

ON THE SET OF PRINCIPAL CONGRUENCES IN A DISTRIBUTIVE CONGRUENCE LATTICE OF AN ALGEBRA

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ABSTRACT. Let Q be a subset of a finite distributive lattice D . An algebra A represents the inclusion $Q \subseteq D$ by principal congruences if the congruence lattice of A is isomorphic to D and the ordered set of principal congruences of A corresponds to Q under this isomorphism. If there is such an algebra for every subset Q containing 0, 1, and all join-irreducible elements of D , then D is said to be *fully (A1)-representable*. We prove that every fully (A1)-representable finite distributive lattice is planar and it has at most one join-reducible coatom. Conversely, we prove that every finite planar distributive lattice with at most one join-reducible coatom is *fully chain-representable* in the sense of a recent paper of G. Grätzer. Combining the results of this paper with another result of the present author, it follows that every fully (A1)-representable finite distributive lattice is “fully representable” even by principal congruences of *finite lattices*. Finally, we prove that every *chain-representable* inclusion $Q \subseteq D$ can be represented by the principal congruences of a finite (and quite small) algebra.

1. INTRODUCTION AND RESULTS

Grätzer [11, Probl. 12] and [12, Probl. 22.1] raised the problem of characterizing lattices and their subsets that can be represented simultaneously as congruence lattices and the sets of principal congruences, respectively, of algebras or lattices. The first steps in this direction were made by Grätzer [14] and Grätzer and Lakser [18]; here we continue their investigations. For a finite lattice L , $J(L)$ denotes the ordered set of nonzero join-irreducible elements of L , $J_0(L)$ stands for $J(L) \cup \{0\}$, and we let $J^+(L) = J(L) \cup \{0, 1\}$. For an algebra A , let $\text{Con}(A)$ be the *congruence lattice* of A , while $\text{Princ}(A)$ will stand for the ordered set of *principal congruences* of A . The algebra A can be infinite but we always assume that $\text{Con}(A)$ is finite. Then every congruence of A is the join of finitely many principal congruences, whereby

$$(1.1) \quad J_0(\text{Con}(A)) \subseteq \text{Princ}(A) \subseteq \text{Con}(A).$$

For a subset Q of a finite lattice D , an algebra A represents the inclusion $Q \subseteq D$ by principal congruences if there exists an isomorphism $\varphi: \text{Con}(A) \rightarrow D$ such that $Q = \varphi(\text{Princ}(A))$. In this case, we also say that the inclusion $Q \subseteq D$ is *represented by the principal congruences of A* . Mostly, we consider only the case where D is distributive. Our first aim is to prove the following statement; condition (1.2) in it is motivated by (1.1). Note that this section does not contain proofs; they are given in the subsequent sections.

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Proposition 1.1. *Let D be a finite distributive lattice. Then the following two conditions are equivalent.*

(a) *For every set Q such that*

$$(1.2) \quad J_0(D) \subseteq Q \subseteq D,$$

the inclusion $Q \subseteq D$ is represented by the principal congruences of some algebra A . (This condition will be called the full (A)-representability of D .)

(b) *The lattice D is a planar and $1_D \in J_0(D)$, that is, $|D| = 1$ or D has exactly one coatom.*

In connection with this statement, see also Corollary 1.7 later.

A finite lattice D will be called *fully (A)-representable* if it satisfies condition (a) from Proposition 1.1. The notation (A) comes from “*algebra*”.

Corollary 1.2. *Let $V = \{0, \alpha, \beta\}$ be the “V-shaped” three-element ordered set with smallest element 0 and maximal elements α and β . Then $\text{Princ}(A) \cong V$ holds for no algebra A .*

It has previously been known that V cannot be represented as $\text{Princ}(L)$ of a lattice L , since we know from Grätzer [11], see also (1.1) in Czédli [2], that $\text{Princ}(L)$ is always a directed ordered set but V is not. Corollary 1.2 indicates why it would be difficult to extend the results of Czédli [1], [2], [3], [4], and [6] and Grätzer [11], [15], and [16] from the representability of ordered sets by principal *lattice* congruences to that by arbitrary principal congruences.

Using (1.1) and that $\text{Princ}(L)$ is a directed ordered set, if $\text{Con}(L)$ is finite, then $J_0(\text{Con}(L))$ has an upper bound in $\text{Princ}(L)$. This upper bound is necessarily the top element of $\text{Con}(L)$. Hence, for every lattice L such that $\text{Con}(L)$ is finite,

$$(1.3) \quad J^+(\text{Con}(L)) \subseteq \text{Princ}(L) \subseteq \text{Con}(L).$$

Our main goal is to prove the following theorem; (1.4) is motivated by (1.3).

Theorem 1.3. *Let D be a finite distributive lattice, and consider the following three conditions on D .*

(i) *For every Q satisfying*

$$(1.4) \quad J^+(D) \subseteq Q \subseteq D,$$

the inclusion $Q \subseteq D$ is represented by the principal congruences of some algebra A ; if this condition holds then D is said to be fully (A1)-representable.

(ii) *For every Q satisfying (1.4), the inclusion $Q \subseteq D$ is represented by the principal congruences of some finite lattice L ; if this condition holds then D is said to be fully (fL)-representable.*

(iii) *D is planar and it has at most one join-reducible coatom.*

Then (i) implies (iii) and the trivial implication (ii) $\Rightarrow$ (i) also holds.

The notation (A1) in Theorem 1.3 comes from *algebra* and $1 \in J^+(D) \subseteq Q$, while (fL) comes from *finite lattice*. The concept of full (fL)-representability and, more generally, the representability of just one subset Q of D by principal congruences of a finite lattice L are taken from Grätzer [14] and Grätzer and Lakser [18].

Remark 1.4. Czédli [7] gave¹ a long proof for the implication (iii) $\Rightarrow$ (ii). Hence, for a finite distributive lattice D , (i), (ii), and (iii) in Theorem 1.3 are equivalent conditions. In particular, (i) $\Rightarrow$ (ii), which seems to be a surprise.

Two-element intervals of a finite lattice are called *prime intervals* or *edges*. We need the following concept, introduced in Grätzer [14].

Definition 1.5. Let D be a finite distributive lattice.

- (i) By a $J(D)$ -colored chain we mean a triplet $\langle C, \text{col}, D \rangle$ such that C is a finite chain, D is a finite distributive lattice, and $\text{col}: \text{Prime}(C) \rightarrow J(D)$ is a surjective map from the set $\text{Prime}(C)$ of all prime intervals of C onto $J(D)$. If $\mathbf{p} \in \text{Prime}(C)$, then $\text{col}(\mathbf{p})$ is the *color* of the edge $\mathbf{p}$.
- (ii) Given a $J(D)$ -colored chain $\langle C, \text{col}, D \rangle$, we define a map denoted by erep from the set $\text{Intv}(C)$ of all intervals of C onto D as follows: for $I \in \text{Intv}(C)$, let

$$(1.5) \quad \text{erep}(I) := \bigvee_{\mathbf{p} \in \text{Prime}(I)} \text{col}(\mathbf{p});$$

the join is taken in D and $\text{erep}(I)$ is called the *element represented* by I .

- (iii) The set

$$\text{SRep}(C, \text{col}, D) := \{\text{erep}(I) : I \in \text{Intv}(C)\}$$

will be called the *set represented* by the $J(D)$ -colored chain $\langle C, \text{col}, D \rangle$. Clearly, by the surjectivity of col , $Q := \text{SRep}(C, \text{col}, D)$ satisfies (1.4) in this case.

- (iv) For a subset Q of D , the inclusion $Q \subseteq D$ is *chain-representable* if there exists a $J(D)$ -colored chain $\langle C, \text{col}, D \rangle$ such that $Q = \text{SRep}(C, \text{col}, D)$. Note that C need not be a subchain of D .
- (v) Finally, a finite distributive lattice D is *fully chain-representable* if for every Q satisfying (1.4), the inclusion $Q \subseteq D$ is chain-representable.

