ABSOLUTE RETRACTS FOR FINITE DISTRIBUTIVE LATTICES AND SLIM SEMIMODULAR LATTICES

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ABSTRACT. Let n denote a positive integer. We describe the absolute retracts for the following five categories of finite lattices: (1) slim semimodular lattices, (2) finite distributive lattices, (3) at most n-dimensional finite distributive lattices, (4) at most n-dimensional finite distributive lattices with cover-preserving $\{0, 1\}$-homomorphisms, and (5) finite distributive lattices with cover-preserving $\{0, 1\}$-homomorphisms. Although the singleton lattice is the only absolute retract for the first category, this result has paved the way to some other classes. For the second category, we prove that the absolute retracts are exactly the finite boolean lattices; this generalizes a 1979 result of J. Schmid. For the third category and also for the fourth, the absolute retracts are the finite boolean lattices of dimension at most n and the direct products of n nontrivial finite chains. For the fifth category, the absolute retracts are the same as those for the second category. Also, we point out that in each of these classes, the algebraically closed lattices and the strongly algebraically closed lattices are the same as the absolute retracts.

   Slim (and necessarily planar) semimodular lattices were introduced by G. Grätzer and E. Knapp in 2007, and they have been intensively studied since then. Algebraically closed and strongly algebraically closed lattices have been investigated by J. Schmid and, in several papers, by A. Molkhasi.

1. INTRODUCTION

Before formulating our targets and results in Subsection 1.5, we give a short historical overview. The history leading to the present work belongs to four topics, which are surveyed in the following four subsections. According to our knowledge, the first three of these four topics have been studied independently so far; one of our goals is to find some connection among them.

1.1. **Strongly algebraically closed algebras in categories of algebras.** By an *equation* in an algebra A we mean a formal expression

$$p(a_1, \ldots, a_m, x_1, \ldots, x_n) \approx q(a_1, \ldots, a_m, x_1, \ldots, x_n)$$

where $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $n \in \mathbb{N}^+ = \mathbb{N}_0 \setminus \{0\}$, p and q are $(m + n)$-ary terms (in the language of A), the elements $a_1, \ldots, a_m$ belong to A and they are called *parameters* (or *coefficients*), and $x_1, \ldots, x_n$ are the *unknowns* of this equation. Although a single equation contains only finitely many unknowns, we allow infinite

systems (that is, sets) of equations and such a system can contain infinitely many unknowns.

By a *category of algebras* we mean a concrete category $\mathcal{X}$ such that the objects of $\mathcal{X}$ are algebras of the same type, every morphism of $\mathcal{X}$ is a homomorphism, and whenever A_1 and A_2 are isomorphic algebras such that A_1 belongs to $\mathcal{X}$, then so does A_2. Note that there can be homomorphisms among the objects of $\mathcal{X}$ that are not morphisms of $\mathcal{X}$. If $\mathcal{X}$ happens to contain all homomorphisms among its objects as morphisms, then $\mathcal{X}$ is a *class of algebras* (*with all homomorphisms*); the parenthesized part of this term is often dropped in the literature. Given a category $\mathcal{X}$ of algebras and objects A, B in $\mathcal{X}$, we say that B is an $\mathcal{X}$-*extension* of A if A is a subalgebra of B and, in addition, the map $\iota\colon A \to B$ defined by $x \mapsto x$ is a morphism in $\mathcal{X}$. (If $\mathcal{X}$ is a class of algebras with all homomorphisms and $A, B \in \mathcal{X}$, then "extension" is the same as "$\mathcal{X}$-extension".)

Note that the concept of "B is an $\mathcal{X}$-extension of A" includes not only A and B, but also the embedding $\iota\colon A \to B$ defined by $x \mapsto x$. Therefore, when we speak of "all $\mathcal{X}$-extensions of A", then the meaning is that all possible embeddings ι are considered. For example, if A is the two-element chain in the class $\mathcal{L}$ of lattices with all homomorphisms, then A has three essentially different $\mathcal{L}$-extensions into a three-element chain.

For a category $\mathcal{X}$ of algebras and an algebra $A \in \mathcal{X}$, we say that A is *strongly algebraically closed in* $\mathcal{X}$ if for every $\mathcal{X}$-extension $B \in \mathcal{X}$ of A and for any system Σ of equations with parameters taken from A, if Σ has a solution in B, then it also has a solution in A. Following Schmid [33], if we replace "any system Σ" by "any finite system Σ", then we obtain the concept of an *algebraically closed algebra* A in $\mathcal{X}$. These two concepts have been studied by many authors; restricting ourselves to lattice theory, we only mention Schmid [33] and Molkhasi [25, 26, 27, 28].

1.2. **Absolute retracts.** Given an algebra B and a subalgebra A of B, we say that A is a *retract* of B if there exists a homomorphism $f\colon B \to A$ such that $f(a) = a$ for all $a \in A$. The homomorphism f in this definition is called a *retraction map* or a *retraction* for short.

Now let A be an algebra belonging to a category $\mathcal{X}$ of algebras. We say that A is an *absolute retract for* $\mathcal{X}$ if for any $\mathcal{X}$-extension B of A, there exists a retraction $B \to A$ among the morphisms of $\mathcal{X}$. Similarly, A is an *absolute* **H**-*retract for* $\mathcal{X}$ if for any $\mathcal{X}$-extension B of A, there exists a retraction $f\colon B \to A$ (but f need not be a morphism of $\mathcal{X}$). The letter **H** in this terminology comes from "homomorphism". Although an absolute **H**-retract is not a purely category theoretical notion, it helps us to state some of our assertions in a stronger form. Observe that

$$\text{an absolute retract for a category } \mathcal{X} \text{ of algebras is also} \atop \text{an absolute } \mathbf{H}\text{-retract for } \mathcal{X}, \text{ and} \qquad (1.1)$$

$$\text{for a class } \mathcal{X} \text{ of algebras with all homomorphisms, ab-} \atop \text{solute } \mathbf{H}\text{-retracts and absolute retracts are the same.} \qquad (1.2)$$

Absolute retracts emerged first in topology, and they appeared in classes of algebras as soon as 1946; see Reinhold [32]. There are powerful tools to deal with homomorphisms and, in particular, retractions in several categories of lattices; we will benefit from these tools in Sections 3 and 4.

1.3. **Slim semimodular lattices.** For a finite lattice L, let $\mathrm{J}(L)$ stand for the set of nonzero join-irreducible elements of L. Note that $\mathrm{J}(L)$ is a poset (i.e., a

partially ordered set) with respect to the order inherited from L. Following Czédli and Schmidt [11], we say that a lattice is *slim* if it is finite and $J(L)$ is the union of two chains. Note that slim lattices are planar; see Lemma 2.2 of Czédli and Schmidt [11]. As usual, a lattice L is (upper) *semimodular* if we have $x \vee z \preceq y \vee z$ for any $x, y, z \in L$ such that y covers or equals x (in notation, $x \preceq y$). Since the pioneering paper Grätzer and Knapp [20], recent years have witnessed a particularly intense activity in studying *slim semimodular lattices*; see Czédli, Dékány, Gyenizse and Kulin [5], Czédli and Grätzer [6, 7], Czédli, Grätzer, and Lakser [8], Czédli and Makay [10], Czédli and Schmidt [11, 12], Grätzer [16, 17, 18, 19], Grätzer and Knapp [21], Grätzer and Nation [22], some additional papers by G. Grätzer, and a dozen other papers written or coauthored by the first author. (Most of these dozen papers are referenced in one of [6], [7], and [9].) For the impact of these lattices on (combinatorial) geometry, see Adaricheva and Bolat [1], Adaricheva and Czédli [2], Czédli [3], and (the surveying) Section 2 of Czédli and Kurusa [9], and see their impact on lattice theory in Ranitović and Tepavčević [30, 31].

1.4. **Finite and n-dimensional distributive lattices.** It is well known that a finite distributive lattice D is determined by the poset $J(D)$ up to isomorphism. Borrowing a definition from Dushnik and Miller [14], the *order dimension* of a poset $P = (P; \leq_P)$, denoted by $\dim_{\mathrm{ord}}(P)$, is the least number n such that the relation $\leq_P$ is the intersection of n linear orderings on P. We know from Milner and Pouzet [24] that $\dim_{\mathrm{ord}}(P)$ is also the least number n such that P has an order embedding into the direct product of n chains. The *width* of a poset P is defined to be the maximum size of an antichain in P; it will be denoted by $\mathrm{width}(P)$. By Dilworth [13, Theorem 1.1], a finite poset P is of width n if and only if P is the union of n (not necessarily disjoint) chains but not a union of fewer chains. As it is pointed out in the first paragraph of page 276 in Rabinovitch and Rival [29], it follows from Dilworth [13] that

$$\text{for a finite distributive lattice } D, \ \dim_{\mathrm{ord}}(D) = \mathrm{width}(J(D)). \qquad (1.3)$$

If $\dim_{\mathrm{ord}}(D) = n$, then D is said to be *n-dimensional*.

1.5. **Targets and results.** First, we are going to prove the following easy proposition. By a finite algebra we mean a finite nonempty set equipped with *finitely many* operations.

Proposition 1.1. *If A is an algebra in a category $\mathcal{X}$ of algebras, then the following two conditions are equivalent.*

    (1) A is strongly algebraically closed in $\mathcal{X}$.
    *(2) A is an absolute **H**-retract for $\mathcal{X}$.*

Furthermore, if $\mathcal{X}$ consists of finite algebras, then each of (1) and (2) is equivalent to

    (3) A is algebraically closed in $\mathcal{X}$.

This proposition will be proved in Section 2. Armed with Proposition 1.1, we are going to prove the following result in Section 3.

Theorem 1.2. *Let L be a slim semimodular lattice and let $\mathcal{S}$ denote the class of all slim semimodular lattices with all homomorphisms. Then the following four conditions are equivalent.*

    (1) L is algebraically closed in $\mathcal{S}$.

    (2) L is strongly algebraically closed in $\mathcal{S}$.
    (3) L is an absolute retract for $\mathcal{S}$.
    (4) L is the one-element lattice, i.e., $|L| = 1$.

Since the singleton lattice does not look too exciting in itself, it is worth noting the following. First, we know neither a really short proof of this theorem nor a proof without using some nontrivial tool from the theory of slim semimodular lattices. Second, Theorem 1.2 together with Molkhasi [25, 26, 27, 28] and Schmid [33] have just motivated a related result with infinitely many absolute retracts for the class of slim semimodular lattices with less morphisms than here; see Czédli [4]. Third and mainly, as it is explained in Subsection 4.1, Theorem 1.2 and the tools needed to prove it have paved the way to Theorem 1.3 of the present paper.

