HOW MANY WAYS CAN TWO COMPOSITION SERIES INTERSECT?

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ABSTRACT. Let $\vec{H}$ and $\vec{K}$ be finite composition series of length h in a group G. The intersections of their members form a lattice $\mathrm{CSL}(\vec{H},\vec{K})$ under set inclusion. Our main result determines the number $N(h)$ of (isomorphism classes) of these lattices recursively. We also show that this number is asymptotically $h!/2$. If the members of $\vec{H}$ and $\vec{K}$ are considered constants, then there are exactly $h!$ such lattices.

Based on recent results of Czédli and Schmidt, first we reduce the problem to lattice theory, concluding that the duals of the lattices $\mathrm{CSL}(\vec{H},\vec{K})$ are exactly the so-called slim semimodular lattices, which can be described by permutations. Hence the results on $h!$ and $h!/2$ follow by simple combinatorial considerations. The combinatorial argument proving the main result is based on Czédli's earlier description of indecomposable slim semimodular lattices by matrices.

1. Introduction

The well-known concept of a composition series in a group goes back to Évariste Galois (1831), see Rotman [25, Thm. 5.9]. The Jordan-Hölder theorem, stating that any two composition series of a finite group have the same length, was also proved in the nineteenth century; see Jordan [21] and Hölder [20]. A stronger statement is obtained from the Schreier Refinement Theorem, see [25, Theorem 5.11]: if a group has a finite composition series, then any two of its composition series have the same length. Let

$$\begin{aligned} &\vec{H}: \quad G = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_h = \{1\},\\ &\vec{K}: \quad G = K_0 \triangleright K_1 \triangleright \cdots \triangleright K_h = \{1\} \end{aligned} \tag{1.1}$$

be composition series of a group G. Here $H_{i-1} \triangleright H_i$ denotes that H_i is a normal subgroup of H_{i-1}; the sequence $\vec{H}$ is a *composition series* if H_i is a maximal normal proper subgroup of H_{i-1}, for $i = 1, \ldots, h$. Denote the set

$$\bigl\{H_i \cap K_j : i, j \in \{0, \ldots, h\}\bigr\}$$

by $\mathrm{CSL}_h(\vec{H},\vec{K})$. The notation comes from "Composition Series Lattice". Under containment, $\mathrm{CSL}_h(\vec{H},\vec{K})$ is an ordered set. Sometimes we write $\mathrm{CSL}(\vec{H},\vec{K})$ for $\mathrm{CSL}_h(\vec{H},\vec{K})$. Since $\mathrm{CSL}_h(\vec{H},\vec{K})$ has a largest element and is closed with respect to

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intersection, $\mathrm{CSL}_h(\vec{H},\vec{K})$ is a finite lattice. The join of $X, Y \in \mathrm{CSL}_h(\vec{H},\vec{K})$ is the intersection of $\{Z : X \subseteq Z,\ Y \subseteq Z$ and $Z \in \mathrm{CSL}_h(\vec{H},\vec{K})\}$. Let $N(h)$ denote the number of isomorphism classes of all lattices $\mathrm{CSL}_h(\vec{H},\vec{K})$ formed from composition series with length h. In other words, $N(h)$ counts the number of lattices of the form $\mathrm{CSL}_h(\vec{H},\vec{K})$; isomorphic lattices are counted only once.

If we view all the H_i and K_j as constants, then $\mathrm{CSL}_h(\vec{H},\vec{K})$ becomes a *multipointed lattice*, which we denote by $\mathrm{CS\ddot{L}}_h(\vec{H},\vec{K})$. If $\vec{H}'$ and $\vec{K}'$ are composition series of length h in a group G', then the multipointed lattices $\mathrm{CS\ddot{L}}_h(\vec{H},\vec{K})$ and $\mathrm{CS\ddot{L}}_h(\vec{H}',\vec{K}')$ are *isomorphic* if there is a lattice isomorphism $\varphi\colon \mathrm{CSL}_h(\vec{H},\vec{K}) \to \mathrm{CSL}_h(\vec{H}',\vec{K}')$ such that $\varphi(H_i) = H_i'$ and $\varphi(K_i) = K_i'$, for $i = 0, \dots, h$. The number of (isomorphism classes of) multipointed lattices $\mathrm{CS\ddot{L}}_h(\vec{H},\vec{K})$ of length h will be denoted by $\ddot{N}(h)$.

Our main goal is to determine $N(h)$ and $\ddot{N}(h)$. Proposition 3.1 gives a simple explicit formula for $\ddot{N}(h)$, and Proposition 7.1 gives a satisfactory asymptotic formula for $N(h)$. Theorem 5.3, our main result, yields only a recursive way to compute $N(h)$. Due to the fact that we count specific lattices, even this recursion is far more efficient than the best known way to compute all finite lattices of a given size s; see Heitzig and Reinhold [19] for $s \le 18$, and the references therein.

We will also consider the abstract class of lattices $\mathrm{CSL}_h(\vec{H},\vec{K})$. This abstract class has recently been characterized by Czédli and Schmidt [11]. To make our approach self-contained and to give a sharper result, we give a direct proof of this characterization; see Proposition 2.3. Also, we prove a join-embedding result, Proposition 2.6, for the multipointed versions of these lattices.

Outline. Sections 2, 3, and 4 are lattice-theoretic, while Sections 5, 6, and 7 are combinatorial. Section 2 deals with the abstract class of lattices $\mathrm{CSL}_h(\vec{H},\vec{K})$. Section 3 proves Proposition 3.1, which asserts that $\ddot{N}(h) = h!$ (h factorial). By recalling and supplementing the main result of Czédli [5], Section 4 translates the problem of determining $N(h)$ to a purely combinatorial problem on certain $0,1$-matrices. Sections 5 formulates the most difficult result in this paper, Theorem 5.3, which is a recursive formula for the exact value of $N(h)$. This section lists some concrete values of $N(h)$, computed by Maple and Mathematica. The main result is proved in Section 6. Finally, Section 7 proves that $N(h)$ is asymptotic to $h!/2$.

2. Composition series and slim semimodular lattices

2.1. **Basic concepts and notation.** The study of semimodular lattices is an important branch of lattice theory; see Stern [28], Grätzer [14] and [15], Nation [23], and Czédli and Schmidt [7] for surveys. Recall that a lattice L is (*upper*) *semimodular* if $a \prec b$ implies $a \vee c \preceq b \vee c$, for all $a, b, c \in L$. Similarly, L is *lower semimodular* or *dually semimodular* if it satisfies the dual property: $a \succ b$ implies $a \wedge c \succeq b \wedge c$, for $a, b, c \in L$. Note that $\mathrm{CSL}(\vec{H},\vec{K})$ will turn out to be lower semimodular but generally is not semimodular. However, it suffices to count their dual lattices, which are semimodular. Therefore, since all the lattice-theoretic results that we reference were formulated for semimodular lattices, it is reasonable to work with semimodular lattices rather than lower semimodular ones.

Except for the lattice $\mathrm{Sub}\, G$ of all subgroups of G (see below), all lattices in this paper are assumed to be of finite length, and mostly they are finite. Following

Grätzer and Knapp [16], a finite lattice L is *slim* if there are no three pairwise incomparable join-irreducible elements in L. A diagram of an ordered set is *planar* if its edges can be incident only at their endpoints. By Czédli and Schmidt [8, Lemma 2.2], every slim lattice is *planar*, that is, it has a planar diagram. Hence slim semimodular lattices are easy to work with. In particular, a visual understanding is provided by Czédli and Schmidt [9], which clearly implies that L in Figure 1 is a slim semimodular lattice.

Slim semimodular lattices have recently proved to be useful in strengthening a classical group theoretical result, namely, the Jordan-Hölder theorem. G. Grätzer and Nation [18] proved that given two composition series of a group, as in (1.1), there is a matching between their quotients such that the corresponding quotients are isomorphic for a very specific reason: they are related by the composite of a down-perspectivity with an up-perspectivity. In Czédli and Schmidt [8], this matching is shown to be unique. Moreover, Czédli and Schmidt [11] have just proved that this matching determines the lattice $\mathrm{CSL}(\vec{H},\vec{K})$. The main role in [8] and [11] is played by slim semimodular lattices. These lattices are also useful in lattice theory, see Czédli [6] and Czédli and Schmidt [10] for the latest results.

The relation "*subnormal subgroup*" is the transitive closure of "normal subgroup". Let G be a group with a finite composition series of length h. Its subnormal subgroups form a sublattice $\mathrm{SnSub}\,G = (\mathrm{SnSub}\,G;\subseteq)$ of the lattice $\mathrm{Sub}\,G$ of all subgroups, by a classical result of Wielandt [29]; see also Schmidt [26, Theorem 1.1.5] and the remark after its proof, or see Stern [28, p. 302]. It is not hard to see that $\mathrm{SnSub}\,G$ is dually semimodular, that is, lower semimodular; see [26, Theorem 2.1.8], or the proof of [28, Theorem 8.3.3], or the proof of Nation [23, Theorem 9.8]. Hence, for $\vec{H}$ and $\vec{K}$ defined in(1.1), $\mathrm{CSL}_h(\vec{H},\vec{K})$ is also lower semimodular by the dual of Czédli and Schmidt [8, Lemma 2.4]. Note that the dual of [8, Lemma 2.4] also asserts that $\mathrm{CSL}_h(\vec{H},\vec{K})$ is a *cover-preserving* meet-subsemilattice of $\mathrm{SnSub}\,G$, that is, if $X,Y\in\mathrm{CSL}_h(\vec{H},\vec{K})$ and $X\prec Y$ in $\mathrm{CSL}_h(\vec{H},\vec{K})$, then $X\prec Y$ in $\mathrm{SnSub}\,G$.

In general, $\mathrm{CSL}_h(\vec{H},\vec{K})$ is distinct from $\mathrm{SnSub}\,G$. This follows easily from (the abelian case of) the description of all finite groups with planar subgroup lattices, given by Schmidt [27], and the fact that $\mathrm{CSL}_h(\vec{H},\vec{K})$ is always a planar lattice by Czédli and Schmidt [8, the dual of Lemma 2.2]. Furthermore, as witnessed by the 8-element elementary 2-group $(\mathbb{Z}_2;+)^3$, $\mathrm{CSL}_h(\vec{H},\vec{K})$ is not even a sublattice of $\mathrm{SnSub}\,G$ in general.

