A DYADIC VIEW OF RATIONAL CONVEX SETS

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Abstract. Let F be a subfield of the field $\mathbb{R}$ of real numbers. Equipped with the binary arithmetic mean operation, each convex subset C of F^n becomes a commutative binary mode, also called idempotent commutative medial (or entropic) groupoid. Let C and C' be convex subsets of F^n. Assume that they are of the same dimension and at least one of them is bounded, or F is the field of all rational numbers. We prove that the corresponding idempotent commutative medial groupoids are isomorphic iff the affine space F^n over F has an automorphism that maps C onto C'. We also prove a more general statement for the case when $C, C' \subseteq F^n$ are considered barycentric algebras over a unital subring of F that is distinct from the ring of integers. A related result, for a subring of $\mathbb{R}$ instead of a subfield F, is given in [4].

1. Introduction and motivation

Let F be a subfield of the field $\mathbb{R}$ of real numbers. Equipped with the arithmetic mean operation $(x, y) \mapsto (x+y)/2$, denoted by $\underline{h}$ (coming from "half"), F^n becomes a groupoid $(F^n, \underline{h})$. This groupoid is idempotent, commutative, medial, and cancellative. In Polish notation, which we use in the paper, these properties mean that, for arbitrary $x, y, z, t \in F^n$, $xx\underline{h} = x$ (idempotence), $xy\underline{h} = yx\underline{h}$ (commutativity), $xy\underline{h}\,zt\underline{h}\,\underline{h} = xz\underline{h}\,yt\underline{h}\,\underline{h}$ (mediality, which is a particular case of entropicity), and $xy\underline{h} = xz\underline{h}$ implies $y = z$ (cancellativity). These groupoids without assuming cancellativity are also called *commutative binary modes* or *CB-modes*, and they were studied in, say, [7] and [11] and [12], and Ježek and Kepka [6].

Let C be a nonempty subset of F^n. If there is a convex subset D of the Euclidean space $\mathbb{R}^n$ in the usual sense such that $C = D \cap F^n$, then C will be called a *geometric convex subset* of F^n. We also say that C is a *geometric convex set over* F. Later we will give an "internal" definition that does not refer to $\mathbb{R}$. Note that C above is simply called a *convex subset* in Romanowska and Smith [12]; however, the adjective "geometric" becomes important soon in a more general situation. For convenience, the empty set will not be called a geometric convex set.

Our initial problem is to characterize those pairs (C_1, C_2) of geometric convex subsets of F^n for which $(C_1, \underline{h})$ and $(C_2, \underline{h})$ are isomorphic groupoids. In the particular case when $F = \mathbb{Q}$, loosely speaking we are interested in what we can see from the "rational world" $\mathbb{Q}^n$ if the only thing we can percept is whether a point equals the arithmetic mean of two other points.

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Similar questions were studied for some particular geometric convex subsets of $\mathbb{D}^2$, where $\mathbb{D} = \{x2^k : x, k \in \mathbb{Z}\}$ is the ring of *rational dyadic numbers*. Namely, the isomorphism problem of line segments and polygons of the rational dyadic plane $\mathbb{D}^2$ were studied in Matczak, Romanowska and Smith [8]. Another problem of deciding whether $(C_1, \underline{h})$ is isomorphic to $(C_2, \underline{h})$ is considered in [3, Ex. 2.6], and [4] also considers a related isomorphism problem.

The isomorphism problem even for intervals of the dyadic line $\mathbb{D}$ is not so evident as one may expect. This explains why our convex sets in the main result, Theorem 2.4, are assumed to have some further properties, including that they are *geometric* over a subfield of $\mathbb{R}$. Further comments on the main result will be given in Section 3.

2. Barycentric algebras over unital subrings of $\mathbb{R}$ and the results

Notation 2.1. The general assumption and notation in the paper are the following.

(i) $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ is the field of real numbers, and $n \in \mathbb{N}$.

(ii) T is a subring of $\mathbb{R}$ such that $1 \in T$ and $T \cap \mathbb{Q} \neq \mathbb{Z}$ (that is, $\mathbb{Z} \subset T \cap \mathbb{Q}$).

(iii) K is the subfield of $\mathbb{R}$ generated by T, and F is a subfield of $\mathbb{R}$ such that $T \subseteq F$. (Clearly, $T \subseteq K \subseteq F \subseteq \mathbb{R}$.)

(iv) The open and the closed unit intervals of T are denoted by $I^o(T) = \{x \in T : 0 < x < 1\}$ and $I^\bullet(T) = \{x \in T : 0 \leq x \leq 1\}$, respectively; $I^o(F)$, $I^\bullet(\mathbb{Q})$, etc. are particular cases. (Notice that T can equal, say, F and F can equal $\mathbb{R}$, etc. Therefore, whatever we define for T or F in what follows, it will automatically make sense for F or $\mathbb{R}$.)

(v) With each $p \in \mathbb{R}$ we associate a binary operation symbol denoted by $\underline{p}$. For $H \subseteq \mathbb{R}$, we let $\underline{H} := \{\underline{p} : p \in H\}$. However, we will write, say, $\underline{I}^o(T)$ instead of $\underline{I^o(T)}$. For $x, y \in \mathbb{R}^n$, $xy\underline{p}$ is defined to be $(1-p)x + py$.

If $p \in I^o(\mathbb{R})$, then $\underline{p}$ is called a *barycentric operation* since $xy\underline{p}$ gives the barycenter of a two-body system with weight $(1-p)$ in the point x and weight p in the point y. For any p, q in $\mathbb{R}$, the operations $\underline{p}$ and $\underline{q}$ commute in $\mathbb{R}^n$, that is $xy\underline{p}\,zt\underline{p}\,\underline{q} = xz\underline{q}\,yt\underline{q}\,\underline{p}$ holds for all $x, y, z, t \in \mathbb{R}$. This property is called the *entropic law*, see [12]. As a particular case, the *medial law* (for $\underline{h}$) means that $\underline{h}$ commutes with itself. Although the present paper is more or less self-contained, for standard general algebraic concepts the reader may want to see Burris and Sankappanavar [1]. He may also want to see Romanowska and Smith [12] for additional information on modes and barycentric algebras. The visual meaning of barycentric operations is revealed by the following lemma; the obvious proof will be omitted. The Euclidean distance $\big((x_1-y_1)^2+\cdots+(x_n-y_n)^2\big)^{1/2}$ of $x, y \in \mathbb{R}^n$ will be denoted by $\mathrm{dist}(x, y)$.