Before formulating Theorem 1.3, we need to define some categories of lattices. Let ω stand for the least infinite cardinal number, let $\mathbb{N}^+ := \{1, 2, 3, 4, \dots\}$, and

$$\text{for } n \in \mathbb{N}^+ \cup \{\omega\}, \text{ let } \mathcal{D}(n) \text{ denote the class of } \textit{finite} \text{ distributive lattices with order dimension at most } n, \text{ with all homomorphisms.} \tag{1.4}$$

In particular $\mathcal{D}(\omega) = \bigcup\{\mathcal{D}(n) : n \in \mathbb{N}^+\}$ is the class of all finite distributive lattices. We also define

$$\text{the class } \mathcal{D}_{\text{all}} \text{ of all (not necessarily finite) distributive lattices with all homomorphisms.} \tag{1.5}$$

For finite lattices A and B, a lattice homomorphism $f \colon A \to B$ is said to be a *cover-preserving* $\{0, 1\}$-*homomorphism* if $f(0) = 0$, $f(1) = 1$, and for all $x, y \in A$ such that $x \prec y$, we have that $f(x) \prec f(y)$. Since any two maximal chains in a finite semimodular lattice are of the same length (this is the so-called *Jordan–Hölder chain condition*), we easily obtain the following observation; see Lemma 4.3 for a bit more information.

$$\text{If } A \text{ and } B \text{ are finite semimodular lattices and there exists a cover-preserving } \{0, 1\}\text{-homomorphism } A \to B, \text{ then } A \text{ and } B \text{ are of the same length.} \tag{1.6}$$

Note that distributive lattices, to which we will apply (1.6), are semimodular.

$$\text{For } n \in \mathbb{N}^+ \cup \{\omega\}, \text{ let } \mathcal{D}_{01\prec}(n) \text{ denote the category consisting of finite distribute lattices of order dimension at most } n \text{ as objects and cover-preserving } \{0, 1\}\text{-homomorphisms as morphisms.} \tag{1.7}$$

By a *nontrivial* lattice we mean a lattice with more than one element. In particular, a *nontrivial chain* is a chain with at least two elements. For $n \in \mathbb{N}^+$, by an n-*dimensional grid* we mean the direct product of n nontrivial finite chains. 2-dimensional grids are simply called *grids*. Note that an n-dimensional grid is a *finite* distributive lattice and, clearly, its order dimension is n.

Now, armed with (1.4)–(1.7), we are in the position to sate our main result.

Theorem 1.3 (Main Theorem). *Let* $n \in \mathbb{N}^+$ *be an integer and let* D *be a finite distributive lattice. Then the following three assertions hold.*

    (1) *If* $D \in \mathcal{D}_{01\prec}(n)$ *and* D *is an absolute* **H**-*retract for* $\mathcal{D}_{01\prec}(n)$, *then* D *is boolean or* D *is an* n-*dimensional grid.*
    (2) *If* D *is boolean, then* D *is an absolute retract for* $\mathcal{D}_{\text{all}}$.
    (3) *if* D *is an* n-*dimensional grid, then* D *is an absolute retract for* $\mathcal{D}(n)$.

Note that for a finite distributive lattice D, part (2) of Theorem 1.3 is stronger than its counterpart in Schmid [33] since he only allows lattice embeddings and homomorphisms that preserve 0 and 1 whenever they exist. Up to our best knowledge, parts (1) and (3) of Theorem 1.3 have no counterparts in the literature.

Next, we are going to formulate some corollaries that accomplish our goal mentioned in the Abstract. To do so economically, we introduce some classes of lattices as follows. For a category $\mathcal{X}$ of lattices, we let

$$\text{AlgC}(\mathcal{X}) := \{L \in \mathcal{X} \text{ and } L \text{ is algebraically closed in } \mathcal{X}\}, \tag{1.8}$$

$$\text{StAlgC}(\mathcal{X}) := \{L \in \mathcal{X} \text{ and } L \text{ is strongly algebraically closed} \\ \text{in } \mathcal{X}\}, \tag{1.9}$$

$$\text{AbRt}(\mathcal{X}) := \{L \in \mathcal{X} \text{ and } L \text{ is an absolute retract for } \mathcal{X}\}, \tag{1.10}$$

$$\text{AHRt}(\mathcal{X}) := \{L \in \mathcal{X} \text{ and } L \text{ is an absolute } \mathbf{H}\text{-retract for } \mathcal{X}\}, \tag{1.11}$$

$$\text{Bool}_{\text{fin}} := \{L : L \text{ is a finite boolean algebra}\}, \tag{1.12}$$

$$\text{Bool}_{\leq}(2^n) := \{L : L \text{ is a finite boolean lattice and } |L| \leq 2^n\}, \tag{1.13}$$

$$\text{Grid}_{=}(n) := \{L : L \text{ is an } n\text{-dimensional grid}\}, \text{ and} \tag{1.14}$$

$$\text{Finite}(\mathcal{X}) := \{X : X \text{ is a finite lattice belonging to } \mathcal{X}\}. \tag{1.15}$$

Armed with (1.4)–(1.15), now we can formulate our corollaries; they will be derived from Theorem 1.3 at the end of Section 4.

Corollary 1.4. *For every positive integer n,*
$$\text{AbRt}(\mathcal{D}(n)) = \text{AlgC}(\mathcal{D}(n)) = \text{StAlgC}(\mathcal{D}(n)) = \text{Bool}_{\leq}(2^n) \cup \text{Grid}_{=}(n).$$

Corollary 1.5. $\text{AbRt}(\mathcal{D}(\omega)) = \text{AlgC}(\mathcal{D}(\omega)) = \text{StAlgC}(\mathcal{D}(\omega)) = \text{Bool}_{\text{fin}}.$

Corollary 1.6. $\text{Finite}(\text{AbRt}(\mathcal{D}_{\text{all}})) = \text{Finite}(\text{StAlgC}(\mathcal{D}_{\text{all}})) = \text{Bool}_{\text{fin}}.$

Corollary 1.7. *For every positive integer n,*
$$\text{AbRt}(\mathcal{D}_{01\prec}(n)) = \text{AHRt}(\mathcal{D}_{01\prec}(n)) = \text{AlgC}(\mathcal{D}_{01\prec}(n))$$
$$= \text{StAlgC}(\mathcal{D}_{01\prec}(n)) = \text{Bool}_{\leq}(2^n) \cup \text{Grid}_{=}(n).$$

Corollary 1.8.
$$\text{AbRt}(\mathcal{D}_{01\prec}(\omega)) = \text{AHRt}(\mathcal{D}_{01\prec}(\omega)) = \text{AlgC}(\mathcal{D}_{01\prec}(\omega))$$
$$= \text{StAlgC}(\mathcal{D}_{01\prec}(\omega)) = \text{Bool}_{\text{fin}}.$$

Corollaries 1.7 and 1.8 show that we can disregards many morphisms from the categories occurring in Corollaries 1.4 and 1.5 so that the absolute retracts remain the same. This is not at all so for the category $\mathcal{S}$ occurring in Theorem 1.2; see Czédli [4] for details.

Recall that a planar lattice is finite by definition.

Corollary 1.9. *The class of planar distributive lattices with all morphisms is $\mathcal{D}(2)$, and we have that*
$$\text{AbRt}(\mathcal{D}(2)) = \text{AlgC}(\mathcal{D}(2)) = \text{StAlgC}(\mathcal{D}(2)) = \text{Bool}_{\leq}(2^2) \cup \text{Grid}_{=}(2).$$

Corollary 1.10. *The category of planar distributive lattices with cover-preserving $\{0,1\}$-homomorphisms is $\mathcal{D}_{01\prec}(2)$, and we have that*
$$\text{AbRt}(\mathcal{D}_{01\prec}(2)) = \text{AHRt}(\mathcal{D}_{01\prec}(2)) = \text{AlgC}(\mathcal{D}_{01\prec}(2))$$
$$= \text{StAlgC}(\mathcal{D}_{01\prec}(2)) = \text{Bool}_{\leq}(2^2) \cup \text{Grid}_{=}(2).$$

2. Proving our proposition

To ease the notation, we give the proof only for lattices; the general proof would be practically the same.

Proof of Proposition 1.1. First, we deal with the implication (1) $\Rightarrow$ (2) and, if $\mathcal{X}$ consists of finite lattices, also with the implication (3) $\Rightarrow$ (2).

Assume that $\mathcal{X}$ is a class of lattices, $A \in \mathcal{X}$, and either A is strongly algebraically closed in $\mathcal{X}$ or $\mathcal{X}$ consists of finite lattices and A is algebraically closed in $\mathcal{X}$. Let $B \in \mathcal{X}$ be an $\mathcal{X}$-extension of A. We need to show the existence of a retraction $f \colon B \to A$. We can assume that A is a proper sublattice of B, because the identity map of B would obviously be a $B \to A$ retraction if $A = B$. The elements of A and those of $B \setminus A$ will be called *old elements* and *new* elements, respectively. For each new element b, we take an unknown x_b. For each pair $(a, b) \in B \times B$ of elements such that at least one of a and b is new, we define an equation $E_{\mathrm{join}}(a, b)$ according to the following six rules.

If a is old, b is new, and $a \vee b$ is old, then $E_{\mathrm{join}}(a, b)$ is $a \vee x_b \approx a \vee b$. (2.1)

If a is new, b is old, and $a \vee b$ is old, then $E_{\mathrm{join}}(a, b)$ is $x_a \vee b \approx a \vee b$. (2.2)

If a and b are new and $a \vee b$ is old, then $E_{\mathrm{join}}(a, b)$ is $x_a \vee x_b \approx a \vee b$. (2.3)

If a is old, b and $a \vee b$ are new, then $E_{\mathrm{join}}(a, b)$ is $a \vee x_b \approx x_{a\vee b}$. (2.4)

If a and $a \vee b$ are new and b is old, then $E_{\mathrm{join}}(a, b)$ is $x_a \vee b \approx x_{a\vee b}$. (2.5)

If a, b, and $a \vee b$ are all new, then $E_{\mathrm{join}}(a, b)$ is $x_a \vee x_b \approx x_{a\vee b}$. (2.6)

Analogously, replacing $\vee$ by $\wedge$, we define the equations $E_{\mathrm{meet}}(a, b)$ for all $(a, b) \in B \times B$ such that at least one of a and b is a new element. Let $\widehat{E}$ be the system of all equations we have defined so far. Note that if $\mathcal{X}$ consists of finite lattices, then $\widehat{E}$ is finite.

Clearly, $\widehat{E}$ has a solution in B. Indeed, we can let $x_b := b$ for all new elements b to obtain a solution of $\widehat{E}$. Since we have assumed that either A is strongly algebraically closed in $\mathcal{X}$ or $\mathcal{X}$ consists of finite lattices and A is algebraically closed in $\mathcal{X}$, it follows that $\widehat{E}$ also has a solution in A. This allows us to fix a solution of $\widehat{E}$ in A. That is, we can choose an element $u_b \in A$ for each new element b such that the equations (2.1)–(2.6) turn into true equalities when the unknowns x_b, for $b \in B \setminus A$, are replaced by the elements u_b.

Next, consider the map

$$f \colon B \to A, \text{ defined by } c \mapsto \begin{cases} c, & \text{if } c \text{ is an old element,} \\ u_c, & \text{if } c \text{ is a new element.} \end{cases}$$

We claim that f is a retraction. Clearly, f acts identically on A. So we need only to show that f is a homomorphism. It suffices to verify that f commutes with joins since the case of meets is analogous. If $a, b \in A$, then $a \vee b$ is also in A, and we have that $f(a) \vee f(b) = a \vee b = f(a \vee b)$, as required. If, say, $a, a \vee b \in A$ and $b \in B \setminus A$, then (2.1) applies and we obtain that $f(a) \vee f(b) = a \vee u_b = a \vee b = f(a \vee b)$, as required. If $a, b, a \vee b$ are all new, then we can use (2.6) to obtain that $f(a) \vee f(b) = u_a \vee u_b = u_{a\vee b} = f(a \vee b)$, as required. The rest of the cases follow similarly from (2.2)–(2.5). Thus, we conclude that f commutes with joins. We obtain analogously that it commutes with meets, whereby f is a homomorphism. So f is a retraction, proving that (1) $\Rightarrow$ (2) and, if $\mathcal{X}$ consists of finite lattices, (3) $\Rightarrow$ (2).