Armed with this definition, we formulate the following statement.

Proposition 1.6. *Let D be a finite distributive lattice. Then D is fully chain-representable if and only if it is planar and it has at most one join-reducible coatom.*

Although Proposition 1.6 is now a consequence of the conjunction of Czédli [7] and Grätzer [14], both [7] and [14] contain long proofs. In the present paper, we give a direct and short proof of Proposition 1.6.

The following corollary will easily be concluded from the previous statements.

Corollary 1.7. *If a finite distributive lattice is fully (A)-representable, then it is fully (fL)-representable.*

Next, we collect some known facts; most of them will be needed in our proofs.

Theorem 1.8 (Summarizing known facts). *Let D be finite distributive lattice and assume that $Q \subseteq D$ satisfies (1.4). Then the following five statements hold.*

- (i) (Grätzer and Lakser [18]²) *If D is fully (fL)-representable, then it is planar.*
- (ii) (G. Grätzer, personal communication) *If D is fully chain-representable, then it is planar.*

¹After <https://arxiv.org/abs/1705.10833v1>, the first version of the present paper. While [7] is mainly for some specialists of lattice theory, the present paper is written for a wider readership.

²Since I am mentioned in the addendum of [18], let me note that only some optimization of G. Grätzer and H. Lakser's original proof of this fact is due to me.

- (iii) (Grätzer [14]) *If the inclusion $Q \subseteq D$ is principal congruence representable by a finite lattice, then it is chain-representable.*
- (iv) (Grätzer [14]) *If D is fully (fL)-representable, then it is fully chain-representable.*
- (v) (Grätzer [14]) *If the inclusion $Q \subseteq D$ is chain-representable and $1_D \in J(D)$, then this inclusion is principal congruence representable by a finite lattice.*

Note that (iv) is a particular case of (iii), (v) is very deep, and the proof of (ii) is similar to that of (i). Note also that there are several results on the representability of an ordered set Q as $\text{Princ}(L)$ (without taking care of D), see Czédli [1], [2], [3], [4], and [6] and Grätzer [11], [13], [15], and [16]. For more about full principal congruence representability, see Grätzer [14] and Grätzer and Lakser [18].

While Theorem 1.3 and Remark 1.4 give a satisfactory description of *full* representability, we know much less on the representability of a *single* inclusion $Q \subseteq D$. Theorem 1.8(iii) and (v), taken from Grätzer [14], reduces the problem to chain-representability, provided that $1_D \in Q$. If $1_D \in Q$ is not assumed then we can prove only the following statement.

Proposition 1.9. *If Q is a subset of a finite distributive lattice D such that the inclusion $Q \subseteq D$ is chain-representable, then this inclusion is principal congruence representable by a finite algebra A . Furthermore, if $Q \subseteq D$ is represented by a $J(D)$ -colored chain $\langle C, \text{col}, D \rangle$, then A can be chosen so that $|A| = |C|$.*

Note that, in the $1_D \in Q$ case, the finite lattice constructed in Grätzer [14] to represent $Q \subseteq D$ has much more elements than $|C|$.

Outline of the rest of the paper. We recall or prove some technical and mostly folkloric statements on planar distributive lattices in Section 2. Section 3 contains the proof of Proposition 1.6. In Section 4, we deal with congruences of arbitrary algebras and, with the exception of Proposition 1.9, prove the rest of our statements formulated in the present section. Finally, Section 5 is devoted to the proof of Proposition 1.9.

2. PROPERTIES OF PLANAR DISTRIBUTIVE LATTICES

In this section, we recall some known facts about planar distributive lattices. For concepts not defined here, see Czédli and Grätzer [8] and see also the monographs Grätzer [10] and [12]. As far as *distributive* lattices are considered, many of the facts below belong to the folklore. Interestingly, a lot of them are valid *not only for distributive* lattices. In the whole section, unless otherwise stated, D denotes a *planar distributive lattice*. Note that a planar lattice is *finite* by definition. It belongs to the folklore, see also Grätzer and Knapp [17], that

- (2.1) each element of D has at most two covers and at most two lower covers.

As usual, we fix a *planar* diagram of D ; adjectives like “left” and “right” are understood modulo this diagram. Assume that

- (2.2) D has two distinct join-reducible coatoms, c_ℓ and c_r , and let $e = c_\ell \wedge c_r$;

the notation is chosen so that c_ℓ is to the left of c_r ; see the first diagram in Figure 1. It follows from (2.1) that c_ℓ and c_r belong to the *left boundary chain* $\text{Bnd}_{\text{left}}(D)$ and to the *right boundary chain* $\text{Bnd}_{\text{right}}(D)$, respectively. Since c_ℓ is incomparable with c_r , these elements are not the least elements of the corresponding boundary

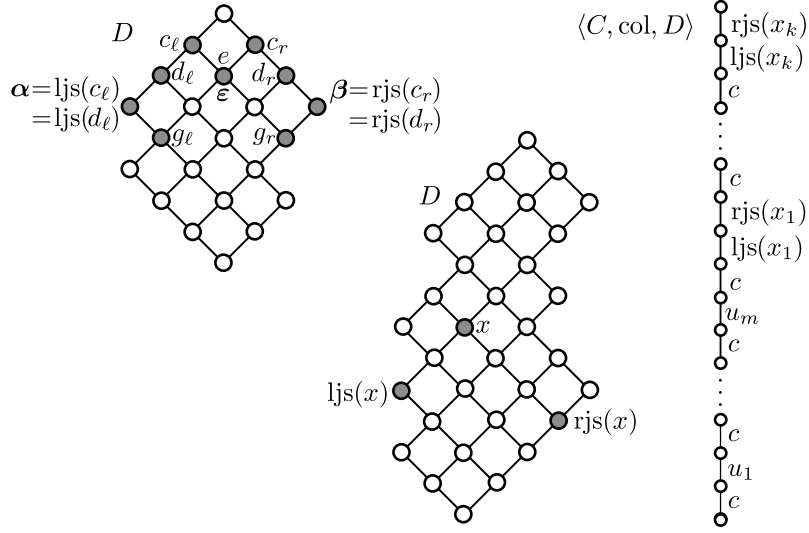


FIGURE 1. Examples

chains. Hence, there are a unique $d_\ell \in \text{Bnd}_{\text{left}}(D)$ and a unique $d_r \in \text{Bnd}_{\text{right}}(D)$ such that $d_\ell \prec c_\ell$ and $d_r \prec c_r$. As usual, for $x \in D$, the ideal $\{y : y \leq x\}$ will be denoted by $\downarrow x$. The following observation belongs to the folklore; note that it holds in every slim semimodular³ lattice, not only in our D .

(2.3) For $x \in D$, the largest element of $\text{Bnd}_{\text{left}}(D) \cap J_0(D) \cap \downarrow x$ is the *left join support* of x ; it is denoted by $\text{ljs}(x)$. The *right join support* $\text{rjs}(x)$ of x is the largest element of $\text{Bnd}_{\text{right}}(D) \cap J_0(D) \cap \downarrow x$. Both $\text{ljs}(x)$ and $\text{rjs}(x)$ are join-irreducible elements (by definition) and we have that $x = \text{ljs}(x) \vee \text{rjs}(x)$;

see the second part of Figure 1. For the slim semimodular case, the equality in (2.3) follows from the inclusion

$$(2.4) \quad J(D) \subseteq \text{Bnd}_{\text{left}}(D) \cup \text{Bnd}_{\text{right}}(D);$$

see Czédli and Schmidt [9, Lemma 6] or Czédli and Grätzer [8]. The following lemma is illustrated by the first part of Figure 1.

Lemma 2.1. *Let D be a finite distributive lattice satisfying (2.2). Then, with the notations from (2.2) to (2.3), $1 = \text{ljs}(d_\ell) \vee \text{rjs}(d_r)$.*

Proof. The following auxiliary statement is a transcript of Czédli [5, Lemma 4.4].