The set of non-zero join-irreducible elements and that of non-unit meet-irreducible elements of a finite lattice L will be denoted by $\mathrm{Ji}\,L$ and $\mathrm{Mi}\,L$, respectively. Let

$$\vec{H}\cup\vec{K}=\{H_i : 0\le i\le h\}\cup\{K_i : 0\le i\le h\}.$$

Since $\mathrm{Mi}\big(\mathrm{CSL}_h(\vec{H},\vec{K})\big)$ is obviously a subset of $\vec{H}\cup\vec{K}$, the set $\mathrm{Mi}\big(\mathrm{CSL}_h(\vec{H},\vec{K})\big)$ contains no three-element antichain. Hence

$$\mathrm{CSL}(\vec{H},\vec{K}) \text{ is a dually slim, dually semimodular lattice.} \tag{2.1}$$

As usual, $\mathbb{N}$ denotes $\{1,2,3,\ldots\}$, and $\mathbb{N}_0$ stands for $\mathbb{N}\cup\{0\}$. The *isomorphism class* of a lattice L, that is, the class $\{L' : L' \cong L\}$, is denoted by $\mathbf{I}(L)$. If $\mathcal{K}(y)$ is a class of lattices depending on a parameter (or a list of parameters) y, then $\mathcal{K}(y)^{\cong}$ stands for the corresponding class $\{\mathbf{I}(L) : L\in\mathcal{K}(y)\}$ of isomorphism classes. Since $\mathcal{K}$ will be treated as a property, to separate the notation above from that for the *dual* *class* $\{L^\delta : L \in \mathcal{K}(y)\}$, the dual class is denoted by $\mathcal{K}^\delta(y)$. We can combine these two notations without extra parentheses; namely, $\mathcal{K}^\delta(y)^\cong = \{\mathbf{I}(L^\delta) : L \in \mathcal{K}(y)\}$.

For a group G of finite composition series length, let $\mathrm{CSL}(G)$ be the class of lattices $\mathrm{CSL}(\vec{H}, \vec{K})$ such that $\vec{H}$ and $\vec{K}$ are composition series of G. Similarly, for $h \in \mathbb{N}_0$, the class of lattices $\mathrm{CSL}_h(\vec{H}, \vec{K})$, where $\vec{H}$ and $\vec{K}$ are composition series of length h, is denoted by $\mathrm{CSL}(h)$. The class of slim semimodular lattices of length h is denoted by $\mathrm{SSL}(h)$. Note that $\mathrm{SSL}^\delta(h)$ is the class of lower semimodular dually slim lattices of length h.

Also, there are self-explanatory "multipointed" variants of the notations introduced above. If L is a slim semimodular lattice with designated maximal chains

$$\begin{aligned} C &= \{0 = c_0 \prec c_1 \prec \cdots \prec c_h = 1\}, \\ D &= \{0 = d_0 \prec d_1 \prec \cdots \prec d_h = 1\} \end{aligned} \tag{2.2}$$

such that $\mathrm{Ji}\, L \subseteq C \cup D$, then the *multipointed lattice* $(L; \vee, \wedge, C, D)$ will be denoted by $\ddot{L}$. The class of these multipointed lattices of length h is denoted by $\mathrm{SS\ddot{L}}(h)$. Note that when we dualize $\ddot{L}$, then c_i and d_j in $\ddot{L}$ correspond to c_{h-i} and d_{h-j} in $\ddot{L}^\delta$, respectively. Generally, if $\ddot{M}$ is a multipointed lattice, then its lattice reduct is denoted by M. If the members of the composition series described in (1.1) are considered constants, then $\mathrm{CSL}_h(\vec{H}, \vec{K})$ turns into a multipointed lattice denoted by $\mathrm{CS\ddot{L}}_h(\vec{H}, \vec{K})$. The class of these multipointed lattices is denoted by $\mathrm{CS\ddot{L}}(G)$ and $\mathrm{CS\ddot{L}}(h)$ for a given group G and for a given length $h \in \mathbb{N}_0$, respectively.

The classes $\mathrm{SSL}(h)^\cong$, $\mathrm{SS\ddot{L}}(h)^\cong$, $\mathrm{SSL}^\delta(h)^\cong$, $\mathrm{SS\ddot{L}}^\delta(h)^\cong$, $\mathrm{CS\ddot{L}}(G)^\cong$, and $\mathrm{CS\ddot{L}}(h)^\cong$ are actually finite sets. With our new notation, $N(h)$ and $\ddot{N}(h)$ are defined by

$$N(h) = |\mathrm{CSL}(h)^\cong| \quad \text{and} \quad \ddot{N}(h) = |\mathrm{CS\ddot{L}}(h)^\cong|. \tag{2.3}$$

2.2. **Another look at slim semimodular lattices.** Semimodular lattices have important links to combinatorics and geometry. We recall one of these links, which is somewhat related to our work. A finite lattice is (*locally*) *upper distributive* if all of its atomistic intervals are boolean. The following theorem is due to Adaricheva, Gorbunov, and Tumanov [1, Theorems 1.7 and 1.9], Dilworth [12], and Monjardet [22]; see also Armstrong [2, Theorem 2.7], Avann [3], and the references given in [22].

Theorem 2.1. *For any finite lattice L, the following conditions are equivalent.*

(i) *L is locally upper distributive.*
(ii) *L is semimodular and it satisfies the meet-semidistributivity law, that is,*

$$x \wedge y = x \wedge z \Rightarrow x \wedge y = x \wedge (y \vee z), \quad \text{for all } x, y, z \in L.$$

(iii) *Every element of L has a unique irredundant decomposition as a meet of meet-irreducible elements.*
(iv) *Every maximal chain of L consists of $1 + |\mathrm{Mi}\, L|$ elements.*
(v) *L is (isomorphic to) the lattice of feasible sets of an antimatroid.*

Czédli and Schmidt [9, Lemma 2] observed that every element in a slim lattice has at most two covers. This implies the following statement.

Corollary 2.2. *The slim semimodular lattices are exactly the locally upper distributive lattices whose elements have at most two upper covers.*

2.3. **Preliminary lemmas.** A cyclic group is nontrivial and simple if and only if it is of prime order. The first part of the following proposition is due to Czédli and Schmidt [11]; the second part strengthens a statement of [11].

Proposition 2.3.

(i) $\mathrm{CSL}(h)^{\cong} = \mathrm{SSL}^{\delta}(h)^{\cong}$ *and* $\mathrm{CS\ddot{L}}(h)^{\cong} = \mathrm{SS\ddot{L}}^{\delta}(h)^{\cong}$*, for all* $h \in \mathbb{N}_0$.

(ii) *If* G *is the direct product of* h *nontrivial simple cyclic groups, then* $\mathrm{CSL}(G)^{\cong} = \mathrm{SSL}^{\delta}(h)^{\cong}$ *and* $\mathrm{CS\ddot{L}}(G)^{\cong} = \mathrm{SS\ddot{L}}^{\delta}(h)^{\cong}$.

(iii) $N(h) = |\mathrm{SSL}(h)^{\cong}|$ *and* $\ddot{N}(h) = |\mathrm{SS\ddot{L}}(h)^{\cong}|$*, for all* $h \in \mathbb{N}_0$.

Before proving Proposition 2.3, which reduces the problem of computing the functions in (2.3) to a lattice-theoretic question, we need some preparation.

Definition 2.4. Let $\ddot{L}$ be as in (2.2). We define two maps, $\pi = \pi(\ddot{L})$ and $\sigma = \sigma(\ddot{L})$, as follows. For $i, j \in \{1, \dots, h\}$, let

$$\begin{aligned} I(i) &= \bigl\{j \in \{1, \dots, h\} : c_{i-1} \vee d_j = c_i \vee d_j\bigr\},\\ \pi(i) &= \text{the smallest element of } I(i),\\ J(j) &= \bigl\{i \in \{1, \dots, h\} : c_i \vee d_{j-1} = c_i \vee d_j\bigr\},\\ \sigma(j) &= \text{the smallest element of } J(j). \end{aligned}$$

The set of permutations acting on $\{1, \dots, h\}$, that is, the set of bijective maps $\{1, \dots, h\} \to \{1, \dots, h\}$, will be denoted by S_h.

Lemma 2.5. $\pi = \pi(\ddot{L})$ *and* $\sigma = \sigma(\ddot{L})$ *belong to* S_h*, provided that the assumption and the notation of Definition 2.4 are in effect. Furthermore,* $\sigma = \pi^{-1}$ *in this case.*

Note that π is the same as the permutation defined in Czédli and Schmidt [11, Def. 2.5]. However, Definition 2.4 serves our goal in a simpler way.

Proof of Lemma 2.5. Clearly, $0 \notin I(i) \cup J(j)$ and $h \in I(i) \cap J(j)$. If j belongs to $I(i)$ and $j < h$, then

$$c_{i-1} \vee d_{j+1} = c_{i-1} \vee d_j \vee d_{j+1} = c_i \vee d_j \vee d_{j+1} = c_i \vee d_{j+1}$$

shows that $j + 1 \in I(i)$. Since the same argument works for $J(j)$, we conclude that, for $i, j \in \{1, \dots, h\}$, both $I(i)$ and $J(j)$ are (order) filters of $\{1, \dots, h\}$. For $i \in \{1, \dots, h\}$, let $j = \pi(i)$. Since $j - 1 \notin I(i)$ and $j \in I(i)$, we obtain

$$c_{i-1} \vee d_{j-1} < c_i \vee d_{j-1} \le c_i \vee d_j = c_{i-1} \vee d_j. \tag{2.4}$$

Semimodularity implies $c_{i-1} \vee d_{j-1} \preceq c_{i-1} \vee d_j$. This and (2.4) yield $c_i \vee d_{j-1} = c_i \vee d_j$. Hence $i \in J(j)$, and we obtain $\sigma(j) \le i$. If we had $\sigma(j) < i$, then $i-1 \in J(j)$ would imply $c_{i-1} \vee d_{j-1} = c_{i-1} \vee d_j$, contradicting (2.4). Hence $i = \sigma(j) = \sigma(\pi(i))$, that is, $\sigma \circ \pi$ is the identity map on $\{1, \dots, h\}$. By symmetry, so is $\pi \circ \sigma$. $\square$

For a set A, the *powerset lattice* $\operatorname{Pow} A$ of A consists of all subsets of A. Sometimes, especially when we need a notation for the covering relation, we write $x \le y$ instead of $x \subseteq y$, for $x, y \in \operatorname{Pow} A$. By De Morgan's laws, $\operatorname{Pow} A$ is a self-dual lattice. It is well-known, see Nation [23, the dual of Thm. 2.2], that for each lattice M, the join-semilattice $(M; \vee)$ has an embedding into $\bigl(\operatorname{Pow} M; \cup\bigr)$. In other words, M has a join-embedding into the powerset lattice $\operatorname{Pow} M$. Since $h < |L|$ in general, the following proposition gives a more economical embedding for slim semimodular lattices.