To prove the implication, $(2) \Rightarrow (1)$, assume that $A \in \mathcal{X}$ is an absolute **H**-retract for $\mathcal{X}$, $B \in \mathcal{X}$ is an $\mathcal{X}$-extension of A, and a system $\widehat{G}$ of equations with constants taken from A has a solution in B.

Let $x, y, z, \ldots$ denote the unknowns occurring in $\widehat{G}$ (possibly, infinitely many), and let $b_x, b_y, b_z, \cdots \in B$ form a solution of $\widehat{G}$. Since we have assumed that A is an absolute **H**-retract for $\mathcal{X}$, we can take a retraction $f \colon B \to A$. We define $d_x := f(b_x)$, $d_y := f(b_y)$, $d_z := f(b_z)$, $\ldots$; they are elements of A. Let $p(a_1, \ldots, a_k, x, y, z, \ldots) = q(a_1, \ldots, a_k, x, y, z, \ldots)$ be one of the equations of $\widehat{G}$; here p and q are lattice terms, the constants $a_1, \ldots, a_k$ are in A, and only finitely many unknowns occur in this equation, of course. Using that f commutes with lattice terms and, at $=^*$, using also that $b_x, b_y, b_z, \ldots$ form a solution of the equation in question, we obtain that

$$
\begin{aligned}
p(a_1, \ldots, a_k, d_x, d_y, d_z, \ldots) &= p(f(a_1), \ldots, f(a_k), f(b_x), f(b_y), f(b_z), \ldots) \\
&= f(p(a_1, \ldots, a_k, b_x, b_y, b_z, \ldots)) =^* f(q(a_1, \ldots, a_k, b_x, b_y, b_z, \ldots)) = \\
q(f(a_1), \ldots, f(a_k), f(b_x), f(b_y), f(b_z), \ldots) &= q(a_1, \ldots, a_k, d_x, d_y, d_z, \ldots).
\end{aligned}
$$

This shows that $d_x, d_y, d_z, \cdots \in A$ form a solution of $\widehat{G}$ in A. Therefore, A is strongly algebraically closed in $\mathcal{X}$, showing the validity of $(2) \Rightarrow (1)$.

Finally, the implication $(1) \Rightarrow (3)$ is trivial, completing the proof of Proposition 1.1. $\square$

3. Proving Theorem 1.2

First, we recall briefly from Czédli and Schmidt [12] what we need to know about slim semimodular lattices. Let us repeat that slim lattices are finite by definition; every lattice in this section is assumed to be *finite*. For a slim semimodular lattice L, we always assume that a planar diagram of L is fixed. A cover-preserving four-element boolean sublattice of L is called a 4-*cell*. For $m, n \in \mathbb{N}^+$, the direct product of an $(m+1)$-element chain and an $(n+1)$-element chain is called a *grid* or, when we want to be more precise, an *m-by-n grid*; note that this grid has exactly mn 4-cells.

We can add a *fork* to a 4-cell of a slim semimodular lattice as it is shown in Figure 5 of [12]; this is also shown here in Figure 1, where we have added a fork to the light-grey 4-cell of $S_7^{(1)}$ to obtain $S_7^{(2)}$, and in Figure 2, where we can obtain R from the grid G by adding a fork to the upper 4-cell of G. *Corners* are particular doubly irreducible elements on the boundary of L, see Figure 2 in [12], but we do not need their definition here. Instead of the exact definition of slim rectangular lattices, it suffices to know their characterization, which is given by (the last sentence of) Theorem 11 and Lemma 22 in [12] as follows:

> L is a *slim rectangular* lattice if and only if it can be obtained from a grid by adding forks, one by one, in a finite (possibly zero) number of steps.   (3.1)

We know from Lemma 21 of [12] that

> a lattice L is a slim semimodular lattice if and only if $|L| \leq 2$ or L can be obtained from a slim rectangular lattice by removing finitely many corners, one by one.   (3.2)
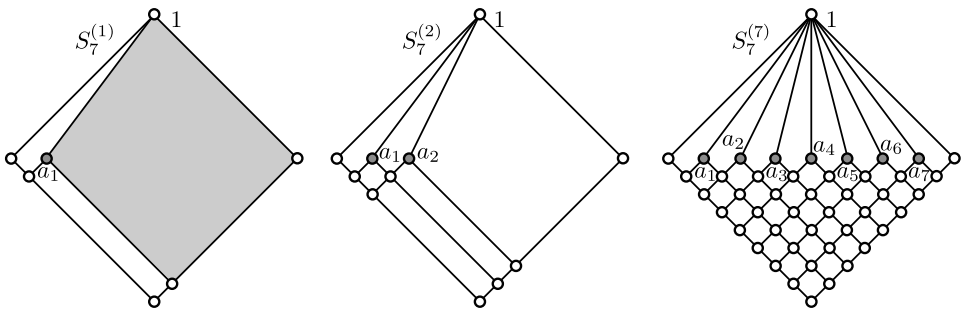
FIGURE 1. $S_7^{(1)}$, $S_7^{(2)}$, and $S_7^{(7)}$

Proof of Theorem 1.2. Since slim semimodular lattices are finite by definition, the equivalence of (1) and (2) follows trivially from Proposition 1.1. Also, Proposition 1.1 yields the equivalence of (2) and (3). Since the one-element lattice is an absolute retract for any class of lattices containing it, the implication (4) ⇒ (3) is trivial.

Thus, it suffices to prove the implication (3) ⇒ (4). To do so, it is sufficient to prove that whenever $L \in \mathcal{S}$ and $|L| \geq 2$, then L is not an absolute retract for $\mathcal{S}$. So let L be a slim semimodular lattice with at least two elements. By (3.2) (or trivially if $|L| = 2$), we can pick a slim rectangular lattice R such that L is a sublattice of R. It follows from (3.1) that there exist $m, n \in \mathbb{N}^+$ such that R can be obtained from an m-by-n grid G by adding forks, one by one. Let $t \in \mathbb{N}^+$ denote the smallest number such that $m + n + 1 \leq t$ and $|L| < t$.

To present an example that helps the reader follow the proof, let L be the 9-element slim semimodular lattice on the top left of Figure 2. For this L, we define R and G by the top right diagram and the bottom right diagram of Figure 2, respectively, and we have that $m = 2$, $n = 1$, and $t = 10$.

We define the lattices $S_7^{(i)}$ for $i \in \mathbb{N}^+$ by induction as follows; see Figure 1 for $i \in \{1, 2, 7\}$, and see the diagram in the middle of Figure 2 for $i = 10$ if we disregard the black-filled elements. (That is, $S_7^{(i)} = K \setminus \{\text{black-filled elements}\}$ in this diagram.) Resuming the definition of the lattices $S_7^{(i)}$, we obtain $S_7^{(1)}$ by adding a fork to the only 4-cell of the four-element boolean lattice. From $S_7^{(i)}$, we obtain $S_7^{(i+1)}$ by adding a fork to the rightmost 4-cell of $S_7^{(i)}$ that contains 1, the largest element of $S_7^{(i)}$. (Note that we have also defined a fixed planar diagram of $S_7^{(i)}$ in this way.) The elements of $S_7^{(i)}$ (or those of a planar lattice diagram) not on the boundary of the diagram are called *inner elements*. Let $a_1, a_2, \ldots, a_i$ be the inner coatoms of $S_7^{(i)}$, listed from left to right. In our diagrams, they are grey-filled. From now on, we only need $S_7^{(t)}$. It follows from (3.1) that $S_7^{(t)}$ is a slim semimodular (in fact, a slim rectangular) lattice. The meet $a_1 \wedge \cdots \wedge a_t$ of its inner coatoms will be denoted by b, as it is indicated in Figure 2.

Since $m+n+1 \leq t$, the interval $[b, a_{m+1}]$ of $S_7^{(t)}$ includes an m-by-n grid G' with top element a_{m+1}. In our example, G' is indicated by the light-grey area in the sense that G' consists of those six elements of $S_7^{(t)} = S_7^{(10)}$ that are on the boundary of the light-grey rectangle. (Remember that $S_7^{(10)} = K \setminus \{\text{black-filled elements}\}$ in the middle of Figure 2.) Since the grids G' and G have the same "sizes", they
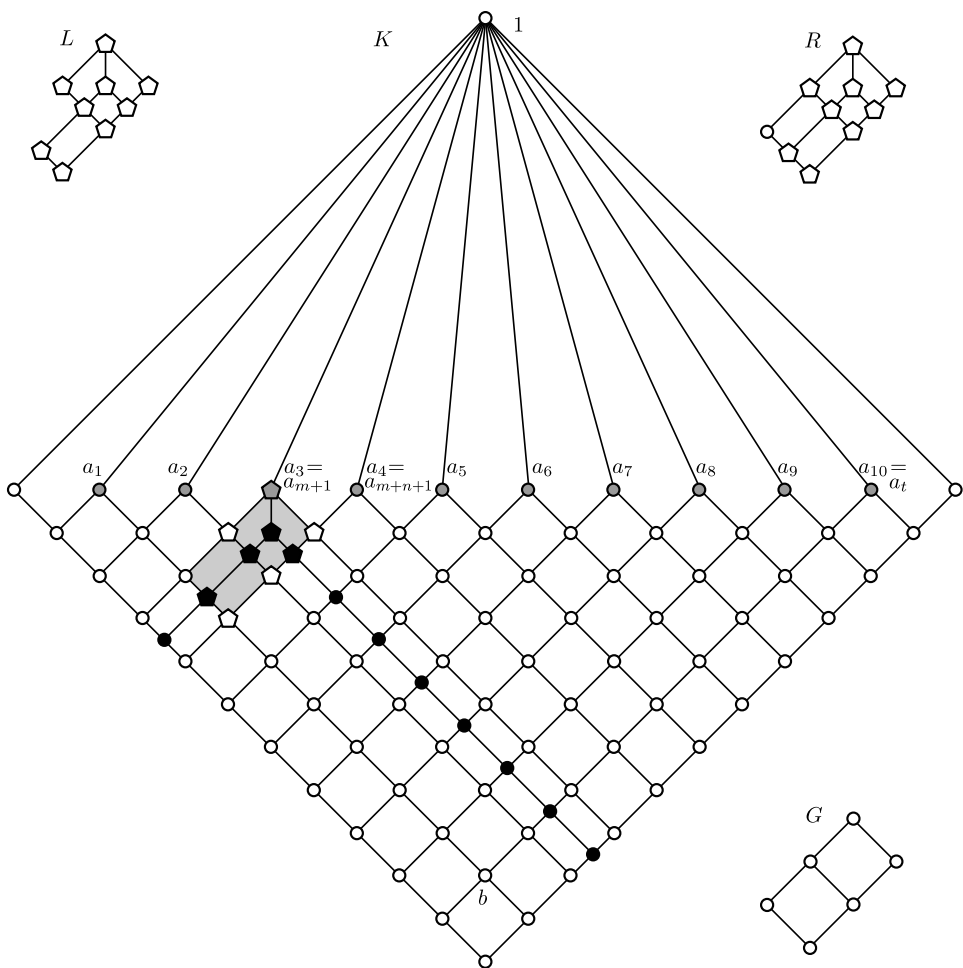
FIGURE 2. Illustrating the proof of Theorem 1.2

are isomorphic. Thinking of the diagrams, we can even assume that G' and G are geometrically congruent. Hence, when we add forks to G one by one in order to get R, we can simultaneously add forks to G' in the same way and, consequently, also to $S_7^{(t)}$. In this way, we obtain a slim rectangular lattice K from $S_7^{(t)}$; this follows from (3.1). Note that $K \in \mathcal{S}$. In the middle of Figure 2, K consists of the empty-filled elements, the grey-filled elements, and the black-filled elements. In K, the former interval G' has become an interval isomorphic to R. But R is an extension of L, whereby K has a sublattice L' such that L' is isomorphic to L. In the middle of the figure, the elements of L' are the pentagon-shaped larger elements. Note that the original inner coatoms $a_1, \ldots, a_t$ are also inner coatoms of K.