(2.5) For $x, y \in D$, x is strictly to the left of y
iff $\text{ljs}(x) > \text{ljs}(y)$ and $\text{rjs}(x) < \text{rjs}(y)$.

By distributivity (in fact, by lower semimodularity),

$$(2.6) \quad e \prec c_\ell \text{ and } e \prec c_r.$$

We claim that $d_\ell \neq e$. Suppose to the contrary that $d_\ell = e$. Then, since $c_\ell \notin J(D)$ and $e = d_\ell$ belongs to $\text{Bnd}_{\text{left}}(D)$, we obtain that c_ℓ has a lower cover u strictly

³This concept, introduced in Grätzer and Knapp [17] and surveyed in Czédli and Grätzer [8], is not needed here; it suffices to know that every planar distributive lattice is slim and semimodular.

to the right of e . By Kelly and Rival [19, Proposition 1.6], e is strictly on the left and c_r is strictly on the right of a maximal chain through $\{u, c_\ell\}$. This contradicts Kelly and Rival [19, Lemma 1.2] since $e \prec c_r$. Hence, $d_\ell \neq e$ and e is strictly to the right of d_ℓ . Thus, since $c_\ell \in \text{Bnd}_{\text{left}}(D)$ has a lower cover belonging to $\text{Bnd}_{\text{left}}(D)$ and it has at most two lower covers by (2.1), it follows that

$$(2.7) \quad \begin{array}{l} e \text{ is strictly to the right of } d_\ell \text{ and } d_\ell \in \text{Bnd}_{\text{left}}(D). \text{ Similarly,} \\ e \text{ is strictly to the left of } d_r \text{ and } d_r \in \text{Bnd}_{\text{right}}(D); \end{array}$$

see the first part of Figure 1. Next, observe that $\text{ljs}(d_\ell) > \text{ljs}(e)$ and $\text{rjs}(d_r) > \text{rjs}(e)$ by (2.5). So, by the definition of left and right join supports,

$$(2.8) \quad \text{ljs}(d_\ell) \not\leq e \quad \text{and} \quad \text{rjs}(d_r) \not\leq e.$$

Hence $e < \text{ljs}(d_\ell) \vee e \leq d_\ell \vee e \leq c_\ell$, whereby (2.6) gives that $\text{ljs}(d_\ell) \vee e = c_\ell$. Similarly, $\text{rjs}(d_r) \vee e = c_r$. Using the equality from (2.3) and the facts mentioned in the present paragraph, we obtain that

$$\begin{aligned} \text{ljs}(d_\ell) \vee \text{rjs}(d_r) &= \text{ljs}(d_\ell) \vee \text{ljs}(e) \vee \text{rjs}(e) \vee \text{rjs}(d_r) \\ &\stackrel{(2.3)}{=} \text{ljs}(d_\ell) \vee e \vee \text{rjs}(d_r) = (\text{ljs}(d_\ell) \vee e) \vee (\text{rjs}(d_r) \vee e) = c_\ell \vee c_r = 1. \end{aligned}$$

This completes the proof of Lemma 2.1. $\square$

For later reference, note the following well-known consequence of distributivity:

$$(2.9) \quad (u \in J(D) \text{ and } u \leq v_0 \vee \dots \vee v_{m-1}) \implies (\exists i < m)(u \leq v_i).$$

Next, we state and prove a technical lemma. For elements x and y of a lattice, $x \parallel y$ will stand for the conjunction of $x \not\leq y$ and $y \not\leq x$.

Lemma 2.2. *Let D be a finite distributive lattice satisfying (2.2). Then the following four assertions hold.*

- (i) $J(D) \setminus \downarrow e = \{\text{ljs}(c_\ell), \text{rjs}(c_r)\}$ and $\text{ljs}(c_\ell) \neq \text{rjs}(c_r)$.
- (ii) $\{\text{ljs}(c_\ell), \text{rjs}(c_r)\}$ is the set of maximal elements of $J(D)$ and $\text{ljs}(c_\ell) \parallel \text{rjs}(c_r)$.
- (iii) $\text{ljs}(c_\ell) \not\leq e \vee \text{rjs}(c_r)$ and $\text{rjs}(c_r) \not\leq e \vee \text{ljs}(c_\ell)$.
- (iv) $e \not\leq \text{ljs}(c_\ell)$ and $e \not\leq \text{rjs}(c_r)$.

Proof. We use the notation from (2.2) to (2.3). By (2.2), $c_\ell \notin J_0(D)$. Hence, $d_\ell \prec c_\ell$ yields that $\text{Bnd}_{\text{left}}(D) \cap J_0(D) \cap \downarrow c_\ell = \text{Bnd}_{\text{left}}(D) \cap J_0(D) \cap \downarrow d_\ell$. Thus,

$$(2.10) \quad \text{ljs}(c_\ell) = \text{ljs}(d_\ell). \text{ We obtain similarly that } \text{rjs}(c_r) = \text{rjs}(d_r).$$

Lemma 2.1 and (2.10) gives that $1 = \text{ljs}(c_\ell) \vee \text{rjs}(c_r)$, whereby $\text{ljs}(c_\ell) \parallel \text{rjs}(c_r)$. So, since $J(D) \subseteq \downarrow 1 = \downarrow(\text{ljs}(c_\ell) \vee \text{rjs}(c_r))$, (2.9) implies 2.2(ii). If we had that $e \leq \text{ljs}(c_\ell)$, then (2.10) and $\text{ljs}(d_\ell) \leq d_\ell$ would lead to $e \leq d_\ell$, contradicting (2.7). Hence, $e \not\leq \text{ljs}(c_\ell)$ and, similarly, $e \not\leq \text{rjs}(c_r)$. This proves 2.2(iv). For the sake of contradiction, suppose that 2.2(iii) fails. Let, say, $\text{ljs}(c_\ell) \leq e \vee \text{rjs}(c_r)$. Thus, since $\text{ljs}(c_\ell) = \text{ljs}(d_\ell) \not\leq e$ by (2.8) and (2.10), we obtain by (2.9) and (2.10) that $\text{ljs}(d_\ell) = \text{ljs}(c_\ell) \leq \text{rjs}(c_r) = \text{rjs}(d_r)$, contradicting Lemma 2.1. Hence, 2.2(iii) holds. Finally, 2.2(iii) and 2.2(iv) give that $e \parallel \text{ljs}(c_\ell)$, whereby $e < e \vee \text{ljs}(c_\ell) \leq c_\ell$. So (2.6) gives that $e \prec c_\ell = e \vee \text{ljs}(c_\ell)$. By distributivity (in fact, by lower semimodularity), $g_\ell := e \wedge \text{ljs}(c_\ell) \prec \text{ljs}(c_\ell)$. Similarly, $g_r := e \wedge \text{rjs}(c_r) \prec \text{rjs}(c_r)$. Combining these covering relations with 2.2(ii), we obtain that

$$(2.11) \quad \begin{array}{l} \text{With the exception of } \text{ljs}(c_\ell), \text{ all join-irreducible elements of} \\ \text{Bnd}_{\text{left}}(D) \text{ are in } \downarrow e. \text{ Similarly for } \text{rjs}(c_r) \text{ and } \text{Bnd}_{\text{right}}(D). \end{array}$$

Clearly, 2.2(i) follows from (2.4) and (2.11). $\square$

3. CHAIN-REPRESENTABILITY

The first paragraph in the following proof is based on G. Grätzer's idea; see Theorem 1.8(ii).

Proof of Proposition 1.6. In order to prove the necessity of the condition formulated in the proposition, assume that D is fully chain-representable. For the sake of contradiction, suppose that D is not planar. It is well known from, say, Lemma 3-4.1 and the paragraph preceding it in Czédli and Grätzer [8] that

(3.1) every non-planar finite distributive lattice contains a three-element antichain that consists of join-irreducible elements.