Proposition 2.6. *Let $\ddot{L}$ be as in (2.2), and let $A = \{a_1, \dots, a_h\}$ be an h-element set. If $\pi = \pi(\ddot{L})$ and $\sigma = \sigma(\ddot{L})$ are as in Lemma 2.5, then the map $\varphi\colon (L; \vee) \to \big(\mathrm{Pow}\, A; \cup\big)$, defined by*

$$x \mapsto \{a_i : c_i \le x\} \cup \{a_i : d_{\pi(i)} \le x\} = \{a_i : c_i \le x\} \cup \{a_{\sigma(j)} : d_j \le x\}, \tag{2.5}$$

is a cover-preserving join-embedding.

Proof. The equality in (2.5) follows from $\sigma = \pi^{-1}$. We claim that

$$\varphi(c_u \vee d_v) \subseteq \varphi(c_u) \cup \varphi(d_v), \text{ for } u, v \in \{1, \dots, h\}. \tag{2.6}$$

Assume $a_i \in \varphi(c_u \vee d_v)$. This means that $c_i \le c_u \vee d_v$ or $d_{\pi(i)} \le c_u \vee d_v$.

Assume first that $c_i \le c_u \vee d_v$. We may also assume $u < i$, since otherwise $c_i \le c_u$ would imply $a_i \in \varphi(c_u)$. So $c_u \le c_{i-1} < c_i \le c_u \vee d_v$. Taking the joins of these elements with d_v, we obtain $c_{i-1} \vee d_v = c_i \vee d_v$. Hence $v \in I(i)$ implies $\pi(i) \le v$. Thus, $d_{\pi(i)} \le d_v$ yields $a_i \in \varphi(d_v) \subseteq \varphi(c_u) \cup \varphi(d_v)$.

Second, assume $d_{\pi(i)} \le c_u \vee d_v$. Using the notation $j = \pi(i)$, we have $d_j \le c_u \vee d_v$. If $d_j \le d_v$, then $a_i = a_{\sigma(j)} \in \varphi(d_v)$. Hence we may assume $v < j$. Using $d_v \le d_{j-1} < d_j \le c_u \vee d_v$ and taking the joins of these elements with c_u, we obtain $c_u \vee d_{j-1} = c_u \vee d_j$. So $u \in J(j)$, whence $\sigma(j) \le u$. Therefore, $c_i = c_{\sigma(j)} \le c_u$ yields $a_i \in \varphi(c_u) \subseteq \varphi(c_u) \cup \varphi(d_v)$. This proves (2.6).

Next, let $x, y \in L$. Since φ is clearly order-preserving, it follows that $\varphi(x \vee y) \supseteq \varphi(x) \cup \varphi(y)$. So it suffices to show that $\varphi(x \vee y) \subseteq \varphi(x) \cup \varphi(y)$. This is evident if x and y are comparable, since φ is order-preserving. Hence we may assume that x and y are incomparable, which we denote by $x \parallel y$. Since $\mathrm{Ji}\, L \subseteq C \cup D$, we obtain that x is of the form $c_r \vee d_v$ and y is of the form $c_u \vee d_s$. It follows from $x \parallel y$ that either $r < u$ and $s < v$, or $r > u$ and $s > v$; we may assume the former since the latter is analogous. Using (2.6) and the fact that φ is order-preserving, we obtain $\varphi(x \vee y) = \varphi(c_r \vee d_v \vee c_u \vee d_s) = \varphi(c_u \vee d_v) \subseteq \varphi(c_u) \cup \varphi(d_v) \subseteq \varphi(y) \cup \varphi(x) = \varphi(x) \cup \varphi(y)$. This proves that φ is a join-homomorphism.

Finally, we have to show that φ is injective. Suppose to the contrary that there are $x, z \in L$ such that $\varphi(x) = \varphi(z)$ and $z \not\le x$. We have $x < x \vee z$, and we can take an element y such that $x \prec y \le x \vee z$. From $\varphi(x) \subseteq \varphi(y) \subseteq \varphi(x \vee z) = \varphi(x) \cup \varphi(z) = \varphi(x) \cup \varphi(x) = \varphi(x)$, we conclude $\varphi(x) = \varphi(y)$. Let s and t be the largest elements of $\{0, \dots, h\}$ such that $c_s \le x$ and $d_t \le y$. Since $\mathrm{Ji}\, L \subseteq C \cup D$ by (2.2), $x = c_s \vee d_t$. Since each element of L is of the form $c_u \vee d_v$, it follows from $x \prec y$ that $t < h$ and $y = c_s \vee d_{t+1}$, or $s < h$ and $y = c_{s+1} \vee d_t$.

First, assume $y = c_s \vee d_{t+1} = x \vee d_{t+1}$. Let $u = \max\{s, \sigma(t+1)\}$, and observe that $u \in J(t+1)$ since $\sigma(t+1) \in J(t+1)$ and $J(t+1)$ is an order-filter. We have $a_{\sigma(t+1)} \in \varphi(x) = \varphi(y)$ since $d_{t+1} \le y$. So $d_{t+1} \le x$ or $c_{\sigma(t+1)} \le x$. The former violates $x \ne y$. So does the latter, since $u \in J(t+1)$ yields $x = c_{\sigma(t+1)} \vee x = c_{\sigma(t+1)} \vee c_s \vee d_t = c_u \vee d_t = c_u \vee d_{t+1} = c_{\sigma(t+1)} \vee c_s \vee d_{t+1} = c_{\sigma(t+1)} \vee y = y$.

Second, assume $y = c_{s+1} \vee d_t = c_{s+1} \vee x$. Let $v = \max\{t, \pi(s+1)\}$, and observe that $v \in I(s+1)$. We have $a_{s+1} \in \varphi(x) = \varphi(y)$, since $c_{s+1} \le y$. Hence $c_{s+1} \le x$ or $d_{\pi(s+1)} \le x$. The former violates $x \ne y$. So does the latter, since $v \in I(s+1)$ implies $x = x \vee d_{\pi(s+1)} = c_s \vee d_t \vee d_{\pi(s+1)} = c_s \vee d_v = c_{s+1} \vee d_v = c_{s+1} \vee d_t \vee d_{\pi(s+1)} = y \vee d_{\pi(s+1)} = y$.

Both assumptions lead to a contradiction, whence φ is injective. It is also cover-preserving since $\mathrm{length}\, L = \mathrm{length}\big(\mathrm{Pow}\, A\big)$. $\square$

Corollary 2.7. *If L is a slim semimodular lattice of length h and A is a set with $|A| = h$, then there exists a cover-preserving join-embedding $\varphi\colon L \to \operatorname{Pow} A$. Furthermore, we can choose $A = \operatorname{Mi} L$ and $\varphi\colon L \to \operatorname{Pow} A$, $x \mapsto \{a \in A : a \ngeq x\}$.*

This corollary and its nice short proof below were suggested by a referee. Note that we shall use Proposition 2.6 rather than Corollary 2.7 in the proof of Proposition 2.3, because φ should clearly depend on $\pi(\ddot{L})$.

Proof of Corollary 2.7. Obviously, $\varphi(0) = \varnothing$, $\varphi(1) = \operatorname{Mi} L = A$, and $\varphi(x \vee y) = \varphi(x) \cup \varphi(y)$, for all $x, y \in L$. Since $\operatorname{length} L = |\operatorname{Mi} L| = \operatorname{length}\bigl(\operatorname{Pow} A\bigr)$ by Corollary 2.2, φ is cover-preserving and injective. ☐

Proof of Proposition 2.3. Obviously, it suffices to consider only the multipointed version. Clearly, $\mathrm{CS\ddot{L}}(h)^{\cong} \subseteq \mathrm{SS\ddot{L}}^{\delta}(h)^{\cong}$ follows from (2.1). Hence it suffices to prove the converse inclusion in part (ii); then both parts (i) and (ii) will follow. Let $G_1, \dots, G_h$ be nontrivial simple subgroups of an Abelian group G such that G is the (inner) direct product of these subgroups; we have to show that $\mathrm{SS\ddot{L}}^{\delta}(h)^{\cong} \subseteq \mathrm{CS\ddot{L}}(G)^{\cong}$. Let $\mathbf{I}(\ddot{L}^{\delta}) \in \mathrm{SS\ddot{L}}^{\delta}(h)^{\cong}$, that is, $\ddot{L} \in \mathrm{SS\ddot{L}}(h)$ with the notation given in (2.2). Take an h-element set $A = \{a_1, \dots, a_n\}$. The lattice $\operatorname{Sub} G$ of all subgroups of G is well-known to be modular, see, for example, Stern [28, Section 1.6] or Burris and Sankappanavar [4, Ex. I.3.5]. By the definition of a direct product, the subgroups $G_1, \dots, G_h$ form an independent set in $\operatorname{Sub} G$. The definition of an independent set is not important for us; what we need is that these subgroups generate a sublattice isomorphic to the powerset lattice $\operatorname{Pow} A$ by Grätzer [14, Cor. IV.1.10 and Thm. IV.1.11] or [15, Cor. 359 and Thm. 360]. Consequently, we may assume that $\operatorname{Pow} A$ is a sublattice of $\operatorname{Sub} G$. By De Morgan's laws, the map $\psi\colon \operatorname{Pow} A \to \operatorname{Pow} A$, defined by $X \mapsto A \setminus X$, is a dual lattice isomorphism, that is, a bijection such that $\psi(X \cup Y) = \psi(X) \cap \psi(Y)$ and $\psi(X \cap Y) = \psi(X) \cup \psi(Y)$, for all $X, Y \in \operatorname{Pow} A$. Take the map φ defined in Proposition 2.6. Let $\eta\colon L^{\delta} \to L$ denote the identity map, which is a dual isomorphism. Let γ be the composite map $\psi \circ \varphi \circ \eta\colon L^{\delta} \to \bigl(\operatorname{Pow} A; \cap\bigr)$. It is a meet-embedding since

$$\begin{aligned}\gamma(x \wedge_{L^{\delta}} y) &= \psi\bigl(\varphi(\eta(x \wedge_{L^{\delta}} y))\bigr) = \psi\bigl(\varphi(\eta(x) \vee_L \eta(y))\bigr) = \psi\bigl(\varphi(x \vee_L y)\bigr)\\ &= \psi\bigl(\varphi(x) \cup \varphi(y)\bigr) = \psi\bigl(\varphi(x)\bigr) \cap \psi\bigl(\varphi(y)\bigr) = \gamma(x) \cap \gamma(y).\end{aligned}$$

Note that $G = G_1 \dots G_h \triangleright G_1 \dots G_{h-1} \triangleright \dots \triangleright G_1 \triangleright \{1\}$ is a composition series, since the G_i are simple groups. Hence $\operatorname{Sub} G$ and L have the same length, and thus the Jordan-Hölder theorem shows that γ is a cover-preserving embedding. The images of the constants c_i and d_j are the appropriate constants in $\gamma(L^{\delta})$. Therefore, $\ddot{L}^{\delta} \cong \gamma(\ddot{L}^{\delta}) \in \mathrm{CS\ddot{L}}(h)$. Hence $\mathbf{I}(\ddot{L}^{\delta}) \in \mathrm{CS\ddot{L}}(h)^{\cong}$, proving parts (i) and (ii).