Lemma 2.2. *Let y and x be distinct points in $\mathbb{R}^n$, see Figure 1. Then for each b belonging to the open line segment connecting y and x and for each $p \in I^o(R)$,*

$$b = yx\underline{p} \iff x = yb\underline{1/p} \iff y = bx\,\underline{p/(p-1)}.$$

Moreover, $\mathrm{dist}(y, x) = \mathrm{dist}(y, b)/p$.

The algebra $(\mathbb{R}^n; \underline{I}^o(T))$ and all of its subalgebras are particular members of the variety of barycentric algebras over T, or *T-barycentric algebras* for short. (However, as opposed to previous papers and monographs, T is no longer assumed to

FIGURE 1. Illustrating Lemma 2.2 in case $p = 1/3$

be a field.) These particular T-barycentric algebras that we consider are *modes*, that is, idempotent algebras in which any two operations (and therefore any two term functions) commute. Modes and barycentric algebras have intensively been studied in the monographs [10] and [12], see also the extensive bibliography in [3]. It is well-known, see [12], that $(F^n; \underline{h})$ is term-equivalent to $(F^n; \underline{I}^o(\mathbb{D}))$, whence the same holds for its subalgebras. This allows us to translate the initial problem to the language of $\mathbb{D}$-barycentric algebras, and then it is natural to extend it to T-barycentric algebras.

The subalgebras of $(\mathbb{R}^n; \underline{I}^o(T))$ will be called *T-convex subsets* of $\mathbb{R}^n$. The empty set is not considered to be T-convex. (Notice that the adjective "T-convex" in [4] is used only for subsets of T^n.) For $\emptyset \neq X \subseteq \mathbb{R}^n$, the *$T$-convex hull* of X, denoted by $\mathrm{Cnv}_T(X)$, is the subalgebra generated by X in $(\mathbb{R}^n; \underline{I}^o(T))$. It is well-known, see [12], that $\underline{I}^\bullet(T)$ is exactly the set of binary term functions of $(F^n; \underline{I}^o(T))$. Moreover, each $(1+k)$-ary term function of $(F^n; \underline{I}^o(T))$ agrees with a function $\boldsymbol{\tau}\colon (x_0, \dots, x_k) \mapsto \xi_0 x_0 + \dots + \xi_k x_k$ where $\xi_0, \dots, \xi_k \in I^\bullet(T)$ such that $\xi_0 + \dots + \xi_k = 1$. This implies that, for any $\emptyset \neq X \subseteq F^n$,

$$(1) \qquad \mathrm{Cnv}_T(X) = \{x_0 \cdots x_k \boldsymbol{\tau} : k \in \mathbb{N}_0,\ x_0, \dots, x_k \in X \text{ and } \boldsymbol{\tau} \text{ is as above}\}.$$

The full idempotent reduct of the T-module ${}_T F^n$ is a so-called affine module over T; we call it an *affine T-module* and denote it by $\mathrm{Aff}_T(F^n)$. When T is understood or irrelevant, we often write F^n instead of $\mathrm{Aff}_T(F^n)$. In the particular case $T = F$, the affine F-module $\mathrm{Aff}_F(F^n)$ is an n-dimensional *affine F-space*, see more (well-known) details later.

The mere assumption that $C \subseteq F^n$ is a T-convex subset would rarely be sufficient for our purposes, see also [4] for a similar analysis. There are three reasonable ways to make a stronger assumption.

Firstly, we can assume that C is an *F-convex subset*, that is, a subalgebra of $(F^n, \underline{I}^o(F))$.

Secondly, we can assume that C is the intersection of F^n with an $\mathbb{R}$-convex subset of $\mathbb{R}^n$. (That is, with a convex subset of $\mathbb{R}^n$ in the usual geometric meaning.) In this case we say that C is a *geometric convex subset* of F^n. In other words, we say that C is a *geometric convex set over F*. Notice that the geometric convexity of C depends on F, so we can use this concept only for subsets of F^n. (Note also that [4] defines geometric convexity even when $C \subseteq T^n$ and it does it in a different way, which is equivalent to our approach for the case $T = F$.)

To define the third variant of convexity, let $a, b \in F^n$ with $a \neq b$. By the *T-line* generated by $\{a, b\}$ we mean the subalgebra generated by $\{a, b\}$ in the affine T-module $\mathrm{Aff}_T(F^n)$. This T-line is denoted by $\ell_T(a, b)$. It is easy to see that $\ell_T(a, b) = \{ab\underline{p} : p \in T\}$. It follows from cancellativity that for each $x \in \ell_T(a, b)$, there is exactly one $p \in T$ such that $x = ab\underline{p}$. Let $c, d \in \ell_T(a, b)$. Then there are unique $p, r \in T$ such that $c = ab\underline{p}$ and $d = ab\underline{r}$. For $s \in T$, we say that s is between p and r iff $p \leq s \leq r$ or $r \leq s \leq p$. Then

$$[c, d]_{\ell_T(a,b)} := \{ab\underline{s} : s \text{ is between } p \text{ and } r\}$$

is called a *T-segment* of the T-line $\ell_T(a,b)$ with *endpoints* c and d. As opposed to the case when T happens to be a field, a T-segment is usually *not* determined by its endpoints. For example, 0 and 3 are the endpoints of the $\mathbb{D}$-segment $[0,3]_{\ell_{\mathbb{D}}(0,1)}$ and also of the $\mathbb{D}$-segment $[0,3]_{\ell_{\mathbb{D}}(0,3)}$ in $\mathbb{Q}^1$, but $1 \in [0,3]_{\ell_{\mathbb{D}}(0,1)} \setminus [0,3]_{\ell_{\mathbb{D}}(0,3)}$ indicates that these $\mathbb{D}$-segments are distinct. Now, a nonempty subset C of F^n will be called *T-segment convex* if for all $c, d \in C$ and all T-segments S with endpoints c and d, $S \subseteq C$. This definition, is quite "internal" since it does not refer to outer objects like $\mathbb{R}$ (besides that T is a subring of $\mathbb{R}$). The relationship among the three concepts above is clarified by the following statement, to be proved later. A related treatment for analogous concepts is given in [4].

