WHEN DO COALITIONS FORM A LATTICE?

Gábor Czédli and György Pollák

Abstract. Given a finite partially ordered set P, for subsets or, in other words, coalitions X, Y of P let $X \leq Y$ mean that there exists an injection φ: $X \to Y$ such that $x \leq \varphi(x)$ for all $x \in X$. The set $\mathcal{L}(P)$ of all subsets of P equipped with this relation is a partially ordered set. All partially ordered sets P such that $\mathcal{L}(P)$ is a lattice are determined, and this result is extended to quasiordered set P versus q-lattice $\mathcal{L}(P)$ as well. Some elementary properties of distributive lattices $\mathcal{L}(P)$ are also given.

Dedicated to Professors László Leindler on his 60th and Károly Tandori on his 70th birthday

Motivation and preliminaries

In game theory or in the mathematics of human decision making the following situation is frequently considered, cf. e.g. Peleg [5]. Given a finite set P, for example we may think of P as a set of political parties, and each $x \in P$ has a certain strength measured on a numerical scale that we may think of as the number of votes x receives. Subsets of P are called *coalitions*. The strength of a coalition is the sum of strengths of its members. Let $\mathcal{L}(P)$ stand for the set of all coalitions. The relation "stronger or equally strong" is a quasiorder on P and also on $\mathcal{L}(P)$. The quasiorder on P has some influence on the quasiorder on $\mathcal{L}(P)$. Sometimes, like before the election in our example, all we have is a quasiorder or, more frequently, a partial order on P, supplied e.g. by a public opinion poll. Yet, as we will see, this often suffices to build some algebraic structure on $\mathcal{L}(P)$.

From now on, let $P = \langle P, \leq \rangle$ be a fixed finite quasiordered set, i.e., $\leq$ is a reflexive and transitive relation on the finite set P. For $x, y \in P$, $x > y$ means that $y \leq x$ and $x \not\leq y$. For undefined terminology the reader is referred to Grätzer [4]. Even without explicit mentioning, all sets occurring in this paper are assumed to be finite. The set of all subsets, alias coalitions, of P is denoted by $\mathcal{L}(P)$. For $X, Y \in \mathcal{L}(P)$, a map φ: $X \to Y$ is called an *extensive map* if φ is injective and for every $x \in X$ we have $x \leq \varphi(x)$. Let $X \leq Y$ mean that there exists an extensive map $X \to Y$; this definition turns $\mathcal{L}(P)$ into a quasiordered set $\mathcal{L}(P) = \langle \mathcal{L}(P), \leq \rangle$. Using singleton coalitions one can easily see that P is a partially ordered set, in short a poset, iff $\mathcal{L}(P)$ is a poset. Our main result, Thm. 2, describes the posets P

1991 *Mathematics Subject Classification.* Primary 06B99, Secondary 06A99, 90D99.

Key words and phrases. Lattice, q-lattice, quasiorder, partially ordered set, coalition.

This work of both authors was partially supported by the Hungarian National Foundation for Scientific Research (OTKA) grant no. 1903

for which $\mathcal{L}(P)$ is a lattice. However, to achieve more generality without essentially lengthening the proof, Thm. 2 will be concluded from its generalization Thm. 1 for quasiorders.

Definition. A quasiordered set P is called *upper bound free*, in short UBF, if for any $a, b, c \in P$ we have