Finally, part (iii) follows from (2.3), part (i), and the obvious equalities

$$|\mathrm{SSL}^{\delta}(h)^{\cong}| = |\mathrm{SSL}(h)^{\cong}|, \qquad |\mathrm{SS\ddot{L}}^{\delta}(h)^{\cong}| = |\mathrm{SS\ddot{L}}(h)^{\cong}|. \qquad ☐$$

3. Describing the multipointed case by permutations

If $\mathbf{I}(\ddot{L}) \in \mathrm{SSL}(h)^{\cong}$ is as in (2.2), then $\pi(\ddot{L}) \in S_h$ is given in Definition 2.4; see also Lemma 2.5. The permutation $\pi(\ddot{L})$ depends only on $\mathbf{I}(\ddot{L})$, since $\pi(\ddot{K}) = \pi(\ddot{L})$, for all $\ddot{K} \in \mathbf{I}(\ddot{L})$. Next, let $\pi \in S_n$, and denote π^{-1} by σ. Let $A = \{a_1, \dots, a_h\}$ be an h-element set. For $u, v \in \{0, \dots, h\}$, let $\widehat{c}_u = \{a_i : i \le u\}$ and $\widehat{d}_v = \{a_{\sigma(i)} : i \le v\}$. We define $\ddot{L}(\pi)$ such that $L(\pi)$ is $\bigl\{\widehat{c}_u \cup \widehat{d}_v : u, v \in \{0, \dots, h\}\bigr\}$, a join-subsemilattice of the powerset lattice $\operatorname{Pow} A$, and the constants are the $\widehat{c}_u$ and the $\widehat{d}_v$. Although the following statement could be extracted from Czédli and Schmidt [11], it is easier to derive it from the previous section.

Proposition 3.1. *The maps*

$$\gamma_1 : \mathrm{SS\ddot{L}}(h)^{\cong} \to S_h,\ \mathbf{I}(\ddot{L}) \mapsto \pi(\ddot{L}) \quad \textit{and} \quad \gamma_2 : S_h \to \mathrm{SS\ddot{L}}(h)^{\cong},\ \pi \mapsto \mathbf{I}(\ddot{L}(\pi))$$

are reciprocal bijections. Thus $\ddot{N}(h) = h!$.

Proof. Assume $\pi \in S_h$. Assume also that $x, y \in \ddot{L}(\pi)$ such that $x \prec y$. We can write these elements in the form $x = \widehat{c}_u \cup \widehat{d}_v$ and $y = \widehat{c}_s \cup \widehat{d}_t$ such that each of $u, v, s, t \in \{0, \dots, h\}$ are maximal with respect to these equations. Now $u \le s$, $v \le t$, and $(u, v) < (s, t)$. If $u < s$, then $x < \widehat{c}_{u+1} \cup \widehat{d}_v$ by the maximality of u, so $x < \widehat{c}_{u+1} \cup \widehat{d}_v \le y$ and $x \prec y$ imply $y = \widehat{c}_{u+1} \cup \widehat{d}_v$. Similarly, if $v < t$, then $x < \widehat{c}_u \cup \widehat{d}_{v+1}$ by the maximality of v, so $x < \widehat{c}_u \cup \widehat{d}_{v+1} \le y$ and $x \prec y$ imply $y = \widehat{c}_u \cup \widehat{d}_{v+1}$. Hence, in both cases, $y \setminus x$ is a singleton, so y covers x in the powerset lattice $\operatorname{Pow} A$. Thus $L(\pi)$ is a cover-preserving join-subsemilattice of $\operatorname{Pow} A$.

Clearly, $\operatorname{Pow} A$ is semimodular, since it is distributive. Semimodularity depends only on the join operation and the covering relation. Therefore $L(\pi)$, which is a cover-preserving join-subsemilattice of $\operatorname{Pow} A$, is semimodular. Its length is h, the length of $\operatorname{Pow} A$. Furthermore, let $\widehat{C} = \bigl\{\widehat{c}_i : 0 \le i \le h\bigr\}$ and $\widehat{D} = \{\widehat{d}_i : 0 \le i \le h\}$; they are maximal chains, and we have $\mathrm{Ji}\bigl(L(\pi)\bigr) \subseteq \widehat{C} \cup \widehat{D}$. This proves $\ddot{L}(\pi) \in \mathrm{SS\ddot{L}}(h)^{\cong}$.

Applying Definition 2.4 to $\bigl((\ddot{L}(\pi); \cup), \widehat{c}_i, \widehat{d}_j\bigr)$ rather than to $\bigl((L; \vee), c_i, d_j\bigr)$, we obtain $\widehat{I}(i)$, $\widehat{\pi}$, $\widehat{J}(j)$ and $\widehat{\sigma}$. For $i, j \in \{1, \dots, h\}$, we have

$$\begin{aligned} j \in \widehat{I}(i) &\iff \widehat{c}_{i-1} \cup \widehat{d}_j = \widehat{c}_i \cup \widehat{d}_j \iff a_i \in \widehat{d}_j \iff i \in \{\sigma(1), \dots, \sigma(j)\} \\ &\iff \pi(i) \in \{1, \dots, j\} \iff \pi(i) \le j \iff j \in I(i). \end{aligned}$$

Hence $\widehat{I}(i)$ equals $I(i)$, and their minimal elements, $\pi\bigl(\ddot{L}(\pi)\bigr)(i)$ and $\pi(i)$, are also equal. This proves $\pi\bigl(\ddot{L}(\pi)\bigr) = \pi$, implying that $\gamma_1 \circ \gamma_2$ is the identity map $S_h \to S_h$.

Next, assume $\ddot{L} \in \mathrm{SS\ddot{L}}(h)$. Let $\pi = \pi(\ddot{L})$. Let φ be the join-embedding defined in Proposition 2.6, and let $\sigma = \pi^{-1}$. We claim that

$$\varphi(c_u) = \{a_i : 1 \le i \le u\} = \widehat{c}_u \ \text{ and } \ \varphi(d_v) = \{a_{\sigma(j)} : 1 \le j \le v\} = \widehat{d}_v. \tag{3.1}$$

Since $a_i \in \varphi(c_u)$ for $i \le u$ is evident by the definition of φ, assume $a_i \in \varphi(c_u)$. We have to show that $i \le u$. This is clear if $c_i \le c_u$, hence we assume $d_{\pi(i)} \le c_u$. Now $c_u \vee d_{\pi(i)-1} = c_u = c_u \vee d_{\pi(i)}$ yields $u \in J(\pi(i))$. Hence $i = \sigma\bigl(\pi(i)\bigr) \le u$, proving the first equation in (3.1). To prove the other equation, note that $a_{\sigma(j)} \in \varphi(d_v)$ for $j \le v$ is obvious again. Assume $a_i \in \varphi(d_v)$. If $j = \pi(i)$, then $i = \sigma(j)$, and we have to show $j \le v$. This is trivial if $d_j = d_{\pi(i)} \le d_v$. If we assume $c_i \le d_v$, then $c_{i-1} \vee d_v = d_v = c_i \vee d_v$ yields $v \in I(i)$, implying $j = \pi(i) \le v$. This proves (3.1).

Finally, $\varphi(L) \subseteq \ddot{L}\bigl(\pi(\ddot{L})\bigr)$ is trivial. Since $\ddot{L}\bigl(\pi(\ddot{L})\bigr)$ is join-generated by the set $\{\widehat{c}_u : 0 \le u \le h\} \cup \{\widehat{d}_v : 0 \le v \le h\}$, which consists of some φ-images by (3.1), we conclude $\varphi(L) \supseteq \ddot{L}\bigl(\pi(L)\bigr)$. So $\varphi(L) = \ddot{L}\bigl(\pi(\ddot{L})\bigr)$. We know from Proposition 2.6 that $\varphi\colon L \to \varphi(L) = \ddot{L}\bigl(\pi(\ddot{L})\bigr)$ is a join-isomorphism, whence it is a lattice isomorphism. By (3.1), it is an isomorphism $\ddot{L} \to \ddot{L}\bigl(\pi(\ddot{L})\bigr)$. Thus $\gamma_2 \circ \gamma_1 : \mathrm{SS\ddot{L}}(h)^{\cong} \to \mathrm{SS\ddot{L}}(h)^{\cong}$ is the identity map. $\square$

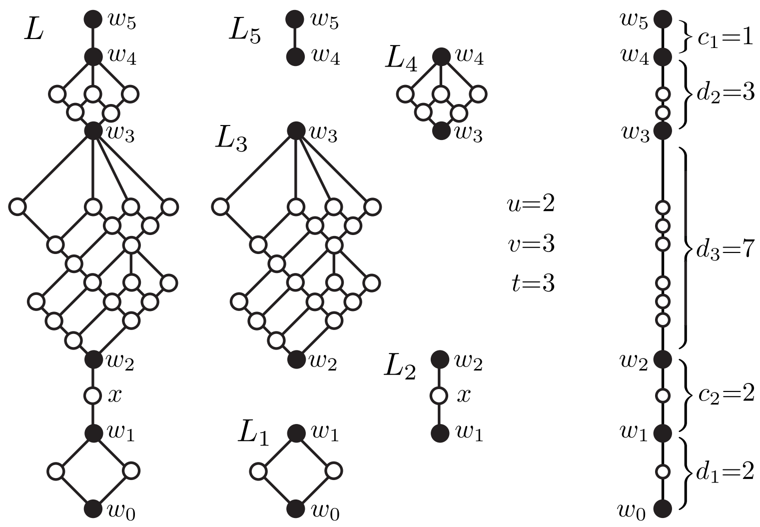

FIGURE 1. A slim semimodular lattice L of length 15 and its decomposition

4. Description by matrices

If an element x of a lattice L is comparable with all $y \in L$, then x is a *narrows* or a *universal element* of L. This terminology is from Grätzer and Quackenbush [17]; however, as opposed to [17], we also define 0 and 1 as narrows of L. In Figure 1, the narrows of $L, L_1, \dots, L_5$ are the black-filled elements and x. We say that L is *indecomposable* if $|L| \geq 3$ and 0 and 1 are the only narrows of L. So an indecomposable lattice is of length at least 2, and it is not a chain. For finite lattices L_1 and L_2, we obtain the *glued sum* of L_1 and L_2 by putting L_2 atop L_1 and identifying 1_{L_1} with 0_{L_2}. Figure 1 indicates that each slim semimodular lattice (like L in the figure) can uniquely be decomposed into a glued sum of maximal chain intervals (here L_2 and L_5) and indecomposable slim semimodular lattice summands (here L_1, L_3 and L_4). Chains are quite simple objects, and the indecomposable summands will be characterized by certain matrices. Let C and D be two finite chains with $C = \{c_0 \prec c_1 \prec \cdots \prec c_m\}$ and $D = \{d_0 \prec d_1 \prec \cdots \prec d_n\}$, and let $G = C \times D$ be their direct product. That is, for $(c_i, d_j), (c_s, d_t) \in C \times D$, $(c_i, d_j) \leq (c_s, d_t)$ means that $i \leq s$ and $j \leq t$. Assume