Proposition 2.3. *Let C be a nonempty subset of F^n.*

- (i) *If C is T-segment convex, then it is T-convex.*
- (ii) *C is a geometric convex subset of F^n iff it is F-convex.*
- (iii) *If C is F-convex, then it is T-segment convex.*
- (iv) *If T generates F (that is, if $F = K$), then C is F-convex iff it is T-segment convex.*

Besides (i), each of the conditions (ii)–(iv) above clearly implies T-convexity. Remember that $\mathbb{Z} \subset T \cap \mathbb{Q}$ means that $\mathbb{Z} \neq T \cap \mathbb{Q}$ and $\mathbb{Z} \subseteq T \cap \mathbb{Q}$. If $X \subseteq F^n$ and $\{\mathrm{dist}(x,y) : x, y \in X\}$ is a bounded subset of $\mathbb{R}$, then X is called a *bounded* set. For $X \subseteq \mathbb{R}^n$, the affine *F-subspace spanned* by X will be denoted by $\mathrm{Span}_F^{\mathrm{aff}}(X)$. As usual, by the affine *F-dimension* of X, denoted by $\dim_F^{\mathrm{aff}}(X)$, we mean the affine F-dimension of $\mathrm{Span}_F^{\mathrm{aff}}(X)$. We are now in the position to formulate the main result.

Theorem 2.4. *Assume that $n \in \mathbb{N}$, F is a subfield of $\mathbb{R}$, T is a subring of F, and $\mathbb{Z} \subset T \cap \mathbb{Q}$. Let C and C' be F-convex subsets* (*equivalently, geometric convex subsets*) *of F^n. Assume also that*

(a) *$F = \mathbb{Q}$,*

or

(b) *C and C' have the same affine F-dimension and at least one of them is bounded.*

Then the following three conditions are equivalent.

- (i) *$(C, \underline{I}^o(T))$ and $(C', \underline{I}^o(T))$ are isomorphic T-barycentric algebras.*
- (ii) *The affine F-space $\mathrm{Aff}_F(F^n)$ has an automorphism ψ such that $\psi(C) = C'$.*
- (iii) *The affine real space $\mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^n)$ has an automorphism ψ such that $\psi(C) = C'$.*

Corollary 2.5. *If C and C' are geometric convex subsets of F^n satisfying* (b) *above, then $(C, \underline{h}) \cong (C', \underline{h})$ iff* (ii) *of Theorem 2.4 holds iff* (iii) *of Theorem 2.4 holds. Furthermore, if D and D' are isomorphic subalgebras of $(\mathbb{Q}^n, \underline{h})$, then D is a geometric convex subset of $\mathbb{Q}^n$ iff so is D'.*

3. Examples and comments

Before proving our results, we present four examples to illustrate and comment them. The first example below is a variant of [3, Ex. 1.5]. While [3] is insufficient to handle it, Theorem 2.4 will apply easily. Remember that h stands for $1/2$.

Example 3.1. Let $C_i = \{(x,y) \in F^2 : x^2 \in I^o(F) \text{ and } |y| < 1 - |x|^i\}$, for $i \in \mathbb{N}$. Are there distinct $i, j \in \mathbb{N}$ such that the groupoids $(C_i, \underline{h})$ and $(C_j, \underline{h})$ are isomorphic?

The answer is negative. Suppose, for a contradiction, that $(C_i, \underline{h}) \cong (C_j, \underline{h})$ and $1 \le j < i$. Then Theorem 2.4 yields an automorphism ψ of $\mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^2)$ such that $\psi(C_i) = C_j$. It is well-known that there exist an invertible 2-by-2 matrix M over $\mathbb{R}$ and a column vector $\vec{c} \in \mathbb{R}^2$ such that

$$\text{for every} \quad \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad \psi(\vec{v}) = M\vec{v} + \vec{c}. \tag{2}$$

The usual topological closure of C_t is denoted by $[C_t]_{\mathbb{R}}^{\mathrm{top}}$, for $t \in \{i, j\}$. Since ψ and ψ^{-1} are continuous by (2), $\psi([C_i]_{\mathbb{R}}^{\mathrm{top}}) = [C_j]_{\mathbb{R}}^{\mathrm{top}}$. Let B_t denote the boundary

$$[C_t]_{\mathbb{R}}^{\mathrm{top}} \setminus C_t = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 1 \text{ and } |y| = 1 - |x|^t\}$$

of C_t, for $t \in \{i, j\}$. Clearly, $\psi(B_i) = B_j$. Depending on the parity of t, B_t consists of two or four algebraic curves. If S_t is a subset of one of these curves, then we can chose the signs in $f_t(x,y) = \pm x^t \pm y - 1$ so that S_t be a subset of $V(f_t) = \{(x,y) \in \mathbb{R}^2 : f_t(x,y) = 0\}$. We chose S_i and S_j so that S_i be infinite and $\psi(S_i) \subseteq S_j$. Since $\pm y - 1$ is an irreducible polynomial in $\mathbb{R}[y]$, the Eisenstein–Schönemann criterion (see Cox [2] for our terminology) yields that f_t is an irreducible polynomial in $\mathbb{R}[x,y]$. Note that the (total) degree of $f_t \in \mathbb{R}[x,y]$, denoted by $\deg(f_t)$, is t. Let $g_j(x,y) = f_j(\psi(x,y))$. It follows from (2) that $g_j \in \mathbb{R}[x,y]$ and that $\deg(g_j) = j$. Since $1 \le \deg(g_j) = j < i = \deg(f_i)$ and f_i is irreducible, the greatest common divisor of f_i and g_j in the unique factorization domain $\mathbb{R}[x,y]$ is 1. Hence, by the classical Bézout's theorem in algebraic geometry (see, for example, Fulton [5]), $|V(f_i) \cap V(g_j)| \le ij$. This is a contradiction, because $S_i \subseteq V(f_i) \cap V(g_j)$ and S_i is infinite.