$$((a \le c) \,\&\, (b \le c)) \implies ((a \le b) \text{ or } (b \le a)).$$

The equivalence classes of the equivalence generated by $\le_P$ will be called the *components* of P. If P is an UBF poset and has only one component then P is called a *tree*. A poset is called a *forest* if its components are trees. Clearly, a finite poset is a forest iff it is UBF. Let $\overline{P} = \langle \overline{P}, \le \rangle$ denote the poset obtained from P in the canonical way, i.e., consider the intersection $\sim$ of $\le_P$ with its inverse, let $\overline{P}$ consist of the classes of the equivalence relation $\sim$, and for $A, B \in \overline{P}$ let $A \le B$ mean that $a \le b$ for some $a \in A$ and $b \in B$. For $x \in P$ the $\sim$-class of x will be denoted by $\bar{x}$. Sometimes, for $x \in P$ and $Y \in \overline{P}$, we write $x \le Y$ or $x > Y$ instead of $\bar{x} \le Y$ or $\bar{x} > Y$, respectively. P is called a *quasilattice* if each two-element subset of P has an infimum and a supremum in P. (The infimum and supremum is defined only up to the equivalence $\sim$!) Equivalently, P is a quasilattice iff $\overline{P}$ is a lattice. Following Chajda [1], cf. also Chajda and Kotrle [2], an algebra $\langle L; \vee, \wedge \rangle$ is called a *q-lattice* if both binary operations are associative and commutative, and the identities $x \vee (x \wedge y) = x \vee x$, $x \vee (y \vee y) = x \vee y$, their duals, and the identity $x \vee x = x \wedge x$ hold. In Chajda [1], the well-known connection between lattices as posets and lattices as algebraic structures is generalized to a similar connection between quasilattices and q-lattices. Hence our first theorem indicates that q-lattices are relevant tools to study coalitions.

Results

Theorem 1. *For a finite quasiordered set P, $\mathcal{L}(P)$ is a quasilattice iff P is upper bound free.*

As indicated in the previous section, this theorem instantly yields

Theorem 2. *For a finite poset P, $\mathcal{L}(P)$ is a lattice iff P is a forest.*

The proof of Thm. 1 gives an effective construction of suprema in $\mathcal{L}(P)$. Proposition 1 below gives a recursive description of infima in $\mathcal{L}(P)$ in the particular case when P is a forest; i.e. $\mathcal{L}(P)$ is a lattice. The quasiorder-theoretic generalization of Proposition 1 would cause considerable technical difficulties even in formulating the result.

Proposition 1. *Let P be a forest, $k \ge 2$, and for $A_1, \dots, A_k \in \mathcal{L}(P)$ let $M = \{b_1 \wedge \dots \wedge b_k\colon b_1 \in A_1, \dots, b_k \in A_k$, and the infimum $b_1 \wedge \dots \wedge b_k$ exists in $P\}$. If M is empty (in particular when one of the A_i is empty) then $\bigwedge_{i=1}^k A_i = \emptyset$. If M is non-empty then choose a maximal element $c = a_1 \wedge \dots \wedge a_k$ in M where the a_i belong to A_i such that, for every i, $c \in A_i \Longrightarrow c = a_i$. Let $A_i' = A_i \setminus \{a_i\}$ for $i = 1, \dots, k$, $P' = P \setminus \{c\}$, and put $C' = \bigwedge_{i=1}^k A_i'$ in $\mathcal{L}(P')$. Then $\bigwedge_{i=1}^k A_i = C' \cup \{c\}$ in $\mathcal{L}(P)$.*

Proposition 2. *For any finite quasiordered set P, $\mathcal{L}(P)$ is selfdual. In fact, the map $\mathcal{L}(P) \to \mathcal{L}(P)$, $X \mapsto P \setminus X$ is a dual automorphism.*

In virtue of Proposition 2 we have

$$A_1 \wedge \ldots \wedge A_k = \overline{\overline{A}_1 \vee \ldots \vee \overline{A}_k}, \tag{1}$$

and dually. This offers a way of deducing infima from suprema and vice versa. In practical computations this can be useful e.g. when the $\overline{A}_i = P \setminus A_i$ have only a few elements. However, Proposition 1 gives a better view of infima for lattices $\mathcal{L}(P)$ than (1), and the authors do not think that (1) would make the proof of Proposition 1 easier.