Next, for the sake of contradiction, suppose that L is an absolute retract for $\mathcal{S}$. Then so is L' since $L' \cong L$. Since $K \in \mathcal{S}$ and L' is a sublattice of K, there exists a retraction $f \colon K \to L'$. Let $\Theta := \{(x, y) \in K^2 : f(x) = f(y)\}$ be the kernel of f. Then Θ is a congruence of K with exactly $|L'|$ blocks. But $t > |L| = |L'|$, whence there are distinct $i, j \in \{1, \ldots, t\}$ such that a_i and a_j belong to the same

Θ-block. Hence, $(a_i, a_j) \in \Theta$, implying that $(a_i, 1) = (a_i \vee a_i, a_j \vee a_i) \in \Theta$. Thus, the Θ-block $1/\Theta$ of 1 contains a_i. By Grätzer's Swing Lemma, see his paper [17] (alternatively, see Czédli, Grätzer, and Lakser [8] or Czédli and Makay [10] for secondary sources), $\{a_1, \ldots, a_t\} \subseteq 1/\Theta$. Since congruence blocks are sublattices, $b = a_1 \wedge \cdots \wedge a_t \in 1/\Theta$. Therefore, using the facts that congruence blocks are *convex* sublattices, $a_{m+1} \in 1/\Theta$, and G' was originally a subinterval of $[b, a_{m+1}]$ in $S_7^{(t)}$, we obtain that $L' \subseteq [b, a_{m+1}] \subseteq 1/\Theta$ in the lattice K. Hence, for any $x, y \in L'$, we have that $(x, y) \in \Theta$. Consequently, the definition of Θ and that of a retraction yield that, for any $x, y \in L'$, $x = f(x) = f(y) = y$. Therefore, $|L| = |L'| = 1$, which is a contradiction. This contradiction implies that neither L', nor L is an absolute retract for $\mathcal{S}$, completing the proof of Theorem 1.2. □

4. Proving Theorem 1.3 and its corollaries

4.1. **Notes before the proof.** This subsection is to enlighten the way from Theorem 1.2 to Theorem 1.3. The reader is not expected to check the in-line statements in this subsection; what will be needed later will be proved or referenced in due course.

In the proof of Theorem 1.2, forks play a crucial role. This raises the question what happens if forks are excluded from (3.1). It follows easily from Czédli and Schmidt [12, Lemma 15] that the lattices we obtain by means of (3.1) and (3.2) *without* adding forks are exactly the members of $\mathcal{D}(2)$. But $\mathcal{D}(2)$ is the class of *distributive* slim semimodular lattices. Hence, utilizing the theory of slim semimodular lattices, the particular case $n = 2$ of Corollary 1.4 becomes available with little effort. Although this section is more ambitious by allowing $n \in \mathbb{N}^+ \cup \{\omega\}$ and aiming at Theorem 1.3 in addition to its corollaries, the ideas extracted from the theory of slim semimodular lattices and from the proof of Theorem 1.2 have been decisive in reaching Theorem 1.3.

4.2. **Auxiliary lemmas.** Unless otherwise explicitly stated, every lattice in this section is assumed to be *finite*. For an n-dimensional grid G and a maximal element $a \in J(G)$, the principal ideal $\downarrow a$ is a nontrivial chain. Chains of this form will be called the *canonical chains* of G. The following lemma follows trivially from the fact that in a direct product of finitely many finite chains we compute componentwise.

Lemma 4.1. *If $n \in \mathbb{N}^+$ and G is an n-dimensional grid, then the following assertions hold.*

(1) *G has exactly n canonical chains; in the rest of the lemma, they will be denoted by $C_1, \ldots, C_n$.*

(2) *Each element x of G can uniquely be written in the* canonical form

$$x = x[1] \vee \cdots \vee x[n] \quad \text{where } x[1] := x \wedge 1_{C_1} \in C_1, \ldots,$$
$$x[n] := x \wedge 1_{C_n} \in C_n; \text{ the elements } x[1], \ldots, x[n] \text{ are called} \quad (4.1)$$
$$\text{the canonical joinands of } x.$$

(3) *For each $i \in \{1, \ldots, n\}$, the map $\pi_i \colon G \to C_i$ defined by $x \mapsto x[i]$ is a surjective homomorphism.*

(4) *The map $G \to C_1 \times \cdots \times C_n$ defined by $x \mapsto (x[1], \ldots, x[n])$ is a lattice isomorphism.*

The notation $x[1], \ldots, x[n]$ will frequently be used, provided the canonical chains of an n-dimensional grid are fixed. The map π_i above is often called the *i-th*

projection. Note that, for an n-dimensional grid G, $J(G)$ is the disjoint union of $C_1 \setminus \{0\}$, ..., $C_n \setminus \{0\}$. Thus, the *set* $\{C_1, \ldots, C_n\}$ of the canonical chains is uniquely determined, and only the *order* of these chains needs fixing. We also need the following lemma; the sublattices of a chain are called *subchains*.

Lemma 4.2. *Assume that $n \in \mathbb{N}^+$, L and K are n-dimensional grids, and L is a sublattice of K. Then there are nontrivial subchains E_1, ..., E_n of the canonical chains C_1, ..., C_n of K, respectively, such that*

$$L = \{x \in K : x[1] \in E_1, \ldots, x[n] \in E_n\}. \tag{4.2}$$

The visual meaning of Lemma 4.2 is that an n-dimensional grid cannot be embedded into another n-dimensional grid in a "skew way".

Proof of Lemma 4.2. Assume that $n \in \mathbb{N}^+$, L and K are n-dimensional grids, and L is a sublattice of K. Then there are integers $t_1 \geq 2$, ..., $t_n \geq 2$ and chains $H_i = \{0, 1, \ldots, t_i - 1\}$ (with the natural ordering of integer numbers) such that we can pick an isomorphism $\varphi \colon H_1 \times \cdots \times H_n \to L$. The canonical chains of K will be denoted by C_1, ..., C_n. The least element of $H_1 \times \cdots \times H_n$ and that of L are $\vec{0} := (0, \ldots, 0)$ and $0_L = \varphi(\vec{0})$, respectively. For $(i_1, \ldots, i_n) \in H_1 \times \cdots \times H_n$, we write $\varphi(i_1, \ldots, i_n)$ rather than the more precise $\varphi((i_1, \ldots, i_n))$. For $j \in \{1, \ldots, n\}$ and $i \in H_j \setminus \{0\}$, we are going to use the notation

$$a_i^{(j)} := (\underbrace{0, \ldots, 0}_{j-1 \text{ zeros}}, i, \underbrace{0, \ldots, 0}_{n-j \text{ zeros}}), \tag{4.3}$$

Clearly,

$$J(H_1 \times \cdots \times H_n) = \{a_i^{(j)} : j \in \{1, \ldots, n\} \text{ and } i \in H_i \setminus \{0\}\}. \tag{4.4}$$

It is also clear that the atoms of $H_1 \times \cdots \times H_n$ are $a_1^{(1)}$, ..., $a_1^{(n)}$. With the notation given in (4.1), for $j \in \{1, \ldots, n\}$ we let

$$I_j := \{i \in \{1, \ldots, n\} : \varphi(a_1^{(j)})[i] > 0_L[i]\}. \tag{4.5}$$

Since $a_1^{(j)} > \vec{0}$ and φ is an isomorphism, $I_j \neq \emptyset$. We claim that

$$\text{if } j \neq k \in \{1, \ldots, n\}, \text{ then } I_j \cap I_k = \emptyset. \tag{4.6}$$

For the sake of contradiction, suppose that $j \neq k$ but $i \in I_j \cap I_k$. Then $\varphi(a_1^{(j)})[i] > 0_L[i]$ and $\varphi(a_1^{(k)})[i] > 0_L[i]$. Since j and k play a symmetrical role and the elements $\varphi(a_1^{(j)})[i]$ and $\varphi(a_1^{(k)})[i]$ belonging to the same canonical chain C_i of K are comparable, we can assume that $0_L[i] < \varphi(a_1^{(j)})[i] \leq \varphi(a_1^{(k)})[i]$. Hence, using Lemma 4.1(3),

$$\varphi(a_1^{(j)})[i] = \varphi(a_1^{(j)})[i] \wedge \varphi(a_1^{(k)})[i] = \left(\varphi(a_1^{(j)}) \wedge \varphi(a_1^{(k)})\right)[i]$$
$$= \varphi(a_1^{(j)} \wedge a_1^{(k)})[i] = \varphi(\vec{0})[i] = 0_L[i],$$

contradicting (4.5) and proving (4.6). Using that I_1, ..., I_n are nonempty subsets of the finite set $\{1, \ldots, n\}$ and they are pairwise disjoint by (4.6), we have that

$$n \leq |I_1| + \cdots + |I_n| = |I_1 \cup \cdots \cup I_n| \leq |\{1, \ldots, n\}| = n.$$

Hence, none of the I_1, ..., I_j can have more than one element, and we obtain that $|I_1| = \cdots = |I_n| = 1$. Therefore, after changing the order of the direct factors in

$H_1 \times \cdots \times H_n$ and so also the order of the atoms $a_1^{(1)}$, ..., $a_1^{(n)}$ if necessary, we can write that $I_1 = \{1\}$, ..., $I_n = \{n\}$. This means that, for all $j, k \in \{1, \ldots, n\}$,

$$\varphi(a_1^{(j)})[k] \geq 0_L[k], \text{ and } \varphi(a_1^{(j)})[k] > 0_L[k] \iff k = j. \qquad (4.7)$$

Next, we generalize (4.7) by claiming that for $j, k \in \{1, \ldots, n\}$ and $i \in H_j \setminus \{0\}$,

$$\varphi(a_i^{(j)})[k] \geq 0_L[k], \text{ and } \varphi(a_i^{(j)})[k] > 0_L[k] \iff k = j. \qquad (4.8)$$

To prove this, we can assume that $i > 1$ since otherwise (4.7) applies. Using (4.7) together with the fact that π_k and π_j defined in Lemma 4.1(3) are order-preserving, we obtain that $\varphi(a_i^{(j)})[k] \geq \varphi(a_1^{(j)})[k] \geq 0_L[k]$ for all $k \in \{1, \ldots, n\}$, as required, and $\varphi(a_i^{(j)})[j] \geq \varphi(a_1^{(j)})[j] > 0_L[j]$. So all we need to show is that $\varphi(a_i^{(j)})[k] > 0_L[k]$ is impossible if $k \neq j$. For the sake of contradiction, suppose that $k \neq j$, $k, j \in \{1, \ldots, n\}$, and $\varphi(a_i^{(j)})[k] > 0_L[k]$. We also have that $\varphi(a_1^{(k)})[k] > 0_L[k]$ by (4.7). Belonging to the same canonical chain of K, the elements $\varphi(a_i^{(j)})[k]$ and $\varphi(a_1^{(k)})[k]$ are comparable, whence their meet is one of the meetands. Thus, $\varphi(a_i^{(j)})[k] \wedge \varphi(a_1^{(k)})[k] > 0_L[k]$. Hence, using that φ and π_k are homomorphisms and $a_i^{(j)} \wedge a_1^{(k)} = \vec{0}$, we obtain that

$$0_L[k] < \varphi(a_i^{(j)})[k] \wedge \varphi(a_1^{(k)})[k] = \big(\varphi(a_i^{(j)}) \wedge \varphi(a_1^{(k)})\big)[k]$$
$$= \varphi(a_i^{(j)} \wedge a_1^{(k)})[k] = \varphi(\vec{0})[k] = 0_L[k],$$

which is a contradiction proving (4.8).