Example 3.2. Let $n = 1$, $F = \mathbb{Q}(\sqrt{2})$, $T = \mathbb{D}$, and let C be the least T-segment convex subset of $F = F^n$ that includes $\{0, 3\}$. Since $[0,3] \cap \mathbb{Q}$ is T-segment convex and includes $\{0, 3\}$, we conclude that $C \subseteq [0,3] \cap \mathbb{Q}$. Hence $\sqrt{2} \notin C$, and C is not F-convex.

Thus, the assumption $F = K$ in Proposition 2.3(iv) cannot be omitted.

Example 3.3. The rational vector spaces ${}_{\mathbb{Q}}(\mathbb{R} \times \{0\})$ and ${}_{\mathbb{Q}}\mathbb{R}^2$ are well-known to be isomorphic since they have the same dimension. (Recall that any basis of ${}_{\mathbb{Q}}\mathbb{R} \cong {}_{\mathbb{Q}}(\mathbb{R} \times \{0\})$ is called a *Hamel-basis*.) Therefore $C = \mathrm{Aff}_{\mathbb{Q}}(\mathbb{R} \times \{0\})$ and $C' = \mathrm{Aff}_{\mathbb{Q}}(\mathbb{R}^2)$ are isomorphic affine $\mathbb{Q}$-spaces. Thus, $(C, \underline{I}^o(\mathbb{Q}))$ is isomorphic to $(C', \underline{I}^o(\mathbb{Q}))$, and they are both $\mathbb{R}$-convex subsets of $\mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^2)$. However, no automorphism of $\mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^2)$ maps C onto C'.

Let $F = \mathbb{R}$, and observe that $\dim_F^{\mathrm{aff}}(C) = 1 \ne 2 = \dim_F^{\mathrm{aff}}(C')$ and none of C and C' is bounded. This motivates (without explaining fully) the assumption "C and C' have the same affine F-dimension and at least one of them is bounded" in Theorem 2.4.

Example 3.4. A routine application of Hamel bases shows that the unit disc $(C_1, \underline{h}) := (\{(x,y) : x^2 + y^2 < 1\}, \underline{h})$ is isomorphic to another subalgebra $(C_2, \underline{h})$ of $(\mathbb{R}^2, \underline{h})$ such that both C_2 and $\mathbb{R}^2 \setminus C_2$ are everywhere dense in the plane; see [3, Proof of Lemma 2.7] for details. Clearly, C_2 is not $\mathbb{R}$-convex. By the term equivalence of $(C_i, \underline{h})$ and $(C_i, \underline{I}^o(\mathbb{D}))$, we also have that $(C_1, \underline{I}^o(\mathbb{D}))$ is isomorphic to $(C_2, \underline{I}^o(\mathbb{D}))$. However, no automorphism of $\mathrm{Aff}_{\mathbb{R}}(\mathbb{R}^2)$ maps C_1 onto C_2.

With $T = \mathbb{D}$ and $F = \mathbb{R}$, this example motivates the assumption in Theorem 2.4 that C and C' are *geometric convex* subsets of F^n.

This and the previous example show that Theorem 2.4 is not valid for arbitrary T-convex subsets of F^n, so we added some further assumptions. However, it remains an open problem whether one could somehow relax the present assumptions. In particular, we do not know whether they are independent.

4. Auxiliary statements and proofs

It is well-known that given an affine space $V = \mathrm{Aff}_F(V)$, which is the full idempotent reduct of the vector space ${}_FV$, we can obtain the vector space structure back as follows: fix an element $o \in V$, to play the role of 0, define $x+y := x-o+y$ and, for $p \in F$, $px := ox\underline{p}$. This explains some (also well-known) basic facts on affine independence. Namely, a $(1+k)$-element subset $\{a_0, \dots, a_k\}$ of $\mathrm{Aff}_F(V)$ is called *affine F-independent*, if $a_i \notin \mathrm{Span}_F^{\mathrm{aff}}(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_k)$, for $i = 0, \dots, k$. In this case, each element of the affine F-subspace $U := \mathrm{Span}_F^{\mathrm{aff}}(a_0, \dots, a_k)$ can uniquely be written in the form $\xi_0 a_0 + \cdots + \xi_k a_k$ where the so-called *barycentric coordinates* $\xi_0, \dots, \xi_k$ belong to F and their sum equals 1. Moreover, then $U = \mathrm{Aff}_F(U)$ is freely generated by $\{a_0, \dots, a_k\}$; that is, each mapping $\{a_0, \dots, a_k\} \to U$ extends to an endomorphism of $\mathrm{Aff}_F(U)$.

To capture convexity, we need a similar concept: $\{a_0, \dots, a_k\} \subseteq F^n$ will be called *$\underline{I}^o(T)$-independent* if $a_i \notin \mathrm{Cnv}_T(a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_k)$, for $i = 0, \dots, k$. It is not hard to see (and it is stated in [9]) that if $\{a_0, \dots, a_k\} \subseteq F^n$ is affine K-independent, then it is a free generating set of $(\mathrm{Cnv}_T(a_0, \dots, a_k), \underline{I}^o(T))$ and of $(\mathrm{Cnv}_K(a_0, \dots, a_k), \underline{I}^o(K))$. However, as opposed to affine K-independence, $\underline{I}^o(K)$-independence does not imply free $\underline{I}^o(K)$-generation. For example, the vertices $a_0, \dots, a_5$ of a regular hexagon in the real plane form an $\underline{I}^o(\mathbb{R})$-independent subset but $(\mathrm{Cnv}_{\mathbb{R}}(a_0, \dots, a_5), \underline{I}^o(\mathbb{R}))$ is not freely generated since $a_0 a_3 \underline{h} = a_1 a_4 \underline{h}$.