Thus, we can pick a three-element antichain $\{p_0, p_1, p_2\} \subseteq J(D)$. Let $p = p_0 \vee p_1 \vee p_2$ and $Q = J^+(D) \cup \{p\}$. Since D is fully chain-representable, we can take a $J(D)$ -colored chain $\langle C, \text{col}, D \rangle$ such that $Q = \text{SRep}(C, \text{col}, D)$. Take a maximal interval

$$[x, y] = \{x = x_0 \prec x_1 \prec \cdots \prec x_n = y\}$$

in C such that $\text{erep}([x, y]) = p$. Letting $r_i := \text{col}([x_i, x_{i+1}]) \in J(D)$, we have that $\bigvee_{i < n} r_i = p = p_0 \vee p_1 \vee p_2$. By (2.9), each of p_0, p_1 , and p_2 is less than or equal to some of the $r_i, i < n$. By the same reason, each of the $r_i, i < n$, is less than or equal to some of p_0, p_1 , and p_2 . Therefore, since $\{p_0, p_1, p_2\}$ is an antichain, each of p_0, p_1 , and p_2 occurs among the $r_i, i < n$. Without loss of generality, we can assume that p_0 or p_2 occurs before p_1 . Take a maximal subinterval $[x_i, x_j]$ of $[x, y]$ such that $\text{erep}([x_i, x_j]) = p_1$; we have that $0 < i$ since p_0 or p_2 occurs before p_1 . Then $r_{i-1} = \text{col}([x_{i-1}, x_i]) \not\leq p_1$, whereby

$$p_1 < r_{i-1} \vee p_1 = \text{erep}([x_{i-1}, x_j]) \in \text{SRep}(C, \text{col}, D) = Q.$$

Since r_{i-1} is less than or equal to some of p_0, p_1 , and p_2 but $r_{i-1} \not\leq p_1$, we can assume that $r_{i-1} \leq p_0$. Thus, since $\{p_0, p_1, p_2\}$ is an antichain, $r_{i-1} \neq r_{i-1} \vee p_1$. Hence, $r_{i-1} \vee p_1$ is join-reducible, and the choice of Q gives that $r_{i-1} \vee p_1 \in \{p, 1\}$. So $p_2 \leq r_{i-1} \vee p_1$, and (2.9) yields that $p_2 \leq p_1$ or $p_2 \leq r_{i-1} \leq p_0$, which contradicts the fact that $\{p_0, p_1, p_2\}$ is an antichain. This proves that D is planar.

Next, striving for a contradiction, suppose that (2.2) holds. With the notation given from (2.2) to (2.3), let $Q := J^+(D) \cup \{e\}$; this need not be the same Q as above. Since there are two coatoms, $1_D \notin J(D)$. Since D is fully chain-representable, there is a $J(D)$ -colored chain $\langle C, \text{col}, D \rangle$ such that $Q = \text{SRep}(C, \text{col}, D)$. Since $e \in Q$, there is a maximal interval $[x, y]$ in C such that $e = \text{erep}([x, y])$. Since $e \neq 1_D = \text{erep}(C)$, either $0_C < x$, or $y < 1_C$; by duality, we can assume that $y < 1_C$. Let z be the cover of y in C , and let $p := \text{col}([y, z]) \in J(D)$. By the maximality of $[x, y]$, we have that $e < e \vee p$. Since there are only three elements, $1 = c_\ell \vee c_r$, $c_\ell = d_\ell \vee e$, and $c_r = d_r \vee e$ strictly above e and they are join-reducible, $e \vee p \notin J(D)$. Thus, since $e \vee p = \text{erep}([x, z]) \in \text{SRep}(C, \text{col}, D) = Q$, it follows that $e \vee p = 1$. Hence, $\text{ljs}(d_\ell) \leq e \vee p$, whereby (2.8) and (2.9) give that $\text{ljs}(d_\ell) \leq p$. Since $\text{rjs}(d_r) \leq p$ follows similarly, Lemma 2.1 gives that $1_D = p \in J(D)$. This is a contradiction, proving the necessity part of Proposition 1.6.

In order to prove the sufficiency part, assume that D is planar and it has at most one join-reducible coatom. Let $Q \subseteq D$ such that (1.4) holds. We need to find a $J(D)$ -colored chain that represents Q . We can assume that $|D| \geq 2$ since otherwise the one-element $J(D)$ -colored chain represents Q .

If $1 \in J(D)$, then we let $c := 1$; now the principal filter $\uparrow c = \{x \in D : c \leq x\}$ is clearly a subset of Q . If $1 \notin J_0(D)$, then there are exactly two coatoms by (2.1) and at least one of them is join-irreducible by our assumption; in this case, let c be a join-irreducible coatom and we still have that $\uparrow c \subseteq Q$.

Let $u_1, u_2, \dots, u_m$ be a repetition-free list of all elements of $J(D)$. Similarly, let $x_1, x_2, \dots, x_k$ be a repetition-free list of all elements of $Q \setminus J(D)$; this list can be empty. Define $\langle C, \text{col}, D \rangle$ such that $\text{length}(C) = 2m + 3k - 1$ and the colors of the edges, from bottom to top, are as follows:

$$u_1, c, u_2, \dots, c, u_m, c, \text{ljs}(x_1), \text{rjs}(x_1), c, \text{ljs}(x_2), \text{rjs}(x_2), c, \\ \text{ljs}(x_3), \text{rjs}(x_3), c, \text{ljs}(x_4), \text{rjs}(x_4), c, \dots, \text{ljs}(x_k), \text{rjs}(x_k);$$

see the third part of Figure 1 where the map col is given by labeling. The equation in (2.3) makes it clear that $Q \subseteq \text{SRep}(C, \text{col}, D)$. In order to see the converse inclusion, take an arbitrary element of $\text{SRep}(C, \text{col}, D)$, that is, take an arbitrary $I \in \text{Intv}(C)$ and consider $\text{erep}(I)$; we need to show that $\text{erep}(I) \in Q$. This is clear if $\text{length}(I) \leq 1$. For $\text{length}(I) \geq 2$, either $\text{erep}(I)$ is in the filter $\uparrow c$, which is a subset of Q , or $\text{erep}(I) = \text{ljs}(x_i) \vee \text{rjs}(x_i)$ and (2.3) gives that $\text{erep}(I) = x_i \in Q$. Therefore, $Q = \text{SRep}(C, \text{col}, D)$, as required. This completes the proof of Proposition 1.6. $\square$

4. DEALING WITH CONGRUENCES OF AN ALGEBRA

Lemma 4.1. *Every fully (A1)-representable finite distributive lattice is planar.*

Proof. For the sake of contradiction, suppose that D is a *non-planar* fully (A1)-representable finite distributive lattice. Pick a three-element antichain $\{\alpha, \beta, \gamma\} \subseteq J(D)$; this is possible by (3.1). Let $\mu = \alpha \vee \beta \vee \gamma$ and $Q = J^+(D) \cup \{\mu\}$. By the full (A1)-representability of D , we can assume that $D = \text{Con}(A)$ and $Q = \text{Princ}(A)$ for an algebra A . Then μ is a principal congruence of A , whereby we can pick elements $a, b \in A$ such that $\text{con}(a, b)$, the *principal congruence generated* by the pair $\langle a, b \rangle$, is μ ; in notation, $\mu = \text{con}(a, b)$. Using that $\mu = \alpha \vee \beta \vee \gamma$, we can pick a *shortest* sequence $a = x_0, x_1, \dots, x_{n-1}, x_n = b$ of elements of A such that $\langle x_i, x_{i+1} \rangle$ belongs to $\alpha \cup \beta \cup \gamma$ for all (non-negative) $i < n$. That is, for all $i < n$,

$$(4.1) \quad \text{con}(x_i, x_{i+1}) \leq \alpha, \quad \text{or} \quad \text{con}(x_i, x_{i+1}) \leq \beta, \quad \text{or} \quad \text{con}(x_i, x_{i+1}) \leq \gamma.$$

We claim that

$$(4.2) \quad \text{each of } \alpha, \beta, \text{ and } \gamma \text{ occurs in (4.1) for some } i < n.$$

Suppose to the contrary that, say, γ does not occur. Then $\gamma \leq \mu = \alpha \vee \beta$ and (2.9) give that $\gamma \leq \alpha$ or $\gamma \leq \beta$, which is impossible since $\{\alpha, \beta, \gamma\}$ is an antichain. This shows the validity of (4.2), and shows also that $\alpha \vee \beta < \mu$.