Next, after extending the notation given in (4.3) by letting $a_0^{(j)} := \vec{0}$ for $j \in \{1, \ldots, n\}$, we have that

$$\varphi(a_i^{(k)})[k] \geq 0_L[k] \text{ for all } k \in \{1, \ldots, n\} \text{ and } i \in H_k \qquad (4.9)$$

since $a_i^{(k)} \geq a_0^{(k)} = \vec{0}$, φ and π_k are order-preserving maps, and $0_L = \varphi(\vec{0})$. For $j \in \{1, \ldots, n\}$, we define

$$E_j := \{\varphi(a_i^{(j)})[j] : i \in H_j\}. \qquad (4.10)$$

By (4.1), $E_j \subseteq C_j$, that is, E_j is a subchain of C_j for all $j \in \{1, \ldots, n\}$. We are going to show that these E_j satisfy (4.2).

First, assume that $x \in K$ is of the form $x = x[1] \vee \cdots \vee x[n]$ such that $x[j] \in E_j$ for all $j \in \{1, \ldots, n\}$. Then, for each $j \in \{1, \ldots, n\}$, there is an $i(j) \in H_j$ such that $x[j] = \varphi(a_{i(j)}^{(j)})[j]$. Using what we already have, let us compute:

$$x = x[1] \vee \cdots \vee x[n] = \varphi(a_{i(1)}^{(1)})[1] \vee \cdots \vee \varphi(a_{i(n)}^{(n)})[n] \qquad (4.11)$$

$$\overset{(4.9)}{=} \varphi(a_{i(1)}^{(1)})[1] \vee \cdots \vee \varphi(a_{i(n)}^{(n)})[n] \vee 0_L[1] \vee \cdots \vee 0_L[n]$$

$$\overset{(4.8)}{=} \varphi(a_{i(1)}^{(1)})[1] \vee \cdots \vee \varphi(a_{i(n)}^{(n)})[n]$$

$$\vee \varphi(a_{i(1)}^{(1)})[2] \vee \cdots \vee \varphi(a_{i(1)}^{(1)})[n]$$

$$\vee \cdots \vee \varphi(a_{i(n)}^{(n)})[1] \vee \cdots \vee \varphi(a_{i(n)}^{(n)})[n-1]$$

$$\overset{(4.1)}{=} \varphi(a_{i(1)}^{(1)}) \vee \cdots \vee \varphi(a_{i(n)}^{(n)}) = \varphi(a_{i(1)}^{(1)} \vee \cdots \vee a_{i(n)}^{(n)}) \qquad (4.12)$$

Since $\varphi(a_{i(1)}^{(1)} \vee \cdots \vee a_{i(n)}^{(n)}) \in \varphi(H_1 \times \cdots \times H_n) = L$, the computation from (4.11) to (4.12) shows the "$\supseteq$" part of (4.2).

To show the reverse inclusion, assume that $x \in L$. Applying Lemma 4.1(2) to the φ-preimage of x (or trivially since we are in a direct product), we obtain the existence of $i(1), \ldots, i(n)$ such that $x = \varphi(a_{i(1)}^{(1)} \vee \cdots \vee a_{i(n)}^{(n)})$. Reading the computation from (4.12) to (4.11) upward, it follows that $x = \varphi(a_{i(1)}^{(1)})[1] \vee \cdots \vee \varphi(a_{i(n)}^{(n)})[n]$. By the uniqueness part of Lemma 4.1(2), $x[1] = \varphi(a_{i(1)}^{(1)})[1], \ldots, x[n] = \varphi(a_{i(n)}^{(n)})[n]$. Combining this with (4.10), we have that $x[1] \in E_1, \ldots, x[n] \in E_n$. This yields the "$\subseteq$" inclusion for (4.2) and completes the proof of Lemma 4.2. $\square$

The following easy lemma sheds more light on the categories $\mathcal{D}_{01\prec}(n)$, $n \in \mathbb{N}^+ \cup \{\omega\}$. The *length* of a lattice M is denoted by $\mathrm{length}(M)$; for definition (in the finite case) we mention that if C is a maximum-sized chain in M, then $\mathrm{length}(M) + 1 = |C|$.

Lemma 4.3. *Assume that K, L are finite semimodular lattices (in particular, finite distributive lattices) and $f\colon K \to L$ is a map. Then the following three assertions hold.*

(1) *If f is a cover-preserving $\{0,1\}$-homomorphism, then f is a cover-preserving $\{0,1\}$-embedding and $\mathrm{length}(K) = \mathrm{length}(L)$.*
(2) *If f is a lattice embedding and $\mathrm{length}(K) = \mathrm{length}(L)$, then f is a cover-preserving $\{0,1\}$-homomorphism.*
(3) *If L is a sublattice of K such that the map $L \to K$ defined by $x \mapsto x$ is a cover-preserving $\{0,1\}$-embedding and f is a retraction, then f is a lattice isomorphism (and, in particular, f is also a cover-preserving $\{0,1\}$-embedding).*

Proof. First, recall the following concept. A sublattice S of a lattice M is a *congruence-determining sublattice* of M if any congruence α of M is uniquely determined by its restriction $\alpha\lceil_S := \{(x,y) \in S^2 : (x,y) \in \alpha\}$. By Grätzer and Nation [22],

$$\text{every maximal chain of a finite semimodular} \atop \text{lattice is a congruence-determining sublattice.} \qquad (4.13)$$

To prove part (1), let $f\colon K \to L$ be a cover-preserving $\{0,1\}$-homomorphism. We know from (1.6) that $\mathrm{length}(K) = \mathrm{length}(L)$. Let $\Theta := \{(x,y) \in K^2 : f(x) = f(y)\}$ be the kernel of f, and take a maximal chain C in K. For $c, d \in C$ such that $c \prec d$, we have that $(c,d) \notin \Theta$ since $f(c) \prec f(d)$. Hence, using that the blocks of $\Theta\lceil_C$ are convex sublattices of C, it follows that $\Theta\lceil_C = \Delta_C$. Applying (4.13), we have that $\Theta = \Delta_K$. Thus, f is injective, proving part (1).

We prove the "lion's share" of part (2) by way of contradiction. Suppose that in spite of the assumptions, f is not cover-preserving. Pick $a, b \in K$ such that $a \prec b$ but $f(a) \nprec f(b)$. The injectivity of f rules out that $f(a) = f(b)$. Hence, the interval $[f(a), f(b)]$ is of length at least 2. Extend $\{a, b\}$ to a maximal chain $C = \{0 = c_0, c_1, \ldots, c_k = 1\}$ of K such that $a = c_{i-1}$, $b = c_i$, and $c_0 \prec c_1 \prec \cdots \prec c_k$. By the Jordan–Hölder chain condition, $k = \mathrm{length}(K)$. Using the injectivity of φ again and the fact that φ is order-preserving, $\mathrm{length}([f(c_{j-1}), f(c_j)]) \geq 1$ for all

$j \in \{1, \ldots, k\}$. So the summands in

$$\text{length}(L) \geq \sum_{j=1}^{k} \text{length}([f(c_{j-1}), f(c_j)]) \tag{4.14}$$

are positive integers but the i-th summand is at least two. Therefore, this sum and $\text{length}(K) = \text{length}(L)$ are at least $k+1$, which is a contradiction. This shows that f is cover-preserving. In particular, with C as above, we have that $f(0) = f(c_0) \prec f(c_1) \prec \cdots \prec f(c_{k-1}) \prec f(c_k) = f(1)$. Hence, $\text{length}([f(0), f(1)]) = k = \text{length}(K) = \text{length}(L)$, implying that $f(0) = 0$ and $f(1) = 1$. Thus, f is a cover-preserving $\{0, 1\}$-homomorphism, completing the proof of part (2).

Next, to prove part (3), observe that $0_L = 0_K$ and $1_L = 1_K$. Hence, since f is a retraction, $f(0_K) = 0_L$ and $f(1_K) = 1_L$, as required. We are going to show that whenever $a \prec b$ in K, then $f(a) \prec f(b)$ in L. For the sake of contradiction, suppose that $a \prec b$ in K but $f(a) \not\prec f(b)$ in L. Then there are two cases (since f is order-preserving): either we have that $f(a) = f(b)$, or $f(a) < f(b)$ and the length of the interval $[f(a), f(b)]$ is at least 2. For each of these two cases, let Θ denote the kernel $\{(x, y) \in K^2 : f(x) = f(y)\}$ of f, and let $U = \{0 = u_0, u_1, \ldots, u_k = 1\}$ be a maximal chain of L. It is also a maximal chain of K since the embedding $L \to K$ defined by $x \mapsto x$ is a cover-preserving $\{0, 1\}$-homomorphism. We know from the Jordan–Hölder chain condition that $\text{length}(K) = k = \text{length}(L)$.

First, we deal with the first case, $f(a) = f(b)$. Then $(a, b) \in \Theta$ shows that $\Theta \neq \Delta_K$. We have that $\Theta \lceil_U \neq \Delta_U$ since (4.13) applies. Using that the blocks of $\Theta \lceil_U$ are convex sublattices of U, it follows that $(u_{i-1}, u_i) \in \Theta \lceil_U$ for some $i \in \{1, \ldots, k\}$. This means that $f(u_{i-1}) = f(u_i)$. This equality leads to a contradiction since f is a retraction and so $u_{i-1} = f(u_{i-1}) = f(u_i) = u_i$. Since the only conditions tailored to a and b were $a \prec b$ and $f(a) = f(b)$, we have also obtained that

$$\text{if } a' \prec b' \text{ in } K, \text{ then } f(a') \neq f(b'). \tag{4.15}$$

Next, we focus on the case $f(a) < f(b)$ and $\text{length}([f(a), f(b)]) \geq 2$. As in the proof of part (2), we can extend $\{a, b\}$ to a maximal chain C of K. Since U is also a maximal chain of K, the Jordan–Hölder chain condition gives that $\text{length}(C) = \text{length}(K) = \text{length}(U) = k$. This allows us to write that $C = \{0 = c_0, c_1, \ldots, c_k = 1\}$ where $a = c_{i-1}$ and $b = c_i$ for some $i \in \{1, \ldots, k\}$, and $c_0 \prec c_1 \prec \cdots \prec c_k$. By the Jordan–Hölder chain condition, (4.14) is still valid. Each summand in (4.14) is at least 1 by (4.15), but the i-th summand is $\text{length}([f(c_{i-1}), f(c_i)]) = \text{length}([f(a), f(b)]) \geq 2$. Hence, $k = \text{length}(K) = \text{length}(L) \geq k+1$, which is a contradiction again. In this way, we have shown that f is a cover-preserving $\{0, 1\}$-homomorphism.