As usual, maximal independent subsets are called *bases*, or *point bases*. If an affine F-space V has a finite affine F-basis, then all of its bases have the same number of elements, which is 1 plus the so-called (*affine F-*) *dimension* $\dim_F^{\mathrm{aff}}(V)$ of the space. If V is an affine F-space with dimension k, then, for any $\{b_0, \dots, b_k\} \subseteq V$,

(3) $\{b_0, \dots, b_k\}$ spans $\mathrm{Aff}_F(V)$ iff $\{b_0, \dots, b_k\}$ is an affine F-basis of $\mathrm{Aff}_F(V)$.

Lemma 4.1. *Let L be a subfield of $\mathbb{R}$ such that $F \subseteq L$. Assume that $X \subseteq F^n$. Then, for each $d \in F^n \cap \mathrm{Cnv}_L(X)$, there are a $k \in \mathbb{N}_0$, an affine L-(and therefore affine F-) independent subset $\{a_0, \dots, a_k\}$ of X, $\xi_0 \in I^\bullet(F)$, and $\xi_1, \dots, \xi_k \in I^o(F)$ such that $\xi_0 + \cdots + \xi_k = 1$ and $d = \xi_0 a_0 + \cdots + \xi_k a_k$. (Note that ξ_0 is necessarily in $I^o(F)$ if $k \geq 1$). Consequently, $\mathrm{Cnv}_F(X) = F^n \cap \mathrm{Cnv}_L(X)$.*

This lemma belongs to the folklore. For the reader's convenience (and having no reference at hand), we present a proof.

Proof of Lemma 4.1. Since $d \in \mathrm{Cnv}_L(X) \subseteq \mathrm{Cnv}_{\mathbb{R}}(X \cap \mathbb{R}^n)$, we can choose an affine $\mathbb{R}$-subspace $V \subseteq \mathbb{R}^n$ of minimal dimension such that $d \in \mathrm{Cnv}_{\mathbb{R}}(X \cap V)$. The affine $\mathbb{R}$-dimension of V will be denoted by k. By Carathéodory's Fundamental Theorem, there are $a_0, \dots, a_k \in X \cap V$ such that $d \in \mathrm{Cnv}_{\mathbb{R}}(a_0, \dots, a_k)$. The affine $\mathbb{R}$-subspace $\mathrm{Span}_{\mathbb{R}}^{\mathrm{aff}}(a_0, \dots, a_k)$ is V since otherwise a subspace with smaller dimension would do. Hence, using (3), we conclude that $\{a_0, \dots, a_k\}$ is an affine $\mathbb{R}$-basis of V.

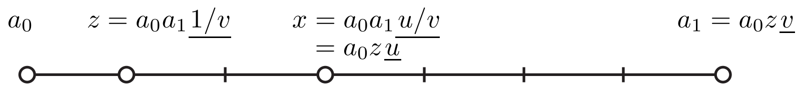

FIGURE 2. The case $k = 1$ and $p = u/v = 3/7$

Therefore, there is a unique $(\xi_0, \dots, \xi_k) \in \mathbb{R}^{1+k}$ such that

$$d = \xi_0 a_0 + \cdots + \xi_k a_k \text{ and } \xi_0 + \cdots + \xi_k = 1. \tag{4}$$

These uniquely determined ξ_i are non-negative since $d \in \mathrm{Cnv}_{\mathbb{R}}(a_0, \dots, a_k)$. We can consider (4) a system of linear equations for $(\xi_0, \dots, \xi_k)$, and this system has a unique solution. Since $d, a_0, \dots, a_k \in F^n$, the rudiments of linear algebra imply that $(\xi_0, \dots, \xi_k) \in F^{1+k}$. This, together with the fact that the affine $\mathbb{R}$-independence of the set $\{a_0, \dots, a_k\} \subseteq F^n$ implies its affine L-independence, proves the first part of the lemma. The second part is a trivial consequence of the first part. □

Proof of Proposition 2.3. Part (i) follows obviously from the fact that $a, b \in C$ with $a \neq b$ implies that $[a, b]_{\ell_T(a,b)} \subseteq C$.

If C is a geometric convex subset of F^n, then it is obviously F-convex. Conversely, if C is F-convex, then it is a geometric convex subset of F^n, because Lemma 4.1 yields that $C = \mathrm{Cnv}_F(C) = F^n \cap \mathrm{Cnv}_{\mathbb{R}}(C)$. This proves part (ii).

Part (iii) is evident.

In order to prove (iv), assume that C is T-segment convex. Let $D := \mathrm{Cnv}_K(C)$. Since D is K-convex and $C \subseteq D$, it suffices to show that $D \subseteq C$. Let x be an arbitrary element of $D = \mathrm{Cnv}_K(C)$. We obtain from Lemma 4.1 that $D = K^n \cap \mathrm{Cnv}_{\mathbb{R}}(C)$. Hence, again by Lemma 4.1, there are a *minimal* $k \in \mathbb{N}_0$, an affine $\mathbb{R}$-independent subset $\{a_0, \dots, a_k\} \subseteq C$, and a $(\xi_0, \dots, \xi_k) \in \big(I^\bullet(K)\big)^{1+k}$ such that

$$x = \xi_0 a_0 + \cdots + \xi_k a_k \text{ and } \xi_0 + \cdots + \xi_k = 1.$$

This allows us to prove the desired containment $x \in C$ by induction on k. If $k = 0$, then $x = a_0 \in C$ is evident. Hence, $k \geq 1$, and the minimality of k implies that $(\xi_0, \dots, \xi_k) \in \big(I^o(K)\big)^{1+k}$.