$$F \subseteq \{1, \dots, m\} \times \{1, \dots, n\} \tag{4.1}$$

such that, for all $(i_1, j_1), (i_2, j_2) \in F$, $i_1 = i_2$ if and only if $j_1 = j_2$. Let $\boldsymbol{\alpha}$ be a join-congruence of G, that is, a congruence of the join-semilattice $(G; \vee)$. The $\boldsymbol{\alpha}$-classes are $\vee$-closed convex subsets. Therefore $(x, y) \in \boldsymbol{\alpha}$ if and only if $(x, x \vee y), (y, x \vee y) \in \boldsymbol{\alpha}$, and we easily obtain a well-known fact: $\boldsymbol{\alpha}$ is determined by the covering pairs it collapses. Hence, to define a join-congruence, it suffices to tell which covering pairs are collapsed. Following Czédli [5, (13), (14) and Cor. 22], we define a join-congruence $\boldsymbol{\beta} = \boldsymbol{\beta}(F)$ of G by

$$\begin{aligned} \big((c_{i-1}, d_j), (c_i, d_j)\big) \in \boldsymbol{\beta} &\iff \text{there is a } v \leq j \text{ such that } (i, v) \in F, \text{ and} \\ \big((c_i, d_{j-1}), (c_i, d_j)\big) \in \boldsymbol{\beta} &\iff \text{there is a } u \leq i \text{ such that } (u, j) \in F. \end{aligned} \tag{4.2}$$

It is not very hard to show, and it is proved in [5, Propositions 17 and 20], that $G/\boldsymbol{\beta}$ is a slim semimodular lattice. What we have to prove here is the following.

Lemma 4.1. *$G/\boldsymbol{\beta}$ is of length $m+n-|F|$.*

Proof. The $\boldsymbol{\beta}$-class of an element x will be denoted by $x/\boldsymbol{\beta}$. Consider the chains

$$\begin{aligned}\widehat{C} &= \bigl\{(c_0,d_0)/\boldsymbol{\beta} \le (c_1,d_0)/\boldsymbol{\beta} \le \dots \le (c_m,d_0)/\boldsymbol{\beta}\bigr\} \text{ and}\\ \widehat{D} &= \bigl\{(c_m,d_0)/\boldsymbol{\beta} \le (c_m,d_1)/\boldsymbol{\beta} \le \dots \le (c_m,d_n)/\boldsymbol{\beta}\bigr\}\end{aligned} \tag{4.3}$$

in $G/\boldsymbol{\beta}$. By Czédli [5, Lemma 1], $x \preceq y$ in G implies $x/\boldsymbol{\beta} \preceq y/\boldsymbol{\beta}$ in $G/\boldsymbol{\beta}$. Therefore, each inequality in (4.3) is a "covers or equals" relation, and $\widehat{C} \cup \widehat{D}$ is a maximal chain in $G/\boldsymbol{\beta}$. Consequently, the length of $G/\boldsymbol{\beta}$ is $m+n$ minus the number of those "$\le$" in (4.3) that are equations. Since $(i,j) \in F$ implies $0 \notin \{i,j\}$, (4.2) yields that all inequalities in $\widehat{C}$ are strict. It also yields that $(c_m,d_{j-1})/\boldsymbol{\beta} = (c_m,d_j)/\boldsymbol{\beta}$ if and only if $(i,j) \in F$ for some i. Hence there are exactly $|F|$ equations in (4.3). $\square$

$0,1$-*matrices* are matrices whose entries lie in $\{0,1\}$. The transpose of a matrix B will be denoted by B^T. To describe F in (4.1), we can consider the m-by-n $0,1$-matrix $A = (a_{ij})_{m\times n}$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in F;\\ 0, & \text{if } (i,j) \notin F.\end{cases}$$

Thus, certain $0,1$-matrices determine slim semimodular lattices: A determines F, and F determines $G/\boldsymbol{\beta}$. It is proved in Czédli [5] (and it follows also from Czédli and Schmidt [11]) that each slim semimodular lattice L is determined by some $0,1$-matrix A. Although A for a given L is not unique, $\{A, A^T\}$ becomes unique for indecomposable slim semimodular lattices if we stipulate additional properties, see Definition 4.2 below. By a *zero matrix* we mean a matrix all of whose entries are zeros; *zero rows* and *zero columns* are understood analogously. Given a matrix A, its k-by-k upper left corner submatrix will be denoted by $\mathrm{Corn}_k\, A$. Sometimes we have to allow the case $k=0$; then $\mathrm{Corn}_0\, A$ is the empty matrix.

Definition 4.2. Let $m,n \in \mathbb{N}$ such that $m \le n$. An m-by-n $0,1$-matrix A is a *slim matrix* if it has the following five properties:

($1^\bullet$) Every row contains at most one unit, and the same holds for every column.
($2^\bullet$) A contains less than m units.
($3^\bullet$) For $k = 1,\dots,m-1$, $\mathrm{Corn}_k\, A$ contains less than k units.
($4^\bullet$) For every $i \in \{1,\dots,m\}$, if the last entry, a_{in}, of the i-th row equals 1, then there is an $i' < i$ such that the i'-th row is a zero row.
($5^\bullet$) For every $j \in \{1,\dots,n\}$, if the last entry, a_{mj}, of the j-th column equals 1, then there is a $j' < j$ such that the j'-th column is a zero column.

By Czédli [5], ($1^\bullet$), ..., ($5^\bullet$) are independent conditions; that is, none of them is implied by the rest. If ($1^\bullet$) and ($2^\bullet$) are assumed, then ($3^\bullet$) means that all the principal upper left minors equal zero. The set of slim matrices is denoted by SM.

For $A \in \mathrm{SM}$, the transpose A^T of A belongs to SM if and only if A is a square matrix. We define an equivalence relation $\sim_T$ on SM as follows. For $A, B \in \mathrm{SM}$, let $A \sim_T B$ mean that $\{A, A^T\} = \{B, B^T\}$. That is, $A \sim_T B$ if and only if $B \in \{A, A^T\}$. In what follows, let $\mathrm{SM}^\sim$ be a full set of representatives of the $\sim_T$-classes. That is, $\mathrm{SM}^\sim$ is a subset of SM such that $|\{A, A^T\} \cap \mathrm{SM}^\sim| = 1$ holds,

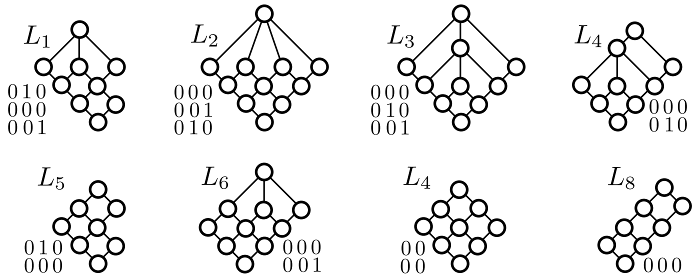

FIGURE 2. Indecomposable slim semimodular lattices of length 4

for all $A \in \mathrm{SM}$. Clearly, all non-square slim matrices and all symmetric slim matrices belong to $\mathrm{SM}^{\sim}$, since they belong to one-element $\sim_T$-classes.

Notation. For $0 \le k < m \le n$, let $\mathrm{SM}(m,n,k)$ denote the set of slim m-by-n matrices containing exactly k units. Let $\mathrm{SM}^{\sim}(m,n,k) = \mathrm{SM}(m,n,k) \cap \mathrm{SM}^{\sim}$. Note that $\mathrm{SM}^{\sim}(m,n,k) = \mathrm{SM}(m,n,k)$ if $m < n$.

Next, we recall the main result of Czédli [5], and supplement it with the statement of Lemma 4.1.

Proposition 4.3 (see [5] for (i) and Lemma 4.1 for (ii)).

(i) *There is a bijective correspondence between* $\mathrm{SM}^{\sim}$ *and the set* $\bigcup_{h=2}^{\infty} \mathrm{SSL}(h)^{\cong}$ *of isomorphism classes of indecomposable slim semimodular lattices.*

(ii) *The restriction of the above-mentioned correspondence yields a bijective correspondence between* $\mathrm{SM}^{\sim}(m,n,k)$ *and* $\mathrm{SSL}(m+n-k)^{\cong}$.

Based on Proposition 4.3, it will be sufficient to count the slim matrices.

5. Formulating the main result

Notation. The set of symmetric slim m-by-m matrices that contain exactly k units is denoted by $\mathrm{SSM}(m,k)$. If a capital letter, possibly with parameters and superscripts, is used to denote a finite set of matrices, then the size of this set will be denoted by the corresponding lowercase letter. For example, $\mathrm{sm}^{\sim}(m,n,k) = |\mathrm{SM}^{\sim}(m,n,k)|$ and $\mathrm{ssm}(m,k) = |\mathrm{SSM}(m,k)|$. We always assume

$$0 \le k < m \le n. \tag{5.1}$$

Clearly,

$$\mathrm{sm}^{\sim}(m,n,k) = \begin{cases} \big(\mathrm{sm}(m,n,k) + \mathrm{ssm}(m,k)\big)/2, & \text{if } m = n \\ \mathrm{sm}(m,n,k), & \text{if } m < n \end{cases}. \tag{5.2}$$

Let $\mathrm{SM}^0(m,n,k)$ and $\mathrm{SSM}^0(m,k)$ denote the set of those members of $\mathrm{SM}(m,n,k)$ and $\mathrm{SSM}(m,k)$, respectively, whose first row is zero. Similarly, let $\mathrm{SM}^1(m,n,k)$ stand for $\mathrm{SM}(m,n,k) \setminus \mathrm{SM}^0(m,n,k)$, and let $\mathrm{SSM}^1(m,k) = \mathrm{SSM}(m,k) \setminus \mathrm{SSM}^0(m,k)$.

Keeping the general assumption (5.1) in mind, we clearly have

$$\mathrm{sm}^0(m,n,0) = 1, \ \ \mathrm{sm}^1(m,n,0) = 0, \ \text{ and } \ \mathrm{ssm}^1(m,0) = \mathrm{ssm}^1(m,1) = 0. \tag{5.3}$$

The main step towards the number of slim semimodular lattices is summarized in the following statement, where $(2t-1)!!$ denotes $1 \cdot 3 \cdot 5 \cdots (2t-1) = (2t)!/(2^t \cdot t!)$. As usual in case of empty products, $(-1)!! = 1$ by definition.

