometric convex subsets of affine $\mathbb{Z}[1/p]$-spaces (among them also affine $\mathbb{D}$-spaces) providing a simple purely algebraic description of their topological closures.

Example 5.8. Consider the closed interval $[0,3]_{\mathbb{R}}$ and the open interval $]0,3[_{\mathbb{R}}$ of $\mathbb{R}$, and similarly, the closed and open intervals $[0,3]_{\mathbb{D}}$ and $]0,3[_{\mathbb{D}}$ of $\mathbb{D}$. The symbol $\langle 0,3\rangle$ denotes the $\underline{I}^o(\mathbb{D})$-subalgebra of $([0,3]_{\mathbb{D}}, \underline{I}^o(\mathbb{D}))$ generated by 0 and 3, and $\langle 0,3\rangle^o = \langle 0,3\rangle \setminus \{0,3\}$ denotes its $\underline{I}^o(\mathbb{D})$-subalgebra. Note that $\mathtt{cl}_{\mathbb{D}}(]0,3[_{\mathbb{D}}) = [0,3]_{\mathbb{D}}$. And since $\langle 0,3\rangle$ is isomorphic with the interval $[0,1]_{\mathbb{D}}$ and $\langle 0,3\rangle^o$ is isomorphic with the interval $]0,1[_{\mathbb{D}}$, it follows that $\mathtt{cl}_{\mathbb{D}}(\langle 0,3\rangle^o) = \langle 0,3\rangle$. On the other hand, the topological closure of both of these algebras in the space $\mathbb{D}$ is equal to $[0,3]_{\mathbb{D}} = [0,3]_{\mathbb{R}} \cap \mathbb{D}$. This shows that for an R-convex set which is not geometric the algebraic closure does not need to coincide with the topological closure. It also shows that two R-convex sets with the same topological closures may have different algebraic closures. (Cf. an example in [7, Lemma 2.7] of two isomorphic convex subsets in an infinite dimensional affine space over a subfield of $\mathbb{R}$, having an isomorphic closure (as defined in [7]) but different topological ones).

Let us also note that for a k-dimensional geometric convex subset C of the affine R-space R^k, its algebraic closure $\mathtt{cl}_R(C)$ is the algebraic closure of only one open geometric convex subset of R^k, namely the interior of $\mathtt{cl}_R(C)$.

Example 5.9. Consider any two open bounded 2-dimensional geometric convex subsets C and D of the affine R-space R^2. By Corollary 3.2, the $\underline{I}^o(R)$-algebras C and D are isomorphic precisely if there is an automorphism of the affine space R^2 that takes C to D. The same automorphism provides an $\underline{I}^o(R)$-isomorphism of the algebraic closures $\mathtt{cl}_R(C)$ and $\mathtt{cl}_R(D)$ which by Theorem 5.6 coincide with the topological closures $\mathtt{cl}_R^{top}(C)$ and $\mathtt{cl}_R^{top}(D)$, respectively. In particular, it maps the boundary of $\mathtt{cl}_R(C)$ onto the boundary of $\mathtt{cl}_R(D)$. So if there is no such an automorphism of R^2, then $\mathtt{cl}_R(C)$ and $\mathtt{cl}_R(D)$ cannot be isomorphic. Moreover, two open convex sets C and D cannot be isomorphic if their closures are non-isomorphic. (Compare this example with Exercise 2.6 of [7] containing an example of non-isomorphic infinite dimensional convex sets.)

The algebraic closure introduced in this paper concerns geometric convex subsets of finite-dimensional affine spaces. The finite dimensionality is essential here, and it was also essential in defining our generalization of convex sets. Results of [7] show that the methods used in this section do not necessarily extend to the case of R-convex subsets of affine R-spaces that are not finite-dimensional, or to arbitrary subreducts of F-convex sets for subfields F of $\mathbb{R}$ containing R. On the other hand, the results of [7] are not powerful enough to yield the results of this section. It is still an open question as to which other classes of $\underline{I}^o(T)$-subreducts of affine spaces over subrings T of $\mathbb{R}$ would admit a (possibly simple) algebraic description of the topological closure of their members.

Another interesting problem concerns a characterization of isomorphic $\underline{I}^o(T)$-subreducts of affine T-spaces and, more generally, their reducts of a fixed type. Section 3 provided some results of this kind. Let us also mention a result of [6] characterizing isomorphic $I^o(T)$-subreducts of F-convex sets for subfields F of $\mathbb{R}$ containing T, in a similar way as in Corollary 3.2. The proof of this result is, however, much more difficult, and requires different methods. Surprisingly, neither of the two results implies the other one.

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