Next, assume that $k = 1$. Then $x = a_0a_1\underline{p}$ where $p = u/v \in I^o(K)$ and $u, v \in T$ with $0 < u < v$. Let $z := a_0a_1\underline{1/v}$, see Figure 2 for $u/v = 3/7$, and we will rely on Lemma 2.2. Then $\ell_T(a_0, z)$ contains $a_0 = a_0z\underline{0}$ and $a_1 = a_0z\underline{v}$ since $0, v \in T$. Hence $x = a_0z\underline{u} \in [a_0, a_1]_{\ell_T(a_0,z)}$, together with T-segment convexity, implies that $x \in C$.

Finally, assume that $k > 1$. Observe that $\xi_i/(1 - \xi_0) \in I^o(K)$ for $i \in \{1, \dots, k\}$, and that $\sum_{i=1}^k \xi_i/(1 - \xi_0) = 1$. Let $b = \sum_{i=1}^k \xi_i/(1 - \xi_0)a_i$; it belongs to C by the induction hypothesis. Hence, $x = \xi_0 a_0 + (1 - \xi_0)b \in C$. □

The next lemma asserts that although $(C, \underline{I}^o(T))$ cannot be generated by an independent set G of points in general, G satisfactorily describes C by means of *existential formulas*. This fact will enable us to use some ideas taken from [8].

Lemma 4.2. *Let $k \in \mathbb{N}_0$ and $\xi_0, \dots, \xi_k \in \mathbb{Q}$ such that $\xi_0 + \cdots + \xi_k = 1$. Then there exists an existential formula $\Phi^{\underline{I}^o(T)}_{\xi_0,\dots,\xi_k}(x_0, \dots, x_k; y)$ in the language of $(F^n, \underline{I}^o(T))$ with the following property: whenever $a_0, \dots, a_k, b \in F^n$, then*

$$b = \xi_0 a_0 + \cdots + \xi_k a_k \textit{ iff } \Phi^{\underline{I}^o(T)}_{\xi_0,\dots,\xi_k}(a_0, \dots, a_k; b) \textit{ holds in } (F^n, \underline{I}^o(T)).$$

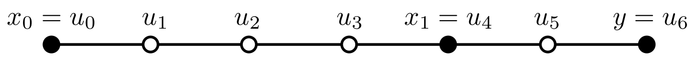

FIGURE 3. Illustrating $\Phi^{\underline{I}^o(T)}_{-2/4,\,6/4}(x_0, x_1; y)$

If, in addition, C is a $\mathbb{Q}$-convex subset of F^n such that C is also T-convex and $\{b, a_0, \dots, a_k\} \subseteq C$, then

$$b = \xi_0 a_0 + \cdots + \xi_k a_k \text{ iff } \Phi^{\underline{I}^o(T)}_{\xi_0,\dots,\xi_k}(a_0, \dots, a_k; b) \text{ holds in } (C, \underline{I}^o(T)).$$

Proof. Let p be the smallest prime number such that $1/p \in T$; there is such a prime since $\mathbb{Z} \subset T \cap \mathbb{Q}$. Note that $i/p \in I^o(T)$ for $i = 1, \dots, p-1$. We proceed by induction on k. If $k = 0$, then $\xi_0 = 1$, so we let $\Phi^{\underline{I}^o(T)}_1(x_0; y)$ to be the formula $y = x_0$.

Assume that $k = 1$. We also assume that at least one of ξ_0 and ξ_1 is greater than 1, because otherwise we can let $\Phi^{\underline{I}^o(T)}_{\xi_0,\,\xi_1}(x_0, x_1; y) := (y = x_0 x_1 \underline{\xi_1})$. (Note that $\underline{\xi_1}$ is a projection if $\xi_1 \in \{0, 1\}$.) Hence, we can assume that $\xi_1 = r/q$ and $\xi_0 = (q - r)/q$ such that $q, r \in \mathbb{N}$ and $p < q < r$. Figure 3 illustrates the particular case $(p, q, r) = (3, 4, 6)$. Let $A(p, r)$ denote the conjunction of the equations $u_{j+i} = u_j u_{j+p} \underline{i/p}$ for all $0 \le j \le r - p$ and $1 \le i \le p - 1$. Clearly, the formula

$$\Phi^{\underline{I}^o(T)}_{(q-r)/q,\, r/q}(x_0, x_1; y) := \\ (\exists u_0) \dots (\exists u_r) \big(x_0 = u_0 \;\&\; x_1 = u_q \;\&\; y = u_r \;\&\; A(p, r)\big)$$

does the job in $(F^n, \underline{I}^o(T))$. If C is a $\mathbb{Q}$-convex subset of F^n, then $\{b, a_0, a_1\} \subseteq C$ implies that the u_i belong to C, and the formula works in $(C, \underline{I}^o(T))$.

Next, assume that $k \ge 2$ and the statement holds for smaller values. If one of the $\xi_0, \dots, \xi_k$ is zero, say $x_i = 0$, then we can obviously let

$$\Phi^{\underline{I}^o(T)}_{\xi_0,\dots,\xi_k}(x_0, \dots, x_k; y) := \Phi^{\underline{I}^o(T)}_{\xi_0,\dots,\xi_{i-1},\xi_{i+1}\dots,\xi_k}(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k; y).$$