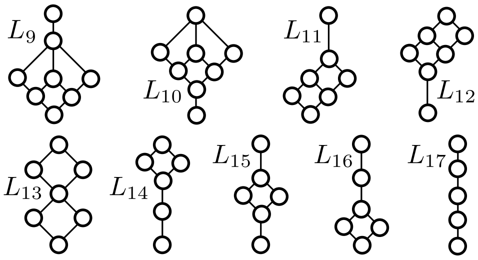

FIGURE 3. Decomposable slim semimodular lattices of length 4

Lemma 5.1. $\mathrm{sm}^{\sim}(m,n,k)$ *is determined by induction based on* (5.1)*,* (5.2)*,* (5.3)*, and the following six formulas:*

$$\mathrm{sm}^0(m,n,k) = \binom{m-1}{k}\cdot\frac{n!}{(n-k)!} - \binom{m-2}{k-1}\cdot\frac{n!}{(n-k+1)!}\,, \tag{5.4}$$

$$\mathrm{sm}^1(m,n,k) = \sum_{j=0}^{k-1} j!\cdot \mathrm{sm}(m-j-1,n-j-1,k-j-1)\cdot(n-j-2)\,, \tag{5.5}$$

$$\mathrm{sm}(m,n,k) = \mathrm{sm}^0(m,n,k) + \mathrm{sm}^1(m,n,k)\,, \tag{5.6}$$

$$\mathrm{ssm}^0(m,k) = \binom{m-1}{k}\cdot\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{k-2j}\cdot(2j-1)!!\,, \tag{5.7}$$

$$\begin{aligned}\mathrm{ssm}^1(m,k) = \sum_{i=0}^{k-2}(m-3-i)\cdot \mathrm{ssm}(m-2-i,k-2-i)\,\times\\ \times\sum_{r=0}^{\lfloor i/2\rfloor}\binom{i}{i-2r}\cdot(2r-1)!!\,, \quad \textit{and}\end{aligned} \tag{5.8}$$

$$\mathrm{ssm}(m,k) = \mathrm{ssm}^0(m,k) + \mathrm{ssm}^1(m,k)\,, \tag{5.9}$$

where, in addition to (5.1)*, we assume* $k\geq 1$ *in* (5.4) *and* (5.5)*, and* $k\geq 2$ *in* (5.8)*.*

For $2\leq h\in\mathbb{N}$, let $\mathrm{ISSL}(h)$ denote the class of indecomposable slim semimodular lattices of length h. The corresponding set of isomorphism classes is $\mathrm{ISSL}(h)^{\cong} = \{\mathbf{I}(L) : L\in\mathrm{ISSL}(h)\}$, and its size is denoted by $N_{\mathrm{issl}}(h) = |\mathrm{ISSL}(h)^{\cong}|$.

Proposition 5.2. *The number of indecomposable slim semimodular lattices of length* h *is*

$$N_{\mathrm{issl}}(h) = \sum_{k=0}^{h-2}\ \sum_{n=\lceil\frac{h+k}{2}\rceil}^{h-1} \mathrm{sm}^{\sim}(h+k-n,n,k). \tag{5.10}$$

Now, we are ready to formulate our main result.

Theorem 5.3. $N(0)=1$ *and, for* $h\in\mathbb{N}$*,*

$$N(h) = N(h-1) + \sum_{j=2}^{h} N_{\mathrm{issl}}(j)\cdot N(h-j).$$

Based on Lemma 5.1 and Proposition 5.2, Theorem 5.3 offers an effective way to compute $N(h)$. For comparison, note that there are several papers on counting other particular lattices; for example, see Erné, Heitzig and Reinhold [13] and Pawar and Waphare [24]. There are also papers on enumerating all finite lattices of a given size s, see Heitzig and Reinhold [19] for $s \le 18$, and see the references listed in [19]. The calculation for $s = 18$ took several days on a parallel supercomputer in 2001.

If we store the previously computed values, then the calculation of $N(h)$ by computer algebra is sufficiently fast. Appropriate programs (Maple 5 and Mathematica 6) are available from the authors' web sites, where $\bigl\{\bigl(h, N_{\mathrm{issl}}(h)\bigr) : h \le 100\bigr\}$ and $\bigl\{\bigl(h, N(h)\bigr) : h \le 100\bigr\}$ are also available. Using a personal computer with Intel Duo CPU 3.00 GHz, 1.98 GHz, and 3.25 GB RAM, it took only four seconds and two minutes, respectively, to obtain the following two values:

$$\begin{aligned} N(50) &= 152067494389203137357189889218916669574887914146900\backslash \\ &\qquad 892747031888674 \approx 0.1520674944 \cdot 10^{65}, \text{ and} \\ N(100) &= 4666300514485158296402274322204901463839367594\backslash \\ &\qquad 2294818488060200326708844394572102663679 22\backslash \\ &\qquad 36922098628302822500133605498186278294103 91\backslash \\ &\qquad 4225784767584940393608 41845 \approx 0.4666300514 \cdot 10^{158}. \end{aligned}$$

The following table was computed in less than 0.1 seconds:

h	0	1	2	3	4	5	6	7	8	9	10	11	12
$N_{\mathrm{issl}}(h)$	0	0	1	2	8	39	242	1 759	14 674	137 127	1 416 430	16 006 403	196 400 810
$N(h)$	1	1	2	5	17	73	397	2 623	20 414	181 607	1 809 104	19 886 032	238 723 606

6. Combinatorial lemmas and proofs

By a *permutation matrix* we mean a k-by-k square $0,1$-matrix satisfying $(1^\bullet)$ and containing k units. The following lemma belongs to folklore.

Lemma 6.1. *The number of symmetric k-by-k permutation matrices is*

$$\sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{k-2j} \cdot (2j-1)!! \,. \tag{6.1}$$

This number is also the size of the set $\{\sigma \in S_k : \sigma = \sigma^{-1}\}$.

Proof. Symmetric permutation matrices correspond to those permutations π on the set $\{1, \dots, k\}$ that are products of pairwise disjoint transpositions. These are exactly those $\pi \in S_k$ that satisfy $\pi = \pi^{-1}$. Express a self-inverse permutation π as $\pi = (u_1\, v_1) \cdots (u_j\, v_j)$ where $j \in \mathbb{N}_0$ and $\{u_s, v_s\} \cap \{u_t, v_t\} = \varnothing$ for $s \ne t$. The order of these transpositions is irrelevant. For a given j, the first factor in (6.1) says how many ways the fixed points of π can be chosen. Let u_1 denote one of the $2j$ non-fixed points. We can choose v_1 in $2j-1$ ways. Denoting by u_2 one of the remaining $2j-2$ points, we can choose v_2 in $2j-3$ ways. Continuing the process, we obtain $(2j-1)!!$, the second factor in (6.1). $\square$

For $i \in \{1, \dots, m\}$, let e_i denote the m-dimensional column vector with 1 in the i-th entry and zeros elsewhere.

Lemma 6.2. (5.4) *holds.*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A' = \begin{pmatrix} \boxed{\mathbf{0}} & \boxed{\mathbf{0}} & \boxed{\mathbf{1}} & \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \\ 0 & 0 & \boxed{\mathbf{0}} & 0 & 0 \\ 0 & 1 & \boxed{\mathbf{0}} & 0 & 0 \\ 0 & 0 & \boxed{\mathbf{0}} & 0 & 1 \\ 0 & 0 & \boxed{\mathbf{0}} & 1 & 0 \end{pmatrix}, \quad A^{\#} = \begin{pmatrix} \boxed{\mathbf{0}} & \boxed{\mathbf{0}} & \boxed{\mathbf{0}} & \boxed{\mathbf{1}} & \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \\ \boxed{\mathbf{0}} & 0 & 0 & \boxed{\mathbf{0}} & 0 & 0 \\ \boxed{\mathbf{0}} & 0 & 1 & \boxed{\mathbf{0}} & 0 & 0 \\ \boxed{\mathbf{1}} & \boxed{\mathbf{0}} & \boxed{\mathbf{0}} & \boxed{\mathbf{0}} & \boxed{\mathbf{0}} & \boxed{\mathbf{0}} \\ \boxed{\mathbf{0}} & 0 & 0 & \boxed{\mathbf{0}} & 0 & 1 \\ \boxed{\mathbf{0}} & 0 & 0 & \boxed{\mathbf{0}} & 1 & 0 \end{pmatrix}$$

FIGURE 4. T-operation and B-operation

Proof. Since $\binom{m-1}{k} = \binom{m-2}{k} + \binom{m-2}{k-1}$ and $\frac{n!}{(n-k)!} - \frac{n!}{(n-k+1)!} = \frac{n!}{(n-k)!} \cdot \frac{n-k}{n-k+1}$, (5.4) is equivalent to

$$\mathrm{sm}^0(m,n,k) = \binom{m-2}{k} \cdot \frac{n!}{(n-k)!} + \binom{m-2}{k-1} \cdot \frac{n!}{(n-k)!} \cdot \frac{n-k}{n-k+1}\,. \tag{6.2}$$

Let $A \in \mathrm{SM}^0(m,n,k)$. It consists of k column vectors from $\{e_2,\dots,e_m\}$ and $n-k$ zero columns. If e_m is excluded, then no matter how we choose k vectors from $\{e_2,\dots,e_{m-1}\}$, we can do this $\binom{m-2}{k}$ ways, and no matter how we order these k distinct vectors and $n-k$ copies of the zero vector, $\frac{n!}{(n-k)!}$ ways, we obviously obtain a matrix in $\mathrm{SM}^0(m,n,k)$. These possibilities give the first summand in (6.2).

We are left with the more complex case when e_m occurs in A. In this case, we select only $k-1$ vectors from $\{e_2,\dots,e_{m-1}\}$, and the product of the first two factors of the second summand of (6.2) tells us how many ways we can select and arrange our vectors. However, not all of these arrangements yield a matrix satisfying ($5^\bullet$). The satisfaction of ($5^\bullet$) depends only on the ordering of e_m and the $n-k$ zero vectors. For a moment, fix the set of the positions of these $n-k+1$ vectors. On this set of positions, only one of the possible $n-k+1$ arrangements violates ($5^\bullet$); namely, where e_m comes first. Hence the ratio of good arrangements to all arrangements is just the third factor in the second summand of (6.2), as desired. □

Lemma 6.3. (5.7) *holds.*

Proof. Let $A \in \mathrm{SSM}^0(m,k)$. Its first row and, by symmetry, its first column contains no unit. Hence ($1^\bullet$) in itself guarantees that A is a slim matrix. The question is how many ways we can ensure ($1^\bullet$) together with symmetry. The first factor of (5.7) says how many ways we can choose (the indices of) the nonzero columns. By symmetry, the same set of indices is obtained if we consider the nonzero rows. Restricting the matrix to these (symmetrically positioned) k rows and k columns, we obtain a symmetric k-by-k permutation matrix B. The number of these B equals the sum in (5.7) by Lemma 6.1. □

Next, we define two matrix operations; see Figure 4. Given an m-by-n matrix A and $i \in \{2,\dots,n\}$, we define the *T-operation between* $i-1$ *and* i, abbreviated to the *T-operation at i*, as follows. First, we insert a new column with zero entries between the $(i-1)$-th and the i-th column. In the next step, we insert a new row right before the first row such the i-th entry of the new row is 1 and the rest of its entries are 0. For example, if A is the matrix given in Figure 4, then the T-operation at 3 yields A' in Figure 4. (The new elements are the boxed boldface ones.)