Applying the already proven part (1) of Lemma 4.3, we obtain that f is a cover-preserving $\{0, 1\}$-embedding. This yields that $|K| \leq |L|$. But we also have that $|L| \leq |K|$ since L is a sublattice of K. Thus, $|K| = |L|$, whence the embedding f is a lattice isomorphism since these lattices are finite. This completes the proof of part (3) and that of Lemma 4.3. □

Lemma 4.4. *If L is a nontrivial finite distributive lattice with order dimension $n \in \mathbb{N}^+$, then there exists a cover-preserving $\{0, 1\}$-embedding of L into an n-dimensional grid G.*

Proof of Lemma 4.4. By (1.3), width($J(L)$) $= n$. It follows from Dilworth [13, Theorem 1.1], mentioned already in Subsection 1.4, that there are chains $C_1, \ldots, C_n$ in $J(L)$ such that $J(L) = C_1 \cup \cdots \cup C_n$. We define $E_1, \ldots, E_n$ by induction as follows:

$$E_1 := C_1 \text{ and, for } i \in \{2, \ldots, n\},\ E_i := C_i \setminus (C_1 \cup \cdots \cup C_{i-1}).$$

We show by an easy induction that

$$\text{for } i \in \{1, \ldots, n\},\ E_1 \cup \cdots \cup E_i = C_1 \cup \cdots \cup C_i, \text{ and the sets } E_1, \ldots, E_i \text{ are pairwise disjoint.} \tag{4.16}$$

Since this is trivial for $i = 1$, assume that $i \in \{2, \ldots, n\}$ and (4.16) holds for $i - 1$. Then $E_1 \cup \cdots \cup E_{i-1} \cup E_i = C_1 \cup \cdots \cup C_{i-1} \cup (C_i \setminus (C_1 \cup \cdots \cup C_{i-1})) = C_1 \cup \cdots \cup C_i$ shows the equality in (4.16) for i. The sets $E_1, \ldots, E_{i-1}$ are pairwise disjoint by the induction hypothesis, while E_i is disjoint from them because of $E_i := C_i \setminus (C_1 \cup \cdots \cup C_{i-1}) = C_i \setminus (E_1 \cup \cdots \cup E_{i-1})$. This shows the validity of (4.16).

Observe that none of $E_1, \ldots, E_n$ is empty. Indeed, $J(L) = C_1 \cup \cdots \cup C_n = E_1 \cup \cdots \cup E_n$ by (4.16), and so if one of $E_1, \ldots, E_n$ was empty, then $J(L)$ would be the union of less than n chains, contradicting Dilworth [13, Theorem 1.1].

Next, with $E_i^+ := E_i \cup \{0\}$ for $i \in \{1, \ldots, n\}$ and $0 = 0_L \notin E_i$, we define $G := E_1^+ \times \cdots \times E_n^+$. Since none of $E_1, \ldots, E_n$ is empty, we have that $|E_i^+| \geq 2$ and so G is an n-dimensional grid. Clearly, $|J(G)| = |E_1| + \cdots + |E_n|$. This equality and (4.16) give that $|J(G)| = |E_1 \cup \cdots \cup E_n| = |C_1 \cup \cdots \cup C_n| = |J(L)|$. We know from the folklore or from Grätzer [15, Corollary 112] that

$$\text{the length of a finite distributive lattice equals the number of its join-irreducible elements,} \tag{4.17}$$

$$\text{whereby } G \text{ and } L \text{ are of the same length.} \tag{4.18}$$

For $x \in L$ and $i \in \{1, \ldots n\}$, let x_i stand for the largest element of $E_i^+ \cap {\downarrow}x$; this makes sense since E_i^+ is a chain of L and $0 \in E_i^+ \cap {\downarrow}x$ shows that $E_i^+ \cap {\downarrow}x \neq \emptyset$. We are going to show that

$$\text{the map } \varphi \colon L \to G \text{ defined by the rule } x \mapsto (x_1, \ldots, x_n) \text{ is a lattice embedding.} \tag{4.19}$$

To prove (4.19), let $x, y \in L$. Denote $x \wedge y$ and $x \vee y$ by u and v, respectively. We have that $\varphi(x) = (x_1, \ldots, x_n)$ and $\varphi(y) = (y_1, \ldots, y_n)$. Here y_i is the largest element of $E_i^+ \cap {\downarrow}y$, and analogous notation applies for $\varphi(u)$ and $\varphi(v)$. Since the lattice operations in the direct product G are computed componentwise, we only need to show that, for every $i \in \{1, \ldots, n\}$, $x_i \wedge y_i = u_i$ and $x_i \vee y_i = v_i$. In fact, we only need to show that $x_i \wedge y_i \leq u_i$ and $x_i \vee y_i \geq v_i$ since the converse inequalities follow from the fact that φ is clearly order-preserving. Since x_i and y_i belong to the same chain, E_i^+, these two elements are comparable. They play a symmetrical role, whence we can assume that $x_i \leq y_i$. Thus, the equalities $x_i = x_i \wedge y_i$ and $y_i = x_i \vee y_i$ reduce our task to show that $x_i \leq u_i$ and $y_i \geq v_i$. Since $x_i \in E_i^+ \cap {\downarrow}x$ and $x_i \leq y_i$ yields that $x_i \in E_i^+ \cap {\downarrow}y$, we have that $x_i \in E_i^+ \cap {\downarrow}x \cap {\downarrow}y = E_i^+ \cap {\downarrow}(x \wedge y) = E_i^+ \cap {\downarrow}u$. Taking into account that u_i is the largest element of $E_i^+ \cap {\downarrow}u$, the required inequality $x_i \leq u_i$ follows. It belongs to the folklore of lattice theory (and it occurs in the last

paragraph of the proof of Theorem 107 in Grätzer [15]) that

> if D is a finite distributive lattice, $t \in \mathbb{N}^+$, $p \in \mathrm{J}(D)$,
> $q_1, \ldots, q_t \in D$, and $p \le q_1 \vee \cdots \vee q_t$, then there is an   (4.20)
> $i \in \{1, \ldots, t\}$ such that $p \le q_i$.

Indeed, if the premise of (4.20) holds, then $p = p \wedge (q_1 \vee \cdots \vee q_t) = (p \wedge q_1) \vee \cdots \vee (p \wedge q_t)$ and $p \in \mathrm{J}(D)$ yield that $p = p \wedge q_i$ for some i, implying the required $p \le q_i$. Resuming our argument for φ, we can assume that $v_i \ne 0$ since otherwise the required $y_i \ge v_i$ is trivial. Then we know that $v_i \le v = x \vee y$ and $v_i \in E_i \subseteq \mathrm{J}(L)$. Hence (4.20) gives that $v_i \le x$ or $v_i \le y$. If $v_i \le x$, then the definition of x_i yields that $v_i \le x_i$, whence $v_i \le y_i$. If $v_i \le y$, then the definition of y_i immediately yields that $v_i \le y_i$. So the required $y_i \ge v_i$ holds in both cases, and we have shown that φ is a lattice homomorphism.

Next, we claim that for each $x \in L$,

$$x = x_1 \vee \cdots \vee x_n. \qquad (4.21)$$

By finiteness, there is a nonempty subset H of $\mathrm{J}(L) \cup \{0\}$ such that $x = \bigvee H$. For each $h \in H$, (4.16) and $\mathrm{J}(L) = C_1 \cup \cdots \cup C_n$ yield an $i \in \{1, \ldots, n\}$ such that $h \in E_i^+$. Then we have that $h \in E_i^+ \cap {\downarrow}x$, whereby $h \le x_i \le x_1 \vee \cdots \vee x_n$. Since this holds for all $h \in H$, we have that $x = \bigvee H \le x_1 \vee \cdots \vee x_n$. The converse inequality is trivial, and we conclude (4.21). Clearly, (4.21) implies the injectivity of φ. Thus, we have shown (4.19).

Finally, (4.18), (4.19), and Lemma 4.3(2) imply that f is a cover-preserving $\{0,1\}$-homomorphism, completing the proof of Lemma 4.4.   □

Lemma 4.5. *If $n \in \mathbb{N}^+$, L is an n-dimensional grid, but L is not a boolean lattice, then L is a sublattice of an $(n+1)$-dimensional grid K such that K and L are of the same length.*

Proof. By the assumption, $L = C_1 \times \cdots \times C_n$ such that $C_1, \ldots, C_n$ are nontrivial chains and at least one of them consists of at least three elements. Up to isomorphism, the order of the direct factors is irrelevant, whereby we can assume that $|C_1| \ge 3$. Let q be the unique coatom of C_1. Then $0 < q \prec 1$ in C_1 and $E_0 := {\uparrow}q = \{q, 1\}$ is a two-element subchain of C_1. The subchain $E_1 := {\downarrow}q$ is still a nontrivial chain. Define $K := E_0 \times E_1 \times C_2 \times \cdots \times C_n$. It is an $(n+1)$-dimensional grid. Since $\mathrm{J}(L)$ consists of the vectors with exactly one nonzero component and similarly for $\mathrm{J}(K)$, $|\mathrm{J}(L)| = (|C_1| - 1) + (|C_2| - 1) + \cdots + (|C_n| - 1) = (|E_0| - 1) + (|E_1| - 1) + (|C_2| - 1) + \cdots + (|C_n| - 1) = |\mathrm{J}(K)|$. Hence, (4.17) gives that L and K are of the same length. We are going to show that L can be embedded into K.

Instead of defining an injective homomorphism $L \to K$ and verifying its properties in a tedious way, recall the following. If H_1 and H_2 are lattices, F_1 is a filter of H_1, I_2 is an ideal of H_2, and $\psi \colon F_1 \to I_2$ is a lattice isomorphism, then the quintuplet $(H_1, H_2, F_1, I_2, \psi)$ uniquely determines a lattice H by identifying x with $\psi(x)$, for all $x \in F_1$, in $H_1 \cup H_2$. This H is the well-known *Hall–Dilworth gluing* of H_1 and H_2 or, to be more precise, the Hall–Dilworth gluing determined by the quintuplet; see, for example, Grätzer [15, Lemma 298] for more details. Furthermore, it is also

well known, see Grätzer [15, Lemma 299], that

> if M is a lattice, M_1 is an ideal of M, M_2 is a filter of M, and $T := M_1 \cap M_2 \neq \emptyset$, then $M_1 \cup M_2$ is a sublattice of M and M is isomorphic to the Hall–Dilworth gluing determined by $(M_1, M_2, T, T, \mathrm{id}_T)$, (4.22)

where $\mathrm{id}_T \colon T \to T$ is the *identity map* defined by $x \mapsto x$.