So we can assume that none of the ξ_i is zero. We have to partition $\{0, 1, \dots, k\}$ into the union of two nonempty disjoint subsets I and J such that the ξ_i, $i \in I$, have the same sign, and the same holds for the ξ_j, $j \in J$. If all the $\xi_0, \dots, \xi_k$ are positive, then any partition will do. Otherwise we can let $\emptyset \ne I = \{i : \xi_i < 0\}$; then $J = \{0, \dots, k\} \setminus I$ is nonempty since $\xi_0 + \cdots + \xi_k = 1 > 0$. To ease our notation, we can assume, without loss of generality, that $I = \{0, \dots, t\}$ and $J = \{t + 1, \dots, k\}$. Let $\kappa_0 = \xi_0 + \cdots + \xi_t$ and $\kappa_1 = \xi_{t+1} + \cdots + \xi_k$. Then $\kappa_0 \ne 0 \ne \kappa_1$ and $\kappa_0 + \kappa_1 = 1$. Define $\eta_i := \xi_i/\kappa_0$ for $i \le t$ and $\tau_j := \xi_j/\kappa_1$ for $j > t$. Clearly, $\eta_0 + \cdots + \eta_t = 1$ and $\tau_{t+1} + \cdots + \tau_k = 1$. Moreover, all the η_i and the τ_j are positive, and the identity

$$\xi_0 x_0 + \cdots + \xi_k x_k = \kappa_0(\eta_0 x_0 + \cdots + \eta_t x_t) + \kappa_1(\tau_{t+1} x_{t+1} + \cdots + \tau_k x_k)$$

clearly holds. Therefore we can let

$$\Phi^{\underline{I}^o(T)}_{\xi_0,\dots,\xi_k}(x_0, \dots, x_k; y) := \Phi^{\underline{I}^o(T)}_{\eta_0,\dots,\eta_t}(x_0, \dots, x_t; z_0) \;\&\; \Phi^{\underline{I}^o(T)}_{\tau_{t+1},\dots,\tau_k}(x_{t+1}, \dots, x_k; z_1) \\ \&\; \Phi^{\underline{I}^o(T)}_{\kappa_0,\kappa_1}(z_0, z_1; y).$$

This formula clearly does the job in $(F^n, \underline{I}^o(T))$. It also works in $(C, \underline{I}^o(T))$, provided that C is $\mathbb{Q}$-convex, since if $a_0, \dots, a_k, b \in C$, then $\eta_0 a_0 + \cdots + \eta_t a_t \in C$ and $\tau_{t+1} a_{t+1} + \cdots + \tau_k a_k \in C$, and the induction hypothesis (for $k - 1$ and then for $k = 1$) applies. $\square$

The following easy lemma is perhaps known (for arbitrary fields). Having no reference at hand, we will give an easy proof.

Lemma 4.3. *Let C be a nonempty subset of F^n. Assume that $\{a_0,\dots,a_k\}$ is a maximal affine F-independent subset of C, and let $V := \mathrm{Span}_F^{\mathrm{aff}}(a_0,\dots,a_k)$. Then*

(i) *$C \subseteq V$ and $V = \mathrm{Span}_F^{\mathrm{aff}}(C)$.*
(ii) *V does not depend on the choice of $\{a_0,\dots,a_k\}$.*
(iii) *All maximal affine F-independent subsets of C consist of $1+k$ elements.*

Proof. We know that $V = \{\xi_0 a_0 + \cdots + \xi_k a_k : \xi_0 + \cdots + \xi_k = 1,\ (\xi_0,\dots,\xi_k) \in F^{1+k}\}$. If we had $C \not\subseteq V$, then $\{a_0,\dots,a_k,a_{k+1}\}$ would be affine F-independent for every $a_{k+1} \in C \setminus V$, which could contradict the maximality of $\{a_0,\dots,a_k\}$. Hence $C \subseteq V$, which gives $\mathrm{Span}_F^{\mathrm{aff}}(C) \subseteq V$. Conversely, $\{a_0,\dots,a_k\} \subseteq C$ implies that $V = \mathrm{Span}_F^{\mathrm{aff}}(a_0,\dots,a_k) \subseteq \mathrm{Span}_F^{\mathrm{aff}}(C)$, proving part (i).

Next, let $\{b_0,\dots,b_t\}$ be another maximal affine F-independent subset of C, and let W be the affine F-subspace it spans. By part (i), $C \subseteq W$. Let $U := V \cap W$. Since $C \subseteq U$, $\{a_0,\dots,a_k\}$ and $\{b_0,\dots,b_t\}$ are affine F-independent in U. This yields that $k \le \dim_F^{\mathrm{aff}}(U)$ and $t \le \dim_F^{\mathrm{aff}}(U)$. On the other hand, $U \subseteq V$ and $U \subseteq W$ give that $\dim_F^{\mathrm{aff}}(U) \le \dim_F^{\mathrm{aff}}(V) = k$ and $\dim_F^{\mathrm{aff}}(U) \le t$. Hence $t = \dim_F^{\mathrm{aff}}(U) = k$, proving part (iii).

Using $\dim_F^{\mathrm{aff}}(U) = \dim_F^{\mathrm{aff}}(V)$ and $U \subseteq V$ we conclude that $U = V$. We obtain $U = W$ similarly, whence $W = V$ proves part (ii). □

Proof of Theorem 2.4. Assume that (ii) holds. Then ψ is of the form $x \mapsto Ax + b$ where $b \in F^n$ is a column vector and A is an invertible n-by-n matrix over F. Then A is also an invertible real matrix and $b \in \mathbb{R}^n$, whence ψ extends to an $\mathbb{R}^n \to \mathbb{R}^n$ automorphism. Thus, (ii) implies (iii).

Since $\underline{I}^o(T) \subseteq \underline{\mathbb{R}}$, the automorphisms of the real affine space preserve the $\underline{I}^o(T)$-structure. Hence (iii) trivially implies (i).

Next, assume that (i) holds, and let $\varphi\colon (C, \underline{I}^o(T)) \to (C', \underline{I}^o(T))$ be an isomorphism. For $x \in C$, $\varphi(x)$ will usually be denoted by x'. If an element of C' is denoted by, say, y', then y will automatically stand for $\varphi^{-1}(y')$. We assume that $|C| > 1$ since otherwise the statement is trivial. Firstly, we show that