Let $C_n = \{c_1 < c_2 < \ldots < c_n\}$ be the n-element chain; then $\mathcal{L}(C_n)$ is a lattice by Theorem 2. Now we give a more informative description of $\mathcal{L}(C_n)$. We define lattices L_n with ideals I_n and dual ideals D_n, and lattice isomorphisms φ_n: $I_n \to D_n$ via induction as follows. Let L_1 be the two-element lattice, $I_1 = \{0\}$, $D_1 = \{1\}$; the meaning of φ_1: $I_1 \to D_1$ is obvious. For $n > 1$, take two disjoint isomorphic copies of L_{n-1}, one of them will be I_n while the other will be D_n, choose an isomorphism φ_n: $I_n \to D_n$, and let $L_n = I_n \cup D_n$. For $x, y \in L_n$ we let $x \le y$ iff one of the following three possibilities holds: either $x \in I_n$, $y \in D_n$, and $x \le d$ in I_n and $\varphi_n(\varphi_{n-1}^{-1}(d)) \le y$ in D_n for some $d \in D_{n-1} \subseteq I_n$, or $x, y \in I_n$ and $x \le y$ in I_n, or $x, y \in D_n$ and $x \le y$ in D_n.

Proposition 3. *For every $n \ge 1$, $\mathcal{L}(C_n) \cong L_n$.*

Proposition 4. *Let $T_1, T_2, \ldots, T_s$ be the components of the quasiordered set P. Then $\mathcal{L}(P) = \langle \mathcal{L}(P), \le \rangle$ is isomorphic to the direct product of the $\mathcal{L}(T_i)$, $1 \le i \le s$.*

Proposition 5. *Let P be a finite forest. Then the lattice $\mathcal{L}(P)$ is distributive iff $\mathcal{L}(P)$ is modular iff every tree of P is a chain.*

Proofs

Proof of Theorem 1. Let us suppose first that $\mathcal{L}(P)$ is a quasilattice, and $a \le c$, $b \le c$ hold for $a, b, c \in P$. Let U be a supremum of $\{a\}$ and $\{b\}$ in $\mathcal{L}(P)$. Since $\{a\} \le \{c\}$ and $\{b\} \le \{c\}$, we have $U \le \{c\}$, whence $|U| \le 1$. On the other hand, $|U| \ge 1$ by $\{a\} \le U$. Thus U is a singleton, say $\{d\}$. From $\{a\} \le U = \{d\}$ and $\{b\} \le U = \{d\}$ we infer $a \le d$ and $b \le d$. Since $\{a, b\}$ is an upper bound of $\{a\}$ and $\{b\}$, we obtain $\{d\} = U \le \{a, b\}$, yielding $d \le b$ or $d \le a$. By transitivity, $a \le b$ or $b \le a$. I.e., P is upper bound free.

To prove the converse, let us assume that P is UBF. Then so is $\overline{P}$. Let $\overline{P}_1$ be the set of maximal elements of the forest $\overline{P}$. If $\overline{P} \setminus \overline{P}_1$ is not empty then let $\overline{P}_2$ denote the set of its maximal elements, etc.; if $\overline{P} \setminus (\overline{P}_1 \cup \ldots \cup \overline{P}_{i-1})$ is not empty then let $\overline{P}_i$ denote the set of its maximal elements. Then $\overline{P}$ is partitioned in finitely many subsets $\overline{P}_1, \ldots, \overline{P}_r$. For $1 \le i \le r$ let $P_i = \{x \in P\colon \bar{x} \in \overline{P}_i\}$; now P is the union of the pairwise disjoint P_i, $1 \le i \le r$. The set $\{x \in P_1 \cup \ldots \cup P_i\colon x \ge B$ holds for no $B \in \overline{P}_i\}$ will be denoted by Q_i.

Now, for given coalitions $A_1, \ldots, A_k$, we intend to define a sequence $\emptyset = C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots \subseteq C_r = C$ of coalitions such that $C_i = C \cap (P_1 \cup \ldots \cup P_i)$ and C

is a supremum of $\{A_1, \dots, A_k\}$. Suppose $i > 0$ and C_{i-1} has already been defined. For given $B \in \overline{P}_i$ and $1 \le j \le k$ we define the following numbers.

$$\begin{aligned}\gamma_i(B) &= |\{x \in C_{i-1}\colon\ x > B\}|,\\ \nu_i(j,B) &= |\{x \in A_j\colon\ x \ge B\}|,\\ \delta_i(j,B) &= \nu_i(j,B) - \gamma_i(B),\\ \lambda_i(B) &= \max\{0, \delta_i(1,B), \delta_i(2,B), \dots, \delta_i(k,B)\}.\end{aligned}$$