In the rest of *this* proof, $\vec{0}$ and $\vec{1}$ will stand for $(0_{C_2}, \ldots, 0_{C_n}) \in C_2 \times \cdots \times C_n$ and $(1_{C_2}, \ldots, 1_{C_n}) \in C_2 \times \cdots \times C_n$, respectively. In L, we let $I_L := {\downarrow}(q, \vec{1})$, $F_L := {\uparrow}(q, \vec{0})$, and $T_L := I_L \cap F_L = [(q, \vec{0}), (q, \vec{1})]$. In K, we let $I_K := {\downarrow}(q, q, \vec{1})$; remember that the first q here is the least element of E_0 while the second q is the largest element of E_1. Still in K, we also let $F_K := {\uparrow}(q, q, \vec{0})$ and $T_K := I_K \cap F_K = [(q, q, \vec{0}), (q, q, \vec{1})]$. Clearly, the map $\rho \colon I_L \to, I_K$ defined by $(x, \vec{y}) \mapsto (q, x, \vec{y})$ is an isomorphism. Let

$$\tau \colon F_L \to F_K \text{ be defined by } (x, \vec{y}) \mapsto (x, q, \vec{y});$$

it is also an isomorphism. We have to check that the restrictions $\rho{\restriction}_{T_L}$ and $\tau{\restriction}_{T_L}$ are the same maps and they are $T_L \to T_K$ isomorphisms. But this is clear since $T_L = \{(q, \vec{y}) : \vec{y} \in C_2 \times \cdots \times C_n\}$ and $T_K = \{(q, q, \vec{y}) : \vec{y} \in C_2 \times \cdots \times C_n\}$. Hence, it follows from (4.22) that $I_K \cup F_K$ is a sublattice of K. It also follows from (4.22) that L, which is the Hall–Dilworth gluing determined by $(I_L, F_L, T_L, T_L, \mathrm{id}_{T_L})$, is isomorphic to this sublattice. Therefore, after replacing K by an isomorphic copy if necessary, we conclude that L is a sublattice of K, proving Lemma 4.5.   □

4.3. **Our lemmas at work.** Armed with our lemmas, we are ready to prove the main theorem of the paper and its corollaries.

Proof of Theorem 1.3. To prove 1.3(1), that is part (1) of Theorem 1.3, assume that $n \in \mathbb{N}^+$ and $D \in \mathcal{D}_{01\prec}(n)$ is an absolute **H**-retract for $\mathcal{D}_{01\prec}(n)$. For the sake of contradiction, suppose that D is neither boolean nor it is an n-dimensional grid. The first task in the proof is to find a *proper* $\mathcal{D}_{01\prec}(n)$-extension K of D. Let $k := \dim_{\mathrm{ord}}(D)$; note that $k \leq n$. By Lemma 4.4, D has a $\mathcal{D}_{01\prec}(n)$-extension L such that L is a k-dimensional grid. There are three cases depending on k and L.

First, assume that $k < n$ and L is boolean. Then $D \neq L$ since D is not boolean. So if we let $K := L$, then

$$K \in \mathcal{D}_{01\prec}(n), \ K \neq D, \text{ and } K \text{ is a } \mathcal{D}_{01\prec}(n)\text{-extension of } D. \tag{4.23}$$

Second, assume that $k < n$ and L is not boolean. Then (1.6) gives that $\mathrm{length}(L) = \mathrm{length}(D)$. Lemma 4.5 allows us to take a $(k+1)$-dimensional grid K such that $\mathrm{length}(K) = \mathrm{length}(L)$ and L is a sublattice of K. So D is a sublattice of K and $\mathrm{length}(D) = \mathrm{length}(K)$. Hence if we apply Lemma 4.3(2) to the map $D \to K$ defined by $x \mapsto x$ and take $\dim_{\mathrm{ord}}(K) = k+1 \leq n$ into account, we obtain that K is a $\mathcal{D}_{01\prec}(n)$-extension of D. Since $\dim_{\mathrm{ord}}(D) = k \neq \dim_{\mathrm{ord}}(K)$, we have that $D \neq K$ and so (4.23) holds again.

Third, assume that $k = n$, that is, $\dim_{\mathrm{ord}}(D) = n$. Then, by Lemma 4.4, D has a $\mathcal{D}_{01\prec}(n)$-extension K such that K is an n-dimensional grid. Since we have assumed that D is not an n-dimensional grid, (4.23) holds again.

We have seen that, in each of the three possible cases, (4.23) holds. Since $D \in \mathcal{D}_{01\prec}(n)$ was assumed to be an absolute **H**-retract for $\mathcal{D}_{01\prec}(n)$, there exists a retraction $f \colon K \to D$. We know from (4.23) that the map $D \to K$ defined by $x \mapsto x$ is a cover-preserving $\{0, 1\}$-embedding. Hence f is an isomorphism by

Lemma 4.3(3), whereby $|K| = |D|$. This contradicts the fact that D is a proper sublattice of K by (4.23), and we have proved 1.3(1).

To prove 1.3(2), assume that a finite boolean lattice D is a sublattice of a not necessarily finite distributive lattice K. We are going to show that there exists a retraction $K \to D$. Since this is trivial if D is a singleton, we can assume that $|D| > 1$. Let $n := \dim_{\mathrm{ord}}(D)$. Combining (1.3) and (4.17) and taking into account that the join-irreducible elements of a finite boolean lattice are exactly its atoms, it follows that D has exactly n atoms and it is of length n. Hence, we can take a maximal chain $C = \{0 = c_0, c_1, \ldots, c_{n-1}, c_n = 1\}$ in D such that $c_{i-1} \prec c_i$ for $i \in \{1, \ldots, n\}$. For $i \in \{1, \ldots, n\}$, the Prime Ideal Theorem allows us to pick a prime ideal I_i of K such that $c_{i-1} \in I_i$ but $c_i \notin I_i$. Since I_i is a prime ideal, the partition $\{I_i, K \setminus I_i\}$ determines a congruence Θ_i of K. This congruence *separates* c_{i-1} and c_i, that is, $(c_{i-1}, c_i) \notin \Theta_i$. Let $\Theta := \bigcap \{\Theta_i : i \in \{1, \ldots n\}\}$. Now Θ is a congruence of K and its restriction $\Theta\lceil_C$ is a congruence of the sublattice C. We claim that $\Theta\lceil_C = \Delta_C$; suppose the contrary. We know from the folklore that any congruence of a finite lattice is determined by the covering pairs it collapses, whence $(c_{i-1}, c_i) \in \Theta\lceil_C$ for some $i \in \{1, \ldots, n\}$. But then $(c_{i-1}, c_i) \in \Theta\lceil_C \subseteq \Theta \subseteq \Theta_i$, contradicting the fact that Θ_i separates c_{i-1} and c_i. This shows that $\Theta\lceil_C = \Delta_C$. Therefore, it follows from (4.13) and $\Theta\lceil_C = (\Theta\lceil_D)\lceil_C$ that

$$\Theta\lceil_D = \Delta_D. \tag{4.24}$$

Observe that

> if α and β are congruences of a not necessarily finite lattice, α has exactly $m \in \mathbb{N}^+$ blocks, and β has exactly $n \in \mathbb{N}^+$ blocks, then $\alpha \cap \beta$ has at most mn blocks. $\tag{4.25}$

Indeed, (4.25) follows from the fact that β cuts each of the m α-blocks into at most n pieces. Since Θ_i has only two blocks, it follows from (4.25) that Θ has at most 2^n blocks. But the elements of D belong to pairwise different Θ-blocks by (4.24), whereby Θ has exactly $2^n = |D|$ blocks. Next, we define a map

$$f \colon K \to D \text{ by the rule } f(x) = d \in D \iff (x, d) \in \Theta. \tag{4.26}$$

For later reference, we note that

> to show that f in (4.26) is a retraction, we will only use that K is a lattice, D is finite a sublattice of K, Θ has exactly $|D|$ blocks, and $\Theta\lceil_D = \Delta_D$. $\tag{4.27}$

Since Θ has exactly 2^n blocks, $|D| = 2^n$ and (4.24) guarantee the properties mentioned in (4.27). The equality $\Theta\lceil_D = \Delta_D$ yields that for each $x \in K$, there is at most one d in (4.26). If there was an $x \in K$ with its Θ-block x/Θ disjoint from D, then Θ would have more than $|D|$-blocks since x/Θ would be different from the pairwise distinct blocks of the elements of D. Thus, for each $x \in K$, there is exactly one $d \in D$ with $(x, d) \in \Theta$, whereby (4.26) defines a map, indeed. If $f(x_1) = d_1$ and $f(x_2) = d_2$, then $(x_1, d_1) \in \Theta$ and $(x_2, d_2) \in \Theta$ yield that $(x_1 \vee x_2, d_1 \vee d_2) \in \Theta$, whence $f(x_1 \vee x_2) = d_1 \vee d_2 \in D$. The same holds for meets, and so f is a homomorphism. By the reflexivity of Θ, $f(d) = d$ for all $d \in D$. Thus, f is a retraction, proving 1.3(2).

Next, to prove 1.3(3), first we observe that

> if E is a finite subchain of a chain C, then E is a retract of C. $\tag{4.28}$

To see this, let $E = \{e_1, e_2, \ldots e_k\}$ such that $e_1 < e_2 < \cdots < e_k$. Understanding the principal ideals below in C, it is trivial that the equivalence Θ with blocks $\downarrow e_1$, $\downarrow e_2 \setminus \downarrow e_1$, $\ldots$, $\downarrow e_{k-1} \setminus \downarrow e_{k-2}$, $C \setminus \downarrow e_{k-1}$ is a congruence of C. Since $\Theta\lceil_E = \Delta_E$ and Θ has $|E|$ blocks, (4.27) implies (4.28).