Our sequence yields that $\text{con}(a, b) \leq \bigvee_{i < n} \text{con}(x_i, x_{i+1})$. Thus, $\alpha \leq \mu = \text{con}(a, b) \leq \bigvee_{i < n} \text{con}(x_i, x_{i+1})$. Hence, (2.9) gives an $i_\alpha < n$ such that $\alpha \leq \text{con}(x_{i_\alpha}, x_{i_\alpha+1})$. Combining this inequality with (4.1) and taking into account that $\{\alpha, \beta, \gamma\}$ is an antichain, we obtain that $\alpha = \text{con}(x_{i_\alpha}, x_{i_\alpha+1})$. Similarly, $\beta = \text{con}(x_{i_\beta}, x_{i_\beta+1})$ and $\gamma = \text{con}(x_{i_\gamma}, x_{i_\gamma+1})$ for some $i_\beta < n$ and $i_\gamma < n$. Since α, β , and γ play a symmetric role, we can assume that i_α is between i_β and i_γ . Let j denote the smallest non-negative number such that $j \leq i_\alpha$ and $\text{con}(x_s, x_{s+1}) \leq \alpha$ for all $s \in [j, i_\alpha] := \{j, j+1, \dots, i_\alpha\}$. By $\text{con}(x_{i_\alpha}, x_{i_\alpha+1}) = \alpha$, this j exists. Since $\text{con}(x_{i_\beta}, x_{i_\beta+1}) = \beta \not\leq \alpha$, $\text{con}(x_{i_\gamma}, x_{i_\gamma+1}) = \gamma \not\leq \alpha$, and i_α is between i_β and i_γ , it follows that $j > 0$. So we can consider $\text{con}(x_{j-1}, x_j)$, which is not in

$\downarrow \alpha := \{\xi \in D : \xi \leq \alpha\}$. By (4.1) and since the role of β and γ is symmetric, we can assume that $\text{con}(x_{j-1}, x_j) \leq \beta$. The minimality of the length n of our sequence implies that $x_{j-1} \neq x_{j+1}$. Hence

$$(4.3) \quad 0 < \text{con}(x_{j-1}, x_{j+1}) \leq \text{con}(x_{j-1}, x_j) \vee \text{con}(x_j, x_{j+1}) \leq \beta \vee \alpha < \mu$$

and the choice of $Q = \text{Princ}(A)$ imply that the principal congruence $\text{con}(x_{j-1}, x_{j+1})$ is join-irreducible. Hence, applying (2.9) to (4.3), we obtain that $\text{con}(x_{j-1}, x_{j+1}) \leq \beta$ or $\text{con}(x_{j-1}, x_{j+1}) \leq \alpha$. Thus, omitting x_j from our sequence, we obtain a shorter sequence that still satisfies (4.1). This contradicts the minimality of n and proves the lemma. $\square$

Lemma 4.2. *Every fully (A1)-representable finite distributive lattice has at most one join-reducible coatom.*

Proof. For the sake of contradiction, suppose that D is a fully (A1)-representable finite distributive lattice that has at least two join-reducible coatoms. By Lemma 4.1, D is planar. It follows from (2.1) that D has exactly two coatoms, whereby (2.2) holds. However, since we are going to compute with congruences, let $\varepsilon = e$, $\alpha = \text{ljs}(c_\ell)$, and $\beta = \text{rjs}(c_r)$; see the diagram on the left of Figure 1. Let $Q = J^+(D) \cup \{\varepsilon\}$. The full (A1)-representability of D allows us to assume that $D = \text{Con}(A)$ and $Q = \text{Princ}(A)$ hold for an algebra A . We claim that for every $u \neq v \in A$,

$$(4.4) \quad \begin{array}{l} \text{there exists a finite sequence } u = w_0, w_1, \dots, w_k = v \text{ of} \\ \text{elements of } A \text{ such that } \text{con}(w_i, w_{i+1}) \in J(D) \text{ for all } i < k. \end{array}$$

Note that this is valid for an arbitrary finite lattice D , not only for a planar distributive one. We prove (4.4) by induction on $\mathfrak{h}(\text{con}(u, v))$, where $\mathfrak{h}$ is the *height function* on D , that is, for $x \in D$, $\mathfrak{h}(x)$ is the length of $\downarrow x$. Since $u \neq v$, the smallest possible value of $\mathfrak{h}(\text{con}(u, v))$ is 1. So, to deal with the base of the induction, assume that $\mathfrak{h}(\text{con}(u, v)) = 1$. Then $\text{con}(u, v)$ is an atom, whereby $\text{con}(u, v) \in J(D)$ and (4.4) holds with $k = 1$, $w_0 = u$, and $w_1 = v$. Next, to perform the induction step, assume that $\mathfrak{h}(\text{con}(u, v)) > 1$. We can also assume that $\text{con}(u, v) \notin J(D)$, because otherwise $\langle k, w_0, \dots, w_k \rangle := \langle 1, u, v \rangle$ does the job again. Then there are $\gamma, \delta \in D = \text{Con}(A)$ such that $\mathfrak{h}(\gamma) < \mathfrak{h}(\text{con}(u, v))$, $\mathfrak{h}(\delta) < \mathfrak{h}(\text{con}(u, v))$, and $\text{con}(u, v) = \gamma \vee \delta$. Therefore, there is a *shortest* sequence $u = s_0, s_1, \dots, s_{m-1}, s_m = v$ of elements of A such that $\langle s_j, s_{j+1} \rangle \in \gamma \cup \delta$ for all $j < m$. That is, for all $j < m$, $\text{con}(s_j, s_{j+1}) \leq \gamma$ or $\text{con}(s_j, s_{j+1}) \leq \delta$. Thus, $\mathfrak{h}(\text{con}(s_j, s_{j+1})) \leq \max\{\mathfrak{h}(\gamma), \mathfrak{h}(\delta)\} < \mathfrak{h}(\text{con}(u, v))$ for all $j < m$. Furthermore, $s_j \neq s_{j+1}$ for all $j < m$ since we took a shortest sequence. Hence, the induction hypothesis yields a sequence of type (4.4) from s_j to s_{j+1} for all $j < m$. Concatenating these sequences, we obtain a sequence from u to v as required by (4.4). This completes the induction and proves (4.4).