Let us choose a subset $S_i(B)$ of B such that $|S_i(B)| = \lambda_i(B)$. (We will soon prove that this choice is possible.) We define C_i by

$$C_i = C_{i-1} \cup \bigcup_{B \in \overline{P}_i} S_i(B).$$

Denote $A_j \cap (P_1 \cup \ldots \cup P_i)$ by $A_j^{(i)}$ and consider the following induction hypothesis

$$(H(i)) \qquad A_j^{(i)} \le C_i \text{ for all } j \text{ and } \lambda_i(B) \le |B| \text{ for all } B \in \overline{P}_i.$$

Note that $\lambda_i(B) \le |B|$ is necessary to make the choice of $S_i(B)$ possible.

For $i = 1$, $\gamma_1(B) = 0$ and $\nu_1(j,B) = |A_j \cap B| \le |B|$ imply $\lambda_1(B) \le |B|$. Since $|A_j \cap B| = \nu_1(j,B) = \delta_1(j,B) \le \lambda_1(B) = |S_1(B)|$, we can chose an injection $\psi_B\colon A_j \cap B \to S_1(B)$. Clearly,

$$\bigcup_{B \in \overline{P}_1} \psi_B\colon\ A_j^{(1)} \to C_1$$

is an extensive map. This proves $H(1)$.

Now, for $1 \le i \le r$, suppose $H(i-1)$. For $B \in \overline{P}_i$, the existence of extensive maps $\alpha_j^{(i-1)}\colon\ A_j^{(i-1)} \to C_{i-1}$, which necessarily map $\{x \in A_j\colon\ x > B\}$ into $\{x \in C_{i-1}\colon\ x > B\}$, yields $|\{x \in A_j\colon\ x > B\}| \le |\{x \in C_{i-1}\colon\ x > B\}|$ for any j. Using this inequality we can estimate: $\delta_i(j,B) = \nu_i(j,B) - \gamma_i(B) = |\{x \in A_j\colon\ x \ge B\}| - |\{x \in C_{i-1}\colon\ x > B\}| = |\{x \in A_j\colon\ x > B\} \cup (A_j \cap B)| - |\{x \in C_{i-1}\colon\ x > B\}| = |A_j \cap B| + |\{x \in A_j\colon\ x > B\}| - |\{x \in C_{i-1}\colon\ x > B\}| \le |A_j \cap B| \le |B|$. Therefore $\lambda_i(B) \le |B|$, indeed.

Now, for a fixed j and arbitrary $B \in \overline{P}_i$, we will define an extensive map $\varphi_B = \varphi_{j,B}\colon\ \{x \in A_j\colon\ x \ge B\} \to \{x \in C_i\colon\ x \ge B\}$. Since $|\{x \in A_j\colon\ x \ge B\}| = \nu_i(j,B) = \gamma_i(B) + \delta_i(j,B) \le \gamma_i(B) + \lambda_i(B) = |\{x \in C_{i-1}\colon\ x > B\}| + |C_i \cap B| = |\{x \in C_i\colon\ x > B\} \cup (C_i \cap B)| = |\{x \in C_i\colon\ x \ge B\}|$, i.e.,

$$(2) \qquad |\{x \in A_j\colon\ x \ge B\}| \le |\{x \in C_i\colon\ x \ge B\}|,$$

the restriction of $\alpha_j^{(i-1)}$ to the set $\{x \in A_j\colon\ x \ge B\} \cap A_j^{(i-1)}$ can be extended to an injective map $\varphi_B\colon\ \{x \in A_j\colon\ x \ge B\} \to \{x \in C_i\colon\ x \ge B\}$. For any $y \in \{x \in A_j\colon\ x \ge B\}$ either $y \in A_j^{(i-1)}$ and $\varphi_B(y) = \alpha_j^{(i-1)}(y) \ge y$ or $y \in B$, whence φ_B is an extensive map. Let $\alpha_j^{(i)}$ be the union of $\alpha_j^{(i-1)}$ and all the φ_B, $B \in \overline{P}_i$. Then $\alpha_j^{(i)}\colon\ A_j^{(i)} \to C_i$. Since, by the UBF property, Q_i and the sets

$\{x \in C_i\colon x \geq B\}$, $B \in \overline{P}_i$, are pairwise disjoint, $\alpha_j^{(i)}$ is injective and therefore it is an extensive map. Hence $A_j^{(i)} \leq C_i$, proving $H(i)$.