Armed with (4.28), assume that $n \in \mathbb{N}^+$, D is an n-dimensional grid, $L \in \mathcal{D}(n)$, and D is a sublattice of L. We are going to find a retraction $L \to D$. It follows from Milner and Pouzet [24], see Subsection 1.4 of the present paper, that $\dim_{\mathrm{ord}}(D) \leq \dim_{\mathrm{ord}}(L)$. Combining this inequality with $n = \dim_{\mathrm{ord}}(D)$ and $L \in \mathcal{D}(n)$, we obtain that $\dim_{\mathrm{ord}}(L) = n$. Hence, by Lemma 4.4, there is a cover-preserving $\{0,1\}$-embedding of L into an n-dimensional grid K. Then D is a sublattice of K, and both D and K are n-dimensional grids. Let C_1, $\ldots$, C_n be the canonical chains of K. By Lemma 4.2, these canonical chains have nontrivial subchains E_1, $\ldots$, E_n, respectively, such that (4.2) holds with D in place of L. For $i \in \{1, \ldots, n\}$, $\pi_i \colon K \to C_i$ defined by $x \mapsto x[i]$ is a homomorphism by Lemma 4.1(3). Since $x[i] = 0 \vee \cdots \vee 0 \vee x[i] \vee 0 \vee \cdots \vee 0$ (where $x[i]$ is the i-th joinand on the right), the uniqueness of the canonical form (4.1) gives that $(x[i])[i] = x[i]$. Hence, π_i acts identically on C_i and so π_i is a retraction. Using (4.28), we can take a retraction $g_i \colon C_i \to E_i$. Clearly, the composite map $f_i := g_i \circ \pi_i$ is a retraction $K \to E_i$. For $x \in D$, (4.2) gives that $x[i] \in E_i$. Hence, for $x \in D$ and $i \in \{1, \ldots, n\}$,

$$f_i(x) = g_i(\pi_i(x)) = g_i(x[i]) = x[i]. \tag{4.29}$$

Let Θ_i be the kernel of f_i. Since f_i, as any retraction, is surjective, Θ_i has exactly $|E_i|$ blocks. Therefore, if we let $\Theta := \bigcap_{i=1}^n \Theta_i$, then Θ is a congruence of K with at most $\prod_{i=1}^n |E_i| = |D|$ blocks by (4.25). On the other hand, if $(x, y) \in \Theta$ holds for $x, y \in D$, then $(x, y) \in \Theta_i$ and (4.29) give that $x[i] = f_i(x) = f_i(y) = y[i]$ for all $i \in \{1, \ldots, n\}$, whence it follows from (4.1) that $x = y$. This means that $\Theta\lceil_D = \Delta_D$. Thus, Θ has at least $|D|$ blocks, and obtain that Θ has exactly $|D|$-blocks. Therefore, (4.26) and (4.27) imply that there is a retraction $f \colon K \to D$. Since the restriction $f\lceil_L \colon L \to D$, defined by $x \mapsto f(x)$, is clearly a retraction, we have shown the existence of a retraction $L \to D$, as required. This completes the proof 1.3(3) and that of Theorem 1.3. $\qquad\square$

Proof of Corollary 1.4. For categories $\mathcal{X}$ and $\mathcal{Y}$, we say that $\mathcal{X}$ is a *subcategory* of $\mathcal{Y}$ if every object of $\mathcal{X}$ is an object of $\mathcal{Y}$ and every morphism of $\mathcal{X}$ is a morphism of $\mathcal{Y}$. It is trivial to observe that

> if $\mathcal{X}$ and $\mathcal{Y}$ are categories of lattices such that $\mathcal{X}$ is a sub-category of $\mathcal{Y}$ and a lattice $L \in \mathcal{X}$ is an absolute **H**-retract for $\mathcal{Y}$, then L is also an absolute **H**-retract also for $\mathcal{X}$. $\qquad$(4.30)

Let $n \in \mathbb{N}^+$. It suffices to that $\mathrm{Bool}_\leq(2^n) \cup \mathrm{Grid}_=(n) = \mathrm{AbRt}(\mathcal{D}(n))$ since the rest of equalities follow from (1.2) and Proposition 1.1. If $D \in \mathrm{Bool}_\leq(2^n)$, then $D \in \mathrm{AbRt}(\mathcal{D}_{\mathrm{all}})$ by Theorem 1.3(2), whence (4.30) yields that $D \in \mathrm{AbRt}(\mathcal{D}(n))$. If $D \in \mathrm{Grid}_=(n)$, then $D \in \mathrm{AbRt}(\mathcal{D}(n))$ by Theorem 1.3(3). Thus, we have verified the "$\subseteq$" part of the required equality. Conversely, if $D \in \mathrm{AbRt}(\mathcal{D}(n))$, then (4.30) implies that $D \in \mathrm{AbRt}(\mathcal{D}_{01\prec}(n))$. Hence, $D \in \mathrm{AHRt}(\mathcal{D}_{01\prec}(n))$ by (1.1), and we obtain from Theorem 1.3(1) that $D \in \mathrm{Bool}_\leq(2^n) \cup \mathrm{Grid}_=(n)$. This yields the converse inclusion "$\supseteq$". $\qquad\square$

Proof of Corollary 1.5. In virtue of (1.2) and Proposition 1.1, it suffices to show that $\mathrm{AbRt}(\mathcal{D}(\omega)) = \mathrm{Bool}_{\mathrm{fin}}$. If $D \in \mathrm{Bool}_{\mathrm{fin}}$, then Theorem 1.3(2) and (1.2) give

that $D \in \mathrm{AHRt}(\mathcal{D}_{\mathrm{all}})$. Applying (4.30) and (1.2), we obtain that $D \in \mathrm{AbRt}(\mathcal{D}(\omega))$. This gives the "$\supseteq$" part of the required equality.

Conversely, assume that $D \in \mathrm{AbRt}(\mathcal{D}(\omega))$. Denote $\dim_{\mathrm{ord}}(D)$ by k. Then $D \in \mathcal{D}_{01\prec}(k+1)$, and (1.2) together with (4.30) give that $D \in \mathrm{AHRt}(\mathcal{D}_{01\prec}(k+1))$. Applying Theorem 1.3(1), $D \in \mathrm{Bool}_{\mathrm{fin}}$ or $D \in \mathrm{Grid}_=(k+1)$. But $D \in \mathrm{Grid}_=(k+1)$ is excluded by $\dim_{\mathrm{ord}}(D) = k$. Therefore, $D \in \mathrm{Bool}_{\mathrm{fin}}$, and the "$\subseteq$" inclusion also holds. $\square$

Proof of Corollary 1.6. It follows from (1.2) and Proposition 1.1 that only the equality $\mathrm{Finite}(\mathrm{AbRt}(\mathcal{D}_{\mathrm{all}})) = \mathrm{Bool}_{\mathrm{fin}}$ needs proving. Assume that D belongs to $\mathrm{Finite}(\mathrm{AbRt}(\mathcal{D}_{\mathrm{all}}))$. By (1.2), (4.30), and (1.2) again, $D \in \mathrm{AbRt}(\mathcal{D}(\omega))$. Hence, Corollary 1.5 gives that $D \in \mathrm{Bool}_{\mathrm{fin}}$, showing that $\mathrm{Finite}(\mathrm{AbRt}(\mathcal{D}_{\mathrm{all}})) \subseteq \mathrm{Bool}_{\mathrm{fin}}$. Since Theorem 1.3(2) takes care of the converse inclusion, the proof is complete. $\square$

Proof of Corollary 1.7. Since it follows from Proposition 1.1 that $\mathrm{AHRt}(\mathcal{D}_{01\prec}(n)) = \mathrm{AlgC}(\mathcal{D}_{01\prec}(n)) = \mathrm{StAlgC}(\mathcal{D}_{01\prec}(n))$, it suffices to show that

$$\mathrm{AbRt}(\mathcal{D}_{01\prec}(n)) \subseteq \mathrm{AHRt}(\mathcal{D}_{01\prec}(n)), \tag{4.31}$$

$$\mathrm{AHRt}(\mathcal{D}_{01\prec}(n)) \subseteq \mathrm{Bool}_\leq(2^n) \cup \mathrm{Grid}_=(n), \text{ and} \tag{4.32}$$

$$\mathrm{Bool}_\leq(2^n) \cup \mathrm{Grid}_=(n) \subseteq \mathrm{AbRt}(\mathcal{D}_{01\prec}(n)). \tag{4.33}$$

But (4.31) is a particular case of (1.1) while (4.32) is the same as Theorem 1.3(1), whereby it suffices to prove (4.33). Assume that $D \in \mathrm{Bool}_\leq(2^n) \cup \mathrm{Grid}_=(n)$. Then $D \in \mathrm{AbRt}(\mathcal{D}(n)) = \mathrm{AHRt}(\mathcal{D}(n))$ by Corollary 1.4 and (1.1). Applying (4.30), we obtain that $D \in \mathrm{AHRt}(\mathcal{D}_{01\prec}(n))$. Let K be a $\mathcal{D}_{01\prec}(n)$-extension of D. Since $D \in \mathrm{AHRt}(\mathcal{D}_{01\prec}(n))$, there exists a retraction $f \colon K \to D$. By Lemma 4.3(3), f is a morphism of $\mathcal{D}_{01\prec}(n)$. Thus, $D \in \mathrm{AbRt}(\mathcal{D}_{01\prec}(n))$, proving (4.33) and Corollary 1.7. $\square$

Proof of Corollary 1.8. Denote by (4.31)$[\omega]$, (4.32)$[\omega]$, and (4.33)$[\omega]$ the conditions we obtain from (4.31), (4.32), and (4.33), respectively, by substituting $\mathrm{Bool}_{\mathrm{fin}}$ for $\mathrm{Bool}_\leq(2^n)$, $\emptyset$ for $\mathrm{Grid}_=(n)$, and ω for the remaining occurrences of n. As in the previous proof, it is clear by Proposition 1.1 that it suffices to show (4.31)$[\omega]$, (4.32)$[\omega]$, and (4.33)$[\omega]$. As before, (4.31)$[\omega]$ is a particular case of (1.1). To show (4.32)$[\omega]$, assume that $D \in \mathrm{AHRt}(\mathcal{D}_{01\prec}(\omega))$. Letting $k := \dim_{\mathrm{ord}}(D)$, we have that $D \in \mathrm{AHRt}(\mathcal{D}_{01\prec}(k+1))$. So if we apply the (4.32) part of Corollary 1.7, we obtain that $D \in \mathrm{Bool}_\leq(2^{k+1}) \cup \mathrm{Grid}_=(k+1)$. Since $\dim_{\mathrm{ord}}(D) = k \neq k+1$ excludes that $D \in \mathrm{Grid}_=(k+1)$, we conclude that $D \in \mathrm{Bool}_\leq(2^{k+1}) \subseteq \mathrm{Bool}_{\mathrm{fin}}$, proving (4.32)$[\omega]$.

To show (4.33)$[\omega]$, assume that $D \in \mathrm{Bool}_{\mathrm{fin}}$, and let L be an $\mathcal{D}_{01\prec}(\omega)$-extension of D. Since $\mathrm{Bool}_{\mathrm{fin}} = \bigcup_{k \in \mathbb{N}^+} \mathrm{Bool}_\leq(2^k)$ and $\mathcal{D}_{01\prec}(\omega) = \bigcup_{k \in \mathbb{N}^+} \mathcal{D}_{01\prec}(k)$, we can pick an $n \in \mathbb{N}^+$ such that $D \in \mathrm{Bool}_\leq(2^n)$ and $L \in \mathcal{D}_{01\prec}(n)$. It follows from the (4.33) part of Corollary 1.7 that there is retraction $f \colon L \to D$ such that f is a morphism of $\mathcal{D}_{01\prec}(n)$. Since $\mathcal{D}_{01\prec}(n) \subseteq \mathcal{D}_{01\prec}(\omega)$, f is also a morphism of $\mathcal{D}_{01\prec}(\omega)$. Hence, $D \in \mathrm{AbRt}(\mathcal{D}_{01\prec}(\omega))$, proving (4.33)$[\omega]$ and Corollary 1.8. $\square$

Proof of Corollary 1.9. By Proposition 5.2 of Kelly and Rival [23], a finite lattice is planar if and only if its order dimension is at most 2. Hence, the class of planar distributive lattices is $\mathcal{D}(2)$. Thus, the rest of the statement becomes a particular case of Corollary 1.4. $\square$

Proof of Corollary 1.10. The first half of Corollary 1.9 yields that we are in the category $\mathcal{D}_{01\prec}(2)$. Therefore, the required equalities follow from Corollary 1.7. □

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