Next, we claim that

$$(4.5) \quad \text{for every block } U \text{ of } \varepsilon, \text{ we have that } U^2 \subseteq \alpha \text{ or } U^2 \subseteq \beta.$$

Since this is evident for a singleton U , assume that $|U| > 1$. However, $U \neq A$ since $\varepsilon \neq 1_D = 1_{\text{Con}(A)}$. So we can pick an element $x \in A \setminus U$ and another element $y \in U$. By (4.4), there is a finite sequence of elements from x to y such that any two consecutive elements in this sequence generate a join-irreducible congruence. This sequence begins outside U and terminates in U , whereby there are two consecutive elements in the sequence such that first is outside U but the second is in U . Changing the notation if necessary, we can assume that these two elements are x and y . Thus, $x \in A \setminus U$

and $y \in U$ are chosen so that $\text{con}(x, y) \in J(D)$. It follows from Lemma 2.2(ii) that $\text{con}(x, y) \leq \alpha$ or $\text{con}(x, y) \leq \beta$. Since the role of α and β is symmetric, we can assume that $\text{con}(x, y) \leq \alpha$. Now let z be an arbitrary element of U . Since $\text{con}(x, z)$ is a principal congruence, the choice of $Q = \text{Princ}(A)$ and $x \neq z$ give that $\text{con}(x, z) \in \{1_D, \varepsilon\} \cup J(D)$. If we had that $\text{con}(x, z) = 1_D = 1_{\text{Con}(A)}$, then $\beta \leq 1_D = \text{con}(x, z) \leq \text{con}(x, y) \vee \text{con}(y, z) \leq \alpha \vee \varepsilon$, which would contradict Lemma 2.2(iii). Since z is in U but x is not, $\text{con}(x, z) \notin \downarrow \varepsilon$. In particular, $\text{con}(x, z) \neq \varepsilon$ and we obtain that $\text{con}(x, z) \in J(D)$. In fact, $\text{con}(x, z) \in J(D) \setminus \downarrow \varepsilon$, and it follows from Lemma 2.2(i) that $\text{con}(x, y)$ is α or β . If we had that $\text{con}(x, z)$ is β , then $\beta = \text{con}(x, z) \leq \text{con}(x, y) \vee \text{con}(y, z) \leq \alpha \vee \varepsilon$ would contradict Lemma 2.2(iii). Hence, $\text{con}(x, z) = \alpha$ and $\langle y, z \rangle \in \text{con}(y, x) \vee \text{con}(x, z) = \text{con}(x, y) \vee \alpha = \alpha$. Since z was an arbitrary element of U , the required inclusion $U^2 \subseteq \alpha$ follows by the transitivity of α . This proves (4.5).

Finally, since $\varepsilon \in Q = \text{Princ}(A)$, we can pick $a, b \in A$ such that $\varepsilon = \text{con}(a, b)$. Clearly, the ε -block of a contains b . Applying (4.5) to this block, it follows that $\langle a, b \rangle \in \alpha$ or $\langle a, b \rangle \in \beta$. Thus, $\varepsilon = \text{con}(a, b) \leq \alpha$ or $\varepsilon \leq \beta$, which contradicts Lemma 2.2(iv). This completes the proof of Lemma 4.2. $\square$

Now, we are in the position to prove our main theorem.

Proof of Theorem 1.3. We need to prove only that 1.3(i) implies 1.3(iii); this follows from Lemmas 4.1 and 4.2. $\square$

The following proof relies heavily on Grätzer [14].

Proof of Proposition 1.1. Let D be finite distributive lattice; we can assume that $|D| > 1$. In order to prove that (a) implies (b), assume that D is fully (A)-representable. Then it is fully (A1)-representable, and it follows from Theorem 1.3 that D is planar. We obtain from (2.1) that D has at most two coatoms, and we need to exclude the possibility that D has exactly two coatoms.

(4.6) Suppose to the contrary that D has two coatoms, α and β , and let $Q = D \setminus \{1_D\}$.

We can assume that $D = \text{Con}(A)$ and $Q = \text{Princ}(A)$ for some algebra A , since D is fully (A)-representable. We claim that

$$(4.7) \quad 1_{\text{Con}(A)} = \alpha \cup \beta.$$

In order to see this, let $\langle x, y \rangle \in A^2$. Since $1_{\text{Con}(A)} \notin \text{Princ}(A)$, we have that $\text{con}(x, y) \neq 1_{\text{Con}(A)} = 1_D$. Since D has only two coatoms, $\text{con}(x, y) \leq \alpha$ or $\text{con}(x, y) \leq \beta$. This means that $\langle x, y \rangle \in \alpha$ or $\langle x, y \rangle \in \beta$, implying (4.7).

Next, let U be arbitrary α -block. Since $\alpha \neq 1_D = 1_{\text{Con}(A)}$, we can pick an element $x \in A \setminus U$. For every $y \in U$, we have that $\langle x, y \rangle \notin \alpha$ since x is outside the α -block U of y . Hence, (4.7) gives that $\langle x, y \rangle \in \beta$ for all $y \in U$. So $U^2 \subseteq \beta$ by transitivity, and we conclude that $\alpha \leq \beta$. This is a contradiction since α and β are distinct coatoms. Consequently, (a) implies (b).

Next, in order to prove that (b) implies (a), assume that D has exactly one coatom. Let Q be a subset of D satisfying (1.2). Since $1_D \in J(D)$, we have that $J_0(D) = J^+(D)$, whereby Q satisfies (1.4). Hence, Proposition 1.6 implies that the inclusion $Q \subseteq D$ is chain-representable. Thus, Grätzer [14, Theorem 3], which has been recalled in Theorem 1.8(v), gives that the inclusion in question is

represented by the principal congruences of a finite lattice. Consequently, D is fully (A)-representable and condition (a) in Proposition 1.1 holds, as required. $\square$

Proof of Corollary 1.2. For the sake of contradiction, suppose that A is an algebra such that $\text{Princ}(A) = V$. Every congruence $\gamma \in \text{Con}(A)$ is the join of all principal congruences in $\downarrow \gamma$, whereby $D := \text{Con}(A)$ is the four-element boolean lattice. Thus, A represents the inclusion $V := Q \subseteq D$. By (4.6), this is a contradiction. $\square$

Proof of Corollary 1.7. Assume that D is fully (A)-representable. We can also assume that $|D| > 1$. By Proposition 1.1, D is planar and $1_D \in J(D)$. Proposition 1.6 gives that D is fully chain-representable. Hence, D is fully (fL)-representable by Theorem 1.8(v). $\square$

5. REPRESENTING A SINGLE INCLUSION BY A FINITE ALGEBRA

The aim of this section is to prove Proposition 1.9.

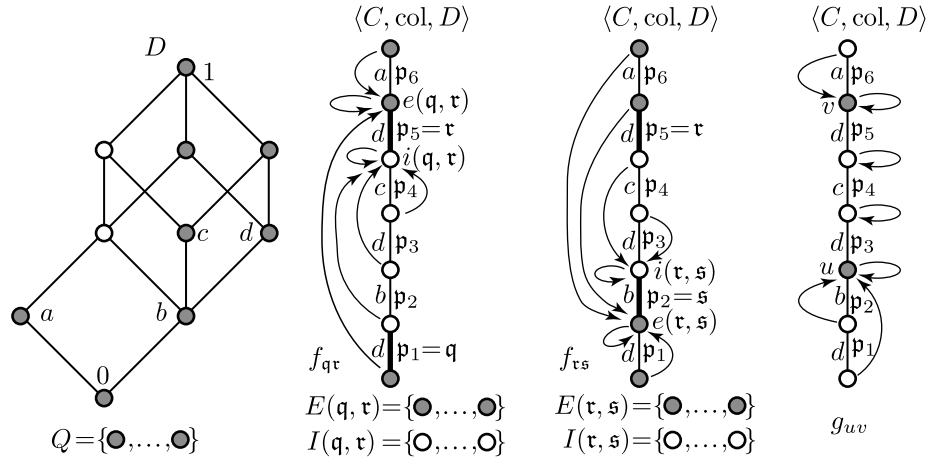


FIGURE 2. From a $J(D)$ -colored chain to an algebra

Proof of Proposition 1.9. Assume that D is a finite distributive lattice and a (finite) $J(D)$ -colored chain $\langle C, \text{col}, D \rangle$ represents an inclusion $Q \subseteq D$. An example is given in Figure 2, where $\langle C, \text{col}, D \rangle$ is drawn thrice. In order to indicate generality, $|C| = 7$ in the figure; note however that a five-element chain whose edges are colored with c, b, d, a , in this order, would also represent $Q \subseteq D$. In the figure, the colors are given by the labels $a, \dots, d$, and the edges of C , that is, the members of $\text{Prime}(C)$, are $p_1, \dots, p_6$. In order to turn C into an algebra, we are going to define two kinds of unary operations on C . First, for $u, v \in C$ with $u < v$, we define a so-called *contraction operation* $g_{uv}: C \rightarrow C$ by the rule

$$g_{uv}(x) := \begin{cases} v, & \text{if } x \geq v, \\ x, & \text{if } u \leq x \leq v, \text{ and} \\ u, & \text{if } x \leq u; \end{cases}$$

see on the right of Figure 2. The name “contraction” comes from the straightforward fact that

$$(5.1) \quad \begin{array}{l} \text{if } w, z \in C \text{ with } w < z \in C \text{ and } g_{uv}(w) \neq g_{uv}(z), \\ \text{then } w \leq g_{uv}(w) < g_{uv}(z) \leq z. \end{array}$$