We have seen that the definition of $C = C_r$ is correct and, by $H(r)$, C is an upper bound of the A_j, $1 \leq j \leq k$.

Now let $D \in \mathcal{L}(P)$ be an arbitrary upper bound of the A_j, $1 \leq j \leq k$. We have to show that $C \leq D$. By the assumption, there are extensive maps $\mu_j\colon A_j \to D$. Let $D_i = D \cap (P_1 \cup \ldots \cup P_i)$. We will define extensive maps $\tau_i\colon C_i \to D_i$ for $i = 1, 2, \ldots, r$ via induction, and $C = C_r \leq D_r = D$ will follow evidently.

For each $B \in \overline{P}_1$ such that $B \cap C = B \cap C_1 = S_1(B)$ is non-empty, choose a j with $|S_1(B)| = \lambda_1(B) = \delta_1(j, B)$. Then $|A_j \cap B| = \nu_1(j, B) - 0 = \delta_1(j, B) = |S_1(B)| = |C_1 \cap B|$. Since μ_j clearly maps $A_j \cap B$ into $D_1 \cap B$, $|C_1 \cap B| = |A_j \cap B| \leq |D_1 \cap B|$. Therefore we can choose an injective map $\beta_B\colon C_1 \cap B \to D_1 \cap B$. Let β_B denote the empty map when $B \cap C = \emptyset$. Define τ_1 as the union of the β_B, $B \in \overline{P}_1$. Clearly, $\tau_1\colon C_1 \to D_1$ is an extensive map.

Now, for $1 < i \leq r$, suppose we already have an extensive map $\tau_{i-1}\colon C_{i-1} \to D_{i-1}$; we define τ_i as follows. For $B \in \overline{P}_i$, if $|C_i \cap B| = \lambda_i(B) = 0$, then let κ_B be the restriction of τ_{i-1} to the set $\{x \in C_{i-1}\colon x > B\} = \{x \in C_i\colon x \geq B\}$. Otherwise choose a j such that $|C_i \cap B| = \lambda_i(B) = \delta_i(j, B)$. Since μ_j maps $\{x \in A_j\colon x \geq B\}$ into $\{x \in D_i\colon x \geq B\}$ and (2) with the j chosen turns into an equality, we conclude that $|\{x \in C_i\colon x \geq B\}| \leq |\{x \in D_i\colon x \geq B\}|$. Further, for all $y \in \{x \in C_i\colon x > B\} = \{x \in C_i\colon x \geq B\} \setminus B$, $\tau_{i-1}(y)$ is defined and belongs to $\{x \in D_i\colon x > B\}$. Therefore there exists an injective map $\kappa_B\colon \{x \in C_i\colon x \geq B\} \to \{x \in D_i\colon x \geq B\}$ such that $\kappa_B(x) = \tau_{i-1}(x)$ if $x \notin B$. Clearly, κ_B is an extensive map. Now let τ_i be the union of τ_{i-1} and the κ_B, $B \in \overline{P}_i$. By the UBF property, Q_i and the sets $\{x \in D_i\colon x \geq B\}$, $B \in \overline{P}_i$, are pairwise disjoint, implying the injectivity of τ_i. Hence τ_i is an extensive map.