By a *dual T-operation* we mean the composite of a transposition, a T-operation, and a transposition again. Given an m-by-n matrix A and $j \in \{2, \dots, m\}$, we define the *B-operation between $j-1$ and j*, or in short the *B-operation at j*, as follows. First, we apply a T-operation at j. Then, in the next step, we apply a dual T-operation at $j+1$. For example, if A is the previous matrix, then the B-operation at 3 yields $A^\#$ in Figure 4. Note that A is a symmetric matrix if and only if $A^\#$ also is a symmetric matrix . Note also that the set of the new elements looks like a $\underline{b}$ird (flying to the northwest); this explains the terminology. Let us always assume automatically that

(6.3) $\qquad 2 \le i \le n$ for any T-operation, and $2 \le j \le m$ for any B-operation.

Definition 6.4. A 0, 1-matrix A is *quasi-slim* if it satisfies ($1^\bullet$), ($2^\bullet$), ($4^\bullet$) and ($5^\bullet$). For an m-by-n quasi-slim matrix A, let $\operatorname{def} A$, the *defect of A*, stand for the largest $j \in \mathbb{N}_0$ such that $\operatorname{Corn}_j A$ contains j units. Observe that $\operatorname{def} A = 0$ if and only if A is slim.

Lemma 6.5. *Let A be an m-by-n matrix. Assume that $i \in \{2, \dots, n\}$ and $j \in \{2, \dots, m\}$. Let A' and $A^\#$ denote the matrices we obtain from A by performing a T-operation at i and a B-operation at j, respectively. The following assertions hold.*

(i) *A' determines A and i. Similarly, $A^\#$ determines A and j.*
(ii) *A is quasi-slim if and only if A' is quasi-slim if and only if $A^\#$ is quasi-slim.*
(iii) *If A is quasi-slim, then A' is slim if and only if $2 + \operatorname{def} A \le i \le n$, and $A^\#$ is slim if and only if $2 + \operatorname{def} A \le j \le m$.*
(iv) *If A is slim, then both A' and $A^\#$ are slim.*

Proof. (i) and (ii) are evident. We prove (iii) only for B-operations; the argument for T-operations is almost the same and easier.

Assume first that $2 + \operatorname{def} A \le j \le m$. Let $s \in \{1, \dots, m\}$, let $D^\# = \operatorname{Corn}_s A^\#$, and denote D the system of those entries of $D^\#$ that belong to A. If $s \le j$, then $D^\#$ has less than s units since its first row is zero. Hence we may assume $s > j$. Now $D = \operatorname{Corn}_{s-2} A$ has less than $s-2$ units since $s - 2 > j - 2 \ge \operatorname{def} A$. Therefore $D^\#$ has less than s units, and $A^\#$ is slim.

Next, to show the converse, assume $j < 2 + \operatorname{def} A$. Let $s = 2 + \operatorname{def} A$. Since the previously defined $D = \operatorname{Corn}_{s-2} A = \operatorname{Corn}_{\operatorname{def} A} A$ has $s-2$ units, $D^\# = \operatorname{Corn}_s A^\#$ has s units. Hence $A^\#$ is not slim, proving (iii).

Finally, (iii) together with (6.3) imply (iv). $\square$

The next two proofs show the importance of the T- and B-operations.

Lemma 6.6. (5.5) *holds.*

Proof. Assume $A' \in \mathrm{SM}^1(m, n, k)$. By Lemma 6.5, there are a unique $(m-1)$-by-$(n-1)$ quasi-slim matrix A and a unique $i \in \{2, \dots, n-1\}$ such that A' is obtained from A by performing a T-operation at i. It suffices to count these A. Let

$j = \operatorname{def} A$. Since A contains exactly $k-1$ units, $\operatorname{def} A \in \{0, \dots, k-1\}$. Hence it is sufficient to show that the j-th summand in (5.5) is the number of those A whose defect equals j.

Let $B = \operatorname{Corn}_j A$. The bottom right $(m-1-j)$-by-$(n-1-j)$ corner of A will be denoted by D. Since $j = \operatorname{def} A$, B is a permutation matrix or the empty matrix. There are $j!$ possibilities to choose B. This yields $j!$ in (5.5). Evidently, D inherits ($1^\bullet$), ($2^\bullet$), ($4^\bullet$) and ($5^\bullet$) from A. Also, since $j = \operatorname{def} A$, ($3^\bullet$) holds for D. Hence $D \in \mathrm{SM}(m-1-j, n-1-j, k-1-j)$, and all members of this set can occur. This gives the next factor in (5.5). Finally, Lemma 6.5 says that i can be chosen from $\{j+2, \dots, n-1\}$. This explains the last factor in (5.5). □

Lemma 6.7. (5.8) *holds.*

Proof. Assume $A^\# \in \mathrm{SSM}^1(m,k)$. By Lemma 6.5, there are a unique $(m-2)$-by-$(m-2)$ quasi-slim symmetric matrix A and a unique $j \in \{2, \dots, m-2\}$ such that $A^\#$ is obtained from A by performing a B-operation at j. Let $i = \operatorname{def} A$. Since A contains exactly $k-2$ units, $i = \operatorname{def} A \in \{0, \dots, k-2\}$. Like in the previous proof, it is sufficient to show that the i-th summand in (5.8) is the number of those A whose defect is i.

Let $B = \operatorname{Corn}_i A = \operatorname{Corn}_{\operatorname{def} A} A$. It is a symmetric permutation matrix, so the second sum in (5.8) counts these B by Lemma 6.1. Let D denote the bottom right $(m-2-i)$-by-$(m-2-i)$ corner of A. From $i = \operatorname{def} A$ we conclude, like in the previous proof, that D is a slim matrix. Hence $D \in \mathrm{SSM}(m-2-i, k-2-i)$, and all members of this set can occur. So, the number of these D is the second factor in (5.8). The first factor equals $|\{i+2, \dots, m-2\}|$, which is the number of all possible j. □

Proof of Lemma 5.1. (5.6) and (5.9) are obvious. The rest are covered by Lemmas 6.2, 6.3, 6.6 and 6.7. □

Proof of Proposition 5.2. By Proposition 4.3(ii), $h = m + n - k$, that is, $m = h+k-n$. The question is how to choose k and n such that $0 \le k < m = h+k-n \le n$.

So, assume $m = h + k - n$ and $0 \le k < m = h + k - n \le n$. Clearly, $n < n+(m-k) = h$, that is, $n \le h-1$. This, together with $k < m \le n$, gives $k \le h-2$. From $h+k = m+n \le n+n = 2n$ we infer $n \ge (h+k)/2$, that is, $n \ge \lceil (h+k)/2 \rceil$. We have obtained

$$0 \le k \le h-2 \text{ and } \left\lceil \frac{h+k}{2} \right\rceil \le n \le h-1. \tag{6.4}$$

On the other hand, if (6.4) holds, then $k = n + k - n < h + k - n = m$ and $m = h + k - n = 2(h+k)/2 - n \le 2\lceil (h+k)/2 \rceil - n \le 2n - n = n$. Hence Proposition 5.2 follows from (6.4). □

Let $\dotplus$ denote the (non-commutative) operation of forming glued sums. Clearly,

Claim 6.8. *Each $L \in \mathrm{SSL}(h)$ uniquely decomposes as a glued sum $L_1 \dotplus \cdots \dotplus L_{t(L)}$ of maximal chain intervals and indecomposable slim semimodular lattices.*

The definition of this decomposition is explained by Figure 1.

Proof of Theorem 5.3. For each $L \in \mathrm{SSL}(h)$, we consider the unique decomposition from Claim 6.8. Let j denote the length of the "bottom summand" L_1, and let $L' = L_2 \dotplus \cdots \dotplus L_{t(L)}$. Note that $t(L)$, $\mathbf{I}(L_1)$, …, $\mathbf{I}(L_{t(L)})$, j, and $\mathbf{I}(L')$ depend only on $\mathbf{I}(L)$. Let $\mathrm{DC}(h)^{\cong} = \{\mathbf{I}(L) \in \mathrm{SSL}(h)^{\cong} : L_1 \text{ is a chain}\}$. (The notation comes from "*D*ecomposable and the bottom summand is a *C*hain".) Similarly, let $\mathrm{DI}(h)^{\cong} = \{\mathbf{I}(L) \in \mathrm{SSL}(h)^{\cong} : L_1$ is an indecomposable slim semimodular lattice$\}$. We denote $|\mathrm{DC}(h)^{\cong}|$ and $|\mathrm{DI}(h)^{\cong}|$ by $N_{\mathrm{dc}}(h)$ and $N_{\mathrm{di}}(h)$, respectively. Since $N(h) = |\mathrm{SSL}(h)^{\cong}|$ and $\mathrm{DC}(h)^{\cong} \cap \mathrm{DI}(h)^{\cong} = \varnothing$, we obtain $N(h) = N_{\mathrm{dc}}(h) + N_{\mathrm{di}}(h)$, for all $h \in \mathbb{N}_0$. We claim that, for all $h \in \mathbb{N}_0$,

$$N_{\mathrm{dc}}(h) = 1 + \sum_{j=1}^{h-2} N_{\mathrm{di}}(h-j) \ \text{ and } \ N_{\mathrm{di}}(h) = \sum_{j=2}^{h} N_{\mathrm{issl}}(j) \cdot N(h-j). \tag{6.5}$$

Assume first that $\mathbf{I}(L) \in \mathrm{DC}(h)^{\cong}$. The simplest possibility is $L_1 = L$; this gives the summand 1 in (6.5). Assume $L \neq L_1$. Now $t(L) > 1$ and L_2 cannot be a chain. Hence $j \in \{1, \dots, h-2\}$, and $\mathbf{I}(L') \in \mathrm{DI}(h-j)^{\cong}$. Since $\mathbf{I}(L') \in \mathrm{DI}(h-j)^{\cong}$ is arbitrary, we obtain the equation for $N_{\mathrm{dc}}(h)$ in (6.5).