Second, let $\mathbf{p}, \mathbf{h} \in \text{Prime}(C)$ be *distinct* edges of C . We define the *interior set* $I(\mathbf{p}, \mathbf{h})$ and the *exterior set* $E(\mathbf{p}, \mathbf{h})$ as follows; see Figure 2 for $\langle \mathbf{p}, \mathbf{h} \rangle \in \{\langle \mathbf{q}, \mathbf{r} \rangle, \langle \mathbf{r}, \mathbf{s} \rangle\}$.

$$I(\mathbf{p}, \mathbf{h}) := \begin{cases} [1_{\mathbf{p}}, 0_{\mathbf{h}}] & \text{if } 1_{\mathbf{p}} \leq 0_{\mathbf{h}}, \\ [1_{\mathbf{h}}, 0_{\mathbf{p}}] & \text{if } 1_{\mathbf{h}} \leq 0_{\mathbf{p}} \end{cases} \quad \text{and} \quad E(\mathbf{p}, \mathbf{h}) := C \setminus I(\mathbf{p}, \mathbf{h}).$$

We also need to define the *interior* and the *exterior target* elements $i(\mathbf{p}, \mathbf{h})$ and $e(\mathbf{p}, \mathbf{h})$, respectively, as follows; see again the figure for $\langle \mathbf{p}, \mathbf{h} \rangle \in \{\langle \mathbf{q}, \mathbf{r} \rangle, \langle \mathbf{r}, \mathbf{s} \rangle\}$.

$$i(\mathbf{p}, \mathbf{h}) := \begin{cases} 0_{\mathbf{h}} & \text{if } 1_{\mathbf{p}} \leq 0_{\mathbf{h}}, \\ 1_{\mathbf{h}} & \text{if } 1_{\mathbf{h}} \leq 0_{\mathbf{p}} \end{cases} \quad \text{and} \quad e(\mathbf{p}, \mathbf{h}) := \begin{cases} 1_{\mathbf{h}} & \text{if } 1_{\mathbf{p}} \leq 0_{\mathbf{h}}, \\ 0_{\mathbf{h}} & \text{if } 1_{\mathbf{h}} \leq 0_{\mathbf{p}}. \end{cases}$$

With the notation given above, we define a unary *forcing operation*

$$f_{\mathbf{p}\mathbf{h}} : C \rightarrow C \quad \text{by} \quad x \mapsto f_{\mathbf{p}\mathbf{h}}(x) := \begin{cases} i(\mathbf{p}, \mathbf{h}) & \text{if } x \in I(\mathbf{p}, \mathbf{h}), \\ e(\mathbf{p}, \mathbf{h}) & \text{if } x \in E(\mathbf{p}, \mathbf{h}). \end{cases}$$

The name “forcing” is motivated by the fact that the presence of this operation “forces” the inequality $\text{con}(0_{\mathbf{p}}, 1_{\mathbf{p}}) \geq \text{con}(0_{\mathbf{h}}, 1_{\mathbf{h}})$. For $\langle \mathbf{p}, \mathbf{h} \rangle \in \{\langle \mathbf{q}, \mathbf{r} \rangle, \langle \mathbf{r}, \mathbf{s} \rangle\}$, the forcing operation is given in the middle part of Figure 2. The unary algebra $\mathfrak{A}$ we need is defined by

$$(5.2) \quad \mathfrak{A} = \langle C; \{g_{uv} : u < v \in C\} \cup \{f_{\mathbf{p}\mathbf{h}} : \mathbf{p} \neq \mathbf{h} \in \text{Prime}(C) \text{ and } \text{col}(\mathbf{p}) \geq \text{col}(\mathbf{h}) \text{ holds in } D\} \rangle.$$

Although $\mathfrak{A}$ does not have a lattice reduct, we will frequently refer to the ordering of the chain C ; for example, when speaking of intervals and edges. For $u, v \in C$, the *symmetrized interval* $[u \wedge v, u \vee v]$ will be denoted by $[u, v]^*$. Since C is a chain, $[u, v]^* = [u, v]$ if $u \leq v$, and $[u, v]^* = [v, u]$ otherwise.

We claim that the map

$$(5.3) \quad \begin{aligned} \varphi : D &\rightarrow \text{Con}(\mathfrak{A}), \quad \text{defined by} \\ x &\mapsto \{\langle u, v \rangle : \text{col}(\mathbf{p}) \leq x \text{ for all } \mathbf{p} \in \text{Prime}([u, v]^*)\}, \end{aligned}$$

is a lattice isomorphism.

First, let $x \in D$; we need to show that $\varphi(x)$ is a congruence of $\mathfrak{A}$. Since we use a symmetrized interval in (5.3), $\varphi(x)$ is reflexive and symmetric. The transitivity of $\varphi(x)$ follows from the rule $[u, w]^* \subseteq [u, v]^* \cup [v, w]^*$. It is clear from (5.1) that every contraction operation preserves $\varphi(x)$. Let $f_{\mathbf{p}\mathbf{h}}$ be a forcing operation of $\mathfrak{A}$; this means that $\mathbf{p} \neq \mathbf{h} \in \text{Prime}(C)$ and $\text{col}(\mathbf{p}) \geq \text{col}(\mathbf{h})$ in D . In order to show that $f_{\mathbf{p}\mathbf{h}}$ preserves $\varphi(x)$, assume that $\langle u, v \rangle \in \varphi(x)$ and $f_{\mathbf{p}\mathbf{h}}(u) \neq f_{\mathbf{p}\mathbf{h}}(v)$. Then $\{f_{\mathbf{p}\mathbf{h}}(u), f_{\mathbf{p}\mathbf{h}}(v)\} = \{0_{\mathbf{h}}, 1_{\mathbf{h}}\}$, whereby $[f_{\mathbf{p}\mathbf{h}}(u), f_{\mathbf{p}\mathbf{h}}(v)]^* = \mathbf{h}$. Hence, to show that $\langle f_{\mathbf{p}\mathbf{h}}(u), f_{\mathbf{p}\mathbf{h}}(v) \rangle \in \varphi(x)$, we need to show that $\text{col}(\mathbf{h}) \leq x$. Since $f_{\mathbf{p}\mathbf{h}}(u) \neq f_{\mathbf{p}\mathbf{h}}(v)$, one of u and v is in $E(\mathbf{p}, \mathbf{h})$ while the other one is in $I(\mathbf{p}, \mathbf{h})$. Hence, u and v in the chain C are “separated” by at least one of the edges $\mathbf{p}$ and $\mathbf{h}$. If they are separated by $\mathbf{h}$, then $\text{col}(\mathbf{h}) \leq x$ by the definition of $\varphi(x)$, as required. So we can assume that u and v are separated by $\mathbf{p}$. Then $\text{col}(\mathbf{p}) \leq x$ and, by (5.2), $\text{col}(\mathbf{p}) \geq \text{col}(\mathbf{h})$. By transitivity, we obtain again that $\text{col}(\mathbf{h}) \leq x$. This proves that $\varphi(x) \in \text{Con}(\mathfrak{A})$, whereby (5.3) really defines a map from D to $\text{Con}(\mathfrak{A})$.