We have seen that finitely many (but more than zero) coalitions of $\mathcal{L}(P)$ have a supremum. By finiteness and $\emptyset \in \mathcal{L}(P)$ we infer that $\mathcal{L}(P)$ is a quasilattice. $\square$

Proof of Proposition 1. Since $b_1 \wedge \ldots \wedge b_k$ exists iff all the b_i belong to the same component of P, $\bigwedge_{i=1}^{k} A_i = \emptyset$ when $M = \emptyset$. Suppose therefore that M is not empty, and put $A_i^* = (A_i \setminus \{a_i\}) \cup \{c\}$. First we show that, for any j,

$$\bigwedge_{i=1}^{k} A_i = A_j^* \wedge \bigwedge_{i \neq j} A_i. \tag{3}$$

We have to show that an arbitrary $D \in \mathcal{L}(P)$ is a lower bound of the A_i if and only if it is a lower bound of A_j^* and the A_i, $i \neq j$. Since $c \leq a_j \Longrightarrow A_j^* \leq A_j$, the "if" part is obvious. Suppose $D \in \mathcal{L}(P)$ is a lower bound of the A_i and, w.l.o.g., $c \neq a_j$, i.e., $c \notin A_j$. We have extensive maps $\alpha_i\colon D \to A_i$, $i = 1, \ldots, k$. We may assume that $a_j \in \alpha_j(D)$, for otherwise $D \leq A_j^*$ hardly needs any proof. Then $a_j = \alpha_j(d)$ for some $d \in D$. Now $c \leq a_j$, $d \leq a_j$ and the UBF property yield that c and d are comparable. Since $d \leq \bigwedge_{i=1}^{k} \alpha_i(d) \in M$ and c is maximal in M, we have $d \leq c$. Hence the map

$$D \to A_j^*, \quad x \mapsto \begin{cases} c, & \text{if } x = d, \\ \alpha_j(x), & \text{otherwise} \end{cases}$$

is extensive, whence $D \leq A_j^*$. This proves (3).

Armed with (3), we may assume that $c \in A_i$ for all i; indeed this situation can be achieved by successive applications of (3) without changing the A'_i. Now $A'_i = A_i \setminus \{c\}$. Put $C = C' \cup \{c\}$. Evidently, $C \leq A_i$ for all i. Suppose $D \leq A_i$, witnessed by an extensive map $\beta_i\colon D \to A_i$, for each i. If $D \leq A'_i$ for all i then $D \setminus \{c\} \leq C'$ in P' gives $D \leq C$ easily, so suppose this is not the case. Therefore $\beta_i^{-1}(c)$ exists for certain i. We want to find an element $c' \in D$ such that

$$c' \leq c, \quad D \setminus \{c'\} \subseteq P' \text{ and } D \setminus \{c'\} \leq A'_i \text{ for } i = 1, \dots, k. \tag{4}$$

From (4) the Proposition will follow easily: we have an extensive map $\varphi\colon D\setminus\{c'\} \to C'$, whence

$$\varphi \cup \{\langle c', c\rangle\}\colon D \to C$$

yields $D \leq C$.

To show (4), choose a j such that $\beta_j^{-1}(c)$ is a maximal element among the $\beta_i^{-1}(c)$, and denote $\beta_j^{-1}(c)$ by d. Now we define extensive maps $\gamma_i\colon D \to A_i$ such that $\gamma_i(d) = c$ for $i = 1, \dots, k$. If $\beta_i(d) = c$ then put $\gamma_i = \beta_i$. Otherwise, if $\beta_i^{-1}(c)$ does not exist then put

$$\gamma_i\colon D \to A_i, \quad x \mapsto \begin{cases} c, & \text{if } x = d, \\ \beta_i(x), & \text{otherwise.} \end{cases}$$

If $\beta_i(d) \neq c$ and $\beta_i(e) = c$ for some $e \in D$, then $d \leq c$, $e \leq c$ and the UBF property yield that d and e are comparable. Hence the choice of j gives $e \leq d$, and $e \leq \beta_i(d)$ follows by transitivity. Therefore

$$\gamma_i\colon D \to A_i, \quad x \mapsto \begin{cases} c, & \text{if } x = d, \\ \beta_i(d), & \text{if } x = e, \\ \beta_i(x), & \text{otherwise} \end{cases}$$

is an extensive map.