Next, if $\mathbf{I}(L) \in \mathrm{DI}(h)^{\cong}$, then $j \in \{2, \dots, h\}$. For each of these j, $\mathbf{I}(L_1)$ can be chosen in $N_{\mathrm{issl}}(j)$ ways. If $j < h$, then $\mathbf{I}(L') \in \mathrm{SSL}(h-j)^{\cong}$ can be chosen in $N(h-j)$ ways. If $j = h$, then $N(0) = 1$ causes no trouble. Hence we conclude the $N_{\mathrm{di}}(h)$-part of (6.5). Thus, (6.5) holds.

Since $N(0) = N(1) = 1$, we may assume $2 \leq h$. Then, by (6.5),

$$\begin{aligned} N(h) - N_{\mathrm{di}}(h) = N_{\mathrm{dc}}(h) &= 1 + \sum_{j=1}^{h-2} N_{\mathrm{di}}(h-j) \\ &= N_{\mathrm{di}}(h-1) + 1 + \sum_{j=2}^{h-2} N_{\mathrm{di}}(h-j) \\ &= N_{\mathrm{di}}(h-1) + 1 + \sum_{i=1}^{h-1-2} N_{\mathrm{di}}(h-1-i) \\ &= N_{\mathrm{di}}(h-1) + N_{\mathrm{dc}}(h-1) = N(h-1). \end{aligned}$$

This yields

$$N(h) = N(h-1) + N_{\mathrm{di}}(h) = N(h-1) + \sum_{j=2}^{h} N_{\mathrm{issl}}(j) \cdot N(h-j). \qquad \square$$

7. The asymptotic value of $N(h)$

The aim of this section is to prove the following asymptotic statement.

Proposition 7.1. $\displaystyle\lim_{h\to\infty} \frac{N(h)}{h!} = \frac{1}{2}$.

Analogously to Proposition 3.1, Czédli and Schmidt [11] described the set $\mathrm{SSL}(h)^{\cong}$ by permutations as detailed below. Let $\sigma \in S_h$, and let $I = [u, v] = \{u, \dots, v\}$ be a (non-empty) interval of the chain $\{1 < \cdots < h\}$. If the sets $\{1, \dots, u-1\}$, I, and $\{v+1, \dots, h\}$ are closed with respect to σ, then I is a *section* of σ. (Here I is non-empty but any of $\{1, \dots, u-1\}$ and $\{v+1, \dots, h\}$ may be empty.) Sections of σ that are minimal with respect to set inclusion are *σ-segments*. Let $\mathrm{Seg}(\sigma)$ denote the set of all σ-segments. A σ-segment I is said to be *large* if $|I| \geq 3$; otherwise I

is *small*. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 7 & 4 & 5 & 3 & 6 & 2 & 9 & 8 \end{pmatrix} = (27)(345)(89), \tag{7.1}$$

then $\mathrm{Seg}(\sigma) = \big\{\{1\}, \{2,3,4,5,6,7\}, \{8,9\}\big\}$. Here $[2,7] = \{2,\dots,7\}$ is a large σ-segment, and the other two σ-segments are small. The restriction of σ to a subset I of $\{1,\dots,h\}$ will be denoted by $\sigma\rceil_I$. Let $\sigma, \mu \in S_h$. We say that σ and μ are *sectionally inverse or equal*[1] if $\mathrm{Seg}(\sigma) = \mathrm{Seg}(\mu)$ and, for all $I \in \mathrm{Seg}(\sigma)$, $\mu\rceil_I \in \{\sigma\rceil_I, (\sigma\rceil_I)^{-1}\}$. The corresponding relation is denoted by $\boldsymbol{\varrho}^i_e$; that is, $(\sigma,\mu) \in \boldsymbol{\varrho}^i_e$ means that σ and μ are sectionally inverse or equal. We recall the following statement from Czédli and Schmidt [11] without proof; parts (i) and (ii) are quite easy.

Lemma 7.2 ([11])**.** *For $h \in \mathbb{N}$, and let $\sigma \in S_h$, the following hold.*

(i) $\mathrm{Seg}(\sigma)$ *is a partition on the set* $\{1,\dots,h\}$*.*
(ii) $\boldsymbol{\varrho}^i_e$ *is an equivalence relation on* S_h*.*
(iii) *There is a bijection between* $\mathrm{SSL}(h)^{\cong}$ *and the quotient set* $S_h/\boldsymbol{\varrho}^i_e$*.*

Proof of Proposition 7.1. From Proposition 2.3(iii) and Lemma 7.2(iii), we obtain $N(h) = |S_h/\boldsymbol{\varrho}^i_e|$. For $\sigma \in S_h$, the $\boldsymbol{\varrho}^i_e$-block (in other words, the $\boldsymbol{\varrho}^i_e$-class) of σ will be denoted by $\sigma/\boldsymbol{\varrho}^i_e$. Suppose $\mu \in \sigma/\boldsymbol{\varrho}^i_e$. For each I in $\mathrm{Seg}(\sigma)$, there are two possibilities: either $\sigma\rceil_I \neq (\sigma\rceil_I)^{-1}$ and there are two ways to choose $\mu\rceil_I$, or $\sigma\rceil_I = (\sigma\rceil_I)^{-1}$ and $\mu\rceil_I$ is uniquely determined. (The former possibility implies that I is a large σ-segment.) Hence $|\sigma/\boldsymbol{\varrho}^i_e|$ is a power of 2. For $k \in \mathbb{N}_0$, let

$$A_k(h) = \{\sigma \in S_h : |\sigma/\boldsymbol{\varrho}^i_e| = 2^k\}.$$

Clearly, $A_0(h) = \{\sigma \in S_h : \sigma = \sigma^{-1}\}$. Hence Lemma 6.1 yields

$$\begin{aligned}\frac{|A_0(h)|}{h!} &= \frac{1}{h!}\sum_{j=0}^{\lfloor h/2\rfloor}\binom{h}{h-2j}(2j-1)!! = \frac{1}{h!}\sum_{j=0}^{\lfloor h/2\rfloor}\frac{h!}{(h-2j)!\cdot(2j)!}\cdot\frac{(2j)!}{2^j\cdot j!}\\ &= \sum_{j=0}^{\lfloor h/4\rfloor}\frac{1}{(h-2j)!\cdot 2^j\cdot j!} + \sum_{j=\lfloor h/4\rfloor+1}^{\lfloor h/2\rfloor}\frac{1}{(h-2j)!\cdot 2^j\cdot j!} = \sum{}' + \sum{}''.\end{aligned}$$

In $\sum'$, each denominator is at least $(h - 2\lfloor h/4\rfloor)! \geq \lfloor h/2\rfloor!$, and there are fewer than h summands. Hence $\sum' \leq h\cdot(\lfloor h/2\rfloor!)^{-1} \to 0$. In $\sum''$, each denominator is at least $2^{h/4}$, and there are fewer than h summands, so $\sum'' \leq h\cdot 2^{-h/4} \to 0$. Thus,

$$\lim_{h\to\infty}\frac{|A_0(h)|}{h!} = 0. \tag{7.2}$$

Next, we denote $A_2(h)\cup A_3(h)\cup A_4(h)\cup\cdots$ by $B(h)$, and we assume $\sigma \in B(h)$. Now there are at least two large σ-segments. We define the *pivot element* $p(\sigma)$ of σ as the greatest element of the leftmost large σ-segment. We have $3 \leq p(\sigma) \leq h-3$ since there are at least two large σ-segments. Both the intervals $[1, p(\sigma)] = \{1,\dots,p(\sigma)\}$ and $[p(\sigma)+1, h]$ are unions of σ-segments, whence both are closed with respect to σ. Hence if we denote the restrictions of σ to these intervals by $\lambda = \sigma\rceil_{[1,p(\sigma)]}$ and $\varrho = \sigma\rceil_{[p(\sigma)+1,h]}$, then σ is determined by λ and ϱ. Since $\lambda \in S_{p(\sigma)}$, there are at most $p(\sigma)!$ possible λ. Similarly, there are at most $\big(h - p(\sigma)\big)!$ many

[1]In the definition of this concept, we use segments rather than sections.

ϱ. Using the unimodality of the binomial coefficients $\binom{h}{i}$ as a function of i and counting the permutations according to their pivot elements,

$$\begin{aligned}\frac{|B(h)|}{h!} &\leq \frac{1}{h!}\sum_{k=3}^{h-3} k!\cdot(h-k)! = \sum_{k=3}^{h-3}\frac{k!\cdot(h-k)!}{h!} = \sum_{k=3}^{h-3}\binom{h}{k}^{-1} \\ &\leq^{*} \sum_{k=3}^{h-3}\binom{h}{3}^{-1} \leq h\cdot\frac{6}{h(h-1)(h-2)} \to 0.\end{aligned} \tag{7.3}$$

For $h \geq 6$, $B(h) \neq \varnothing$ and $\{A_0(h), A_1(h), B(h)\}$ is a partition on S_h. If σ and μ are taken from different blocks of this partition, then $(\sigma,\mu) \notin \boldsymbol{\varrho}_e^i$. Hence, using Lemma 7.2(iii) at $=^*$, and (7.2) and (7.3) later, we conclude

$$\begin{aligned}&\Bigl|\frac{N(h)}{h!} - \frac{|\{\sigma/\boldsymbol{\varrho}_e^i : \sigma\in A_1(h)\}|}{h!}\Bigr| \\ &\quad =^{*} \Bigl|\frac{|\{\sigma/\boldsymbol{\varrho}_e^i : \sigma\in S_h\}|}{h!} - \frac{|\{\sigma/\boldsymbol{\varrho}_e^i : \sigma\in A_1(h)\}|}{h!}\Bigr| \\ &\quad = \frac{|\{\sigma/\boldsymbol{\varrho}_e^i : \sigma\in A_0(h)\cup B(h)\}|}{h!} \\ &\quad \leq \frac{|A_0(h)\cup B(h)|}{h!} = \frac{|A_0(h)|}{h!} + \frac{|B(h)|}{h!} \to 0,\end{aligned} \tag{7.4}$$

Utilizing first the definition of $A_1(h)$ and then (7.2) and (7.3), we derive

$$\begin{aligned}&\frac{|\{\sigma/\boldsymbol{\varrho}_e^i : \sigma\in A_1(h)\}|}{h!} - \frac{1}{2} = \frac{|A_1(h)|}{2\cdot h!} - \frac{1}{2} = \frac{|S_h\setminus(A_0(h)\cup B(h))|}{2\cdot h!} - \frac{1}{2} \\ &\quad = \frac{|S_h|}{2\cdot h!} - \frac{|A_0(h)|}{2\cdot h!} - \frac{|B(h)|}{2\cdot h!} - \frac{1}{2} = -\frac{|A_0(h)|}{2\cdot h!} - \frac{|B(h)|}{2\cdot h!} \to 0.\end{aligned} \tag{7.5}$$

Finally, (7.4) and (7.5) complete the proof. ☐

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