Next, to prove the surjectivity of φ , let $\Theta \in \text{Con}(\mathfrak{A})$. Define $x = \psi(\Theta) \in D$ by

$$(5.4) \quad x = \psi(\Theta) := \bigvee \{ \text{col}(\mathfrak{p}) : \mathfrak{p} \in \text{Prime}(C) \text{ and } \langle 0_{\mathfrak{p}}, 1_{\mathfrak{p}} \rangle \in \Theta \};$$

we are going to show that $\varphi(x) = \Theta$. In order to do so, assume that $\langle w, z \rangle \in \Theta$. Since both Θ and $\psi(x)$ are congruences of $\mathfrak{A}$, we can also assume that $w < z$. If $\mathfrak{p} \in \text{Prime}([w, z])$, then $\langle 0_{\mathfrak{p}}, 1_{\mathfrak{p}} \rangle = \langle g_{0_{\mathfrak{p}}1_{\mathfrak{p}}}(w), g_{0_{\mathfrak{p}}1_{\mathfrak{p}}}(z) \rangle \in \Theta$, whereby $\text{col}(\mathfrak{p}) \leq x$. Since this holds for all $\mathfrak{p} \in \text{Prime}([w, z]^*) = \text{Prime}([w, z])$, we obtain by (5.3) that $\langle w, z \rangle \in \varphi(x)$. Thus, $\Theta \leq \varphi(x)$ holds in $\text{Con}(\mathfrak{A})$. Conversely, assume that $\langle w, z \rangle \in \varphi(x)$ and $w < z$. Since $w < z$,

$$(5.5) \quad \text{there are unique elements } t_i \in C \text{ such that } w = t_0 \prec t_1 \prec \cdots \prec t_k = z.$$

By (5.3), $\text{col}([t_i, t_{i+1}]) \leq x$ for all $i < k$. Thus, combining (2.9) and (5.4), we obtain an edge $\mathfrak{p}_i \in \text{Prime}(C)$ such that $\text{col}([t_i, t_{i+1}]) \leq \text{col}(\mathfrak{p}_i)$ and $\langle 0_{\mathfrak{p}_i}, 1_{\mathfrak{p}_i} \rangle \in \Theta$, for every $i < k$. Applying the forcing operation $f_{\mathfrak{p}_i[t_i, t_{i+1}]}$ componentwise to the pair $\langle 0_{\mathfrak{p}_i}, 1_{\mathfrak{p}_i} \rangle$, we obtain $\langle t_{i+1}, t_i \rangle$. Since this operation preserves Θ , we obtain that

$$(5.6) \quad \langle t_i, t_{i+1} \rangle \in \Theta, \quad \text{for all } i < k;$$

whereby $\langle w, z \rangle = \langle t_0, t_k \rangle \in \Theta$ by the transitivity of Θ . Hence, $\varphi(x) \leq \Theta$. Thus,

$$(5.7) \quad \text{for } x = \psi(\Theta) \text{ defined in (5.4), } \varphi(x) = \Theta.$$

This proves the surjectivity of φ .

It is clear from (5.3) that φ is order-preserving, that is, $x \leq y \in D$ implies that $\varphi(x) \leq \varphi(y)$. To show that φ is injective, assume that $x \neq y \in D$. Since $x = \bigvee (J(D) \cap \downarrow x)$ and similarly for y , there exists an $a \in J(D)$ such that $a \leq x$ and $a \not\leq y$, or conversely. So, we can assume that $a \leq x$ and $a \not\leq y$. Since $\text{col} : \text{Prime}(C) \rightarrow J(D)$ is a surjective map by Definition 1.5, there exists an edge $\mathfrak{p} \in \text{Prime}(C)$ such that $\text{col}(\mathfrak{p}) = a$. Then $\langle 0_{\mathfrak{p}}, 1_{\mathfrak{p}} \rangle$ is in $\varphi(x)$ but not in $\varphi(y)$. This proves the injectivity of φ .

Now that we know that φ is a bijection, its clear by (5.4) and (5.7) that the map

$$(5.8) \quad \psi : \text{Con}(\mathfrak{A}) \rightarrow D, \text{ defined by } \Theta \mapsto x \text{ in (5.4), is the inverse of } \varphi.$$

It is obvious by (5.4) that ψ is order-preserving. Consequently, φ is an order isomorphism and, thus, a lattice isomorphism.

We claim that,

$$(5.9) \quad \text{for every } \mathfrak{p} \in \text{Prime}(C), \quad \text{col}(\mathfrak{p}) = \psi(\text{con}(0_{\mathfrak{p}}, 1_{\mathfrak{p}})).$$

Since $\text{con}(0_{\mathfrak{p}}, 1_{\mathfrak{p}})$ obviously collapses $\langle 0_{\mathfrak{p}}, 1_{\mathfrak{p}} \rangle$, we obtain from (5.4) and (5.8) that $\text{col}(\mathfrak{p}) \leq \psi(\text{con}(0_{\mathfrak{p}}, 1_{\mathfrak{p}}))$. Conversely, with the notation $a := \text{col}(\mathfrak{p})$, it is clear by (5.3) that $\varphi(a) \ni \langle 0_{\mathfrak{p}}, 1_{\mathfrak{p}} \rangle$. Hence, $\varphi(a) \geq \text{con}(0_{\mathfrak{p}}, 1_{\mathfrak{p}})$. Using that ψ is order-preserving, we obtain that $\text{col}(\mathfrak{p}) = a = \psi(\varphi(a)) \geq \psi(\text{con}(0_{\mathfrak{p}}, 1_{\mathfrak{p}}))$, proving (5.9).

Next, let $w < z$ in C , and assume that $\langle w, z \rangle \in \Theta = \varphi(x)$ for some $x \in D$. Since (5.6) holds for the elements t_i , see (5.5), $\text{con}(t_i, t_{i+1}) \leq \Theta$ for all $i < k$. This also holds for $\Theta := \text{con}(w, z)$, whereby we obtain easily that

$$(5.10) \quad \text{if } w, z \in C \text{ with } w < z, \text{ then } \text{con}(w, z) = \bigvee_{i < k} \text{con}(t_i, t_{i+1}).$$

Now, we are in the position to show that $\varphi(Q) = \text{Princ}(\mathfrak{A})$. Assume that $x \in Q$. Since $Q = \text{SRep}(C, \text{col}, D)$, there are $w < z$ such that for the elements defined in

(5.6),

$$(5.11) \quad x = \text{erep}([w, z]) \stackrel{(1.5)}{=} \bigvee_{i < k} \text{col}([t_i, t_{i+1}]).$$

It follows from (5.9) that ψ , which is a lattice isomorphism, maps the right-hand side of (5.10) to that of (5.11). Hence, $\psi(\text{con}(w, z)) = x$, which gives that $\varphi(x) = \text{con}(w, z) \in \text{Princ}(\mathfrak{A})$. Consequently, $\varphi(Q) \subseteq \text{Princ}(\mathfrak{A})$.

Finally, we need to exclude that this inclusion is proper. Suppose to the contrary that $\varphi(Q) \subset \text{Princ}(\mathfrak{A})$, and pick a principal congruence from $\text{Princ}(\mathfrak{A}) \setminus \varphi(Q)$. Since $\varphi: D \rightarrow \text{Con}(\mathfrak{A})$ is surjective, this congruence is of the form $\varphi(x)$ where $x \notin Q$. Since $\varphi(x)$ is a principal congruence, there are $w, z \in C$ such that $w < z$ and $\varphi(x)$ is of the form $\text{con}(w, z)$, described in (5.10). Taking the ψ -images of both sides of the equation in (5.10) and using (5.9) in the same way as before, we obtain the validity of (5.11). Hence, $x = \text{erep}([w, z]) \in \text{SRep}(C, \text{col}, D) = Q$, contradicting the choice of x . This proves the equality $\varphi(Q) = \text{Princ}(\mathfrak{A})$ and completes the proof of Proposition 1.9. $\square$

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