Now, if $d = c$ or $c \notin D$ then $c' = d$ fulfils (4). If $d < c \in D$ then consider the extensive maps

$$D \to A_i, \quad x \mapsto \begin{cases} c, & \text{if } x = c, \\ \gamma_i(c), & \text{if } x = d, \\ \gamma_i(x), & \text{otherwise,} \end{cases}$$

and put $c' = c$. □

Proof of Proposition 2. With the notation $\overline{X} = P \setminus X$, it suffices to show that, for $A, B \in \mathcal{L}(P)$, $A \leq B \implies \overline{B} \leq \overline{A}$, for the reverse implication then also follows. First we show that

$$\text{For } |A| = |B|, \quad A \leq B \iff A \setminus B \leq B \setminus A. \tag{5}$$

Indeed, suppose $A \leq B$, and choose an extensive map $\varphi\colon A \to B$ with a maximum number of fixed points. Suppose that $u \in A \cap B$ is not a fixed point of φ. By the assumptions φ is surjective; let a be a preimage of u. We have $a \leq \varphi(a) = u \leq \varphi(u)$ and $|\{a, u, \varphi(u)\}| = 3$. Clearly, the map

$$\varphi'\colon A \to B, \quad x \mapsto \begin{cases} u, & \text{if } x = u, \\ \varphi(u), & \text{if } x = a, \\ \varphi(x), & \text{otherwise} \end{cases}$$

has one more fixed point than φ, a contradiction. Therefore φ acts identically on $A \cap B$ and its restriction to $A \setminus B$ is an extensive map $A \setminus B \to B \setminus A$, yielding $A \setminus B \leq B \setminus A$. The converse is evident.

Now suppose $A \leq B$. Then necessarily $|A| \leq |B|$. If $|A| = |B|$ then, using (5) twice, $\overline{B} \setminus \overline{A} = A \setminus B \leq B \setminus A = \overline{A} \setminus \overline{B}$ implies $\overline{B} \leq \overline{A}$. If $|A| < |B|$ and φ: $A \to B$ is an extensive map then $A \leq \varphi(A)$ yields $\overline{\varphi(A)} \leq \overline{A}$ by the previous case, $\overline{B} \subseteq \overline{\varphi(A)}$ gives $\overline{B} \leq \overline{\varphi(A)}$, and $\overline{B} \leq \overline{A}$ follows by transitivity. □

Proof of Proposition 3. We omit the technical but straightforward induction which shows the intuitively clear fact that the 7-tuple $U_n = \langle L_n, I_n, D_n, \varphi_n, I_{n-1}, D_{n-1}, \varphi_{n-1} \rangle$ (where $I_{n-1} \subseteq I_n$ and $D_{n-1} \subseteq I_n$, etc., of course) is uniquely determined up to isomorphism. Here by an isomorphism ψ: $U \to U^*$ we mean a lattice isomorphism ψ: $L_n \to L_n^*$ which takes I_n to I_n^*, etc., and commutes with φ_n and φ_{n-1} in the natural way (e.g., $\psi(\varphi_n(x)) = \varphi_n^*(\psi(x))$ for all $x \in I_n$). In particular, L_n is uniquely determined up to isomorphism.

In $L_n' = \mathcal{L}(C_n)$, let $I_n' = \{X \in L_n' \colon c_n \notin X\}$, $D_n' = \{X \in L_n' \colon c_n \in X\}$, $I_{n-1}' = \{X \in I_n' : c_{n-1} \notin X\}$, $D_{n-1}' = \{X \in I_n' : c_{n-1} \in X\}$, φ_n': $I_n' \to D_n'$, $X \mapsto X \cup \{c_n\}$, and φ_{n-1}': $I_{n-1}' \to D_{n-1}'$, $X \mapsto X \cup \{c_{n-1}\}$. The easy but tedious task of checking that the primed objects also satisfy the conditions of the recursive definition of L_n is also left to the reader. □

Proof of Proposition 4. It is straightforward to check that

$$\varphi \cdot \mathcal{L}(P) \to \prod_{i=1}^{s} \mathcal{L}(T_i), \quad X \mapsto \langle X \cap T_1, \dots, X \cap T_s \rangle$$

is an isomorphism. □

Proof of Proposition 5. Suppose that one of the trees of P is not a chain, and let a and b be incomparable elements of this tree. Since trees are meet-semilattices, we can take $c = a \wedge b$. Applying the description of joins in the proof of Thm. 1 or even without it we obtain $\{a\} \vee \{b\} = \{a, b\}$. Let $U = \{a\} \wedge \{b, c\}$. Since $\{c\} \leq U \leq \{a\}$, U is a singleton, say $U = \{u\}$, and $c \leq u \leq a$. From $\{u\} \leq \{b, c\}$ we conclude $u \leq b$ or $u \leq c$. But the first possibility implies the second one via $u \leq a \wedge b = c$. Therefore $u = c$ and $\{a\} \wedge \{b, c\} = c$. What we have already calculated is sufficient to see that $\{\{c\}, \{a\}, \{b\}, \{b, c\}, \{a, b\}\}$ is a pentagon sublattice, whence $\mathcal{L}(P)$ is neither modular nor distributive.

In virtue of Proposition 4, the converse will immediately follow if we show that $\mathcal{L}(P)$ is distributive for any chain P. We outline two different arguments showing this.

Firstly, by Proposition 3, it suffices to deal with the lattices L_n via induction. Since L_1 and L_2 are chains, they are distributive. Suppose L_{n-2} and L_{n-1} are distributive. Then so are I_{n-1}, D_{n-1}, I_n $(= I_{n-1} \cup D_{n-1})$ and D_n, being isomorphic to L_{n-2} or L_{n-1}. The sublattice $D_{n-1} \cup \varphi_n(I_{n-1})$, which is isomorphic to the direct product of L_{n-2} and the two-element lattice, is distributive, too. Since L_n can be obtained from I_n, $D_{n-1} \cup \varphi_n(I_{n-1})$ and D_n by using the Hall–Dilworth gluing construction (cf. Hall and Dilworth [3] or, e.g. Grätzer [4, page 31, Ex. 20, 21]) twice, and this gluing is well-known to preserve distributivity, cf. [3] and [4], L_n is also distributive.

Secondly, let $P = \{c_1 < c_2 < \ldots < c_n\}$ and consider the chain $Q = \{c_0 < c_1 < \ldots < c_n\}$. It is easy to show that $S = \{\langle x_1, \dots, x_n \rangle \in Q^n \colon x_1 \leq x_2 \leq \ldots \leq x_n$

and, for all $i > 1$, $x_i \neq c_0 \Longrightarrow x_{i-1} < x_i\}$ is a sublattice of Q^n. Hence S is distributive. The more or less straightforward argument showing that $S \to \mathcal{L}(P)$, $\langle x_1, \dots, x_n\rangle \mapsto \{x_i\colon x_i > c_0\}$ is a lattice isomorphism will not be detailed. □

Acknowledgment. The authors are grateful to the referee for reading the paper very carefully and pointing out some deficiencies.

References

1. I. Chajda, *Lattices in quasiordered sets*, Acta Palack. Univ. Olomouc **31** (1992), 6–12.
2. I. Chajda and M. Kotrle, *Subdirectly irreducible and congruence distributive q-lattices*, Czechoslovak Math. Journal **43 (118)** (1993), 635–642.
3. R. P. Dilworth and M. Hall, *The embedding problem for modular lattices*, Ann. of Math. **45** (1944), 450–456.
4. G. Grätzer, *General Lattice Theory*, Akademie-Verlag, Berlin, 1978.
5. B. Peleg, *Game Theoretic Analysis of Voting in Committees*, Econometric Society Monographs in Pure Theory 7, Cambridge University Press, Cambridge—New York, 1984.

JATE Bolyai Institute, Szeged, Aradi vértanúk tere 1, H–6720 HUNGARY
E-mail address: `CZEDLI@MATH.U--SZEGED.HU`

Mathematical Research Institute, Budapest, HUNGARY
E-mail address: `H4135POL@ELLA.HU`