$$\dim_F^{\mathrm{aff}}(C) = \dim_F^{\mathrm{aff}}(C'). \tag{5}$$

Since this is stipulated in the theorem if $F \ne \mathbb{Q}$, let us assume that $F = \mathbb{Q}$ while proving (5). Let, say $\dim_{\mathbb{Q}}^{\mathrm{aff}}(C) \le \dim_{\mathbb{Q}}^{\mathrm{aff}}(C') =: k$. By Lemma 4.3, we can choose a (maximal) affine F-independent, that is $\mathbb{Q}$-independent, subset $\{a'_0,\dots,a'_k\}$ in C'. It suffices to show that $\{a_0,\dots,a_k\} \subseteq C$ is affine F-independent. By way of contradiction, suppose that this is not the case. Then, apart from indexing, there is a $t \in \{1,\dots,k\}$ such that $\{a_1,\dots,a_t\}$ is affine $\mathbb{Q}$-independent and $a_0 \in \mathrm{Span}_{\mathbb{Q}}^{\mathrm{aff}}(a_1,\dots,a_t)$. Hence there are $\xi_1,\dots,\xi_t \in \mathbb{Q}$ whose sum equals 1 such that $a_0 = \xi_1 a_1 + \cdots + \xi_t a_t$. It follows from Lemma 4.2 that $\Phi^{\underline{I}^o(T)}_{\xi_1,\dots,\xi_t}(a_1,\dots,a_t;a_0)$ holds in $(C, \underline{I}^o(T))$. Consequently, $\Phi^{\underline{I}^o(T)}_{\xi_1,\dots,\xi_t}(a'_1,\dots,a'_t;a'_0)$ holds in $(C', \underline{I}^o(T))$. Hence Lemma 4.2 implies that $a'_0 = \xi_1 a'_1 + \cdots + \xi_t a'_t$, which contradicts the affine F-independence of $\{a'_0,\dots,a'_k\}$. This proves (5).

Next, we let $k = \dim_F^{\mathrm{aff}}(C) = \dim_F^{\mathrm{aff}}(C')$. Clearly, $k \le n$. Let $V := \mathrm{Span}_F^{\mathrm{aff}}(C)$ and $V' := \mathrm{Span}_F^{\mathrm{aff}}(C')$. We claim that for $t = 0,1,\dots,k$ and for an arbitrarily fixed

$a_0 \in C$,

$$\text{there are } a_1, \dots, a_t \in C \text{ such that both } \{a_0, \dots, a_t\} \subseteq C \text{ and } \{a'_0, \dots, a'_t\} = \varphi\big(\{a_0, \dots, a_t\}\big) \subseteq C' \text{ are affine } F\text{-independent.} \tag{6}$$

(This assertion does not follow from the previous paragraph since here we do not assume that $F = \mathbb{Q}$.) Of course, we need (6) only for $t = k$, but we prove it by induction on t. If $t \le 1$, then (6) is trivial. Assume that $1 < t \le k$ and (6) holds for $t - 1$. So we have an affine F-independent subset $\{a_0, \dots, a_{t-1}\}$ such that $\{a'_0, \dots, a'_{t-1}\}$ is also affine F-independent. Let $\mathrm{Span}^{\mathrm{aff}}_F(a_0, \dots, a_{t-1})$ and $\mathrm{Span}^{\mathrm{aff}}_F(a'_0, \dots, a'_{t-1})$ be denoted by V_{t-1} and V'_{t-1}, respectively. Since $t-1 < k = \dim^{\mathrm{aff}}_F(C) = \dim^{\mathrm{aff}}_F(C')$, there exist elements $x \in C \setminus V_{t-1}$ and $y' \in C' \setminus V'_{t-1}$. Then $\{a_0, \dots, a_{t-1}, x\}$ and $\{a'_0, \dots, a'_{t-1}, y'\}$ are affine F-independent. We can assume that $x' \in V'_{t-1}$ and $y \in V_{t-1}$ since otherwise $\{a'_0, \dots, a'_{t-1}, x'\}$ or $\{a_0, \dots, a_{t-1}, y\}$ would be affine F-independent, and we could choose an appropriate a_t from $\{x, y\}$. Take a $p \in I^o(T)$, and define $a_t := yx\underline{p} \in C$. Then $a'_t = y'x'\underline{p}$. Suppose for a contradiction that $a_t \in V_{t-1}$. Then, by Lemma 2.2, $x = ya_t\underline{1/p} \in V_{t-1}$, a contradiction. Hence $a_t \notin V_{t-1}$ and $\{a_0, \dots, a_{t-1}, a_t\}$ is affine F-independent. Similarly, suppose for a contradiction that $a'_t \in V'_{t-1}$. Then, again by Lemma 2.2, $y' = a'_t x' \underline{p/(p-1)} \in V'_{t-1}$ is a contradiction. Hence $a'_t \notin V'_{t-1}$ and $\{a_0, \dots, a_{t-1}, a'_t\}$ is affine F-independent. This completes the proof of (6).

From now on in the proof, (6) allows us to assume that $\{a_0, \dots, a_k\} \subseteq C$ and $\{a'_0, \dots, a'_k\} \subseteq C'$ are affine F-independent subsets with $a'_i = \varphi(a_i)$, for $i = 0, \dots, k$. For $\emptyset \ne X \subseteq F^n$, we define two "relatively rational" parts of X as follows:

$$\mathrm{rr}_{\vec{a}}(X) := X \cap \mathrm{Span}^{\mathrm{aff}}_{\mathbb{Q}}(a_0, \dots, a_k) \text{ and } \mathrm{rr}_{\vec{a}'}(X) := X \cap \mathrm{Span}^{\mathrm{aff}}_{\mathbb{Q}}(a'_0, \dots, a'_k).$$

If $F = \mathbb{Q}$, then Lemma 4.3(i) yields that

$$\mathrm{rr}_{\vec{a}}(C) = C \cap \mathrm{Span}^{\mathrm{aff}}_{\mathbb{Q}}(a_0, \dots, a_k) = C \cap \mathrm{Span}^{\mathrm{aff}}_{\mathbb{Q}}(C) = C,$$

and $\mathrm{rr}_{\vec{a}'}(C') = C'$ follows similarly. Moreover, even if $F \ne \mathbb{Q}$, $\mathrm{rr}_{\vec{a}}(C)$ is dense in C, and $\mathrm{rr}_{\vec{a}'}(C')$ is dense in C' (in topological sense). The restriction of a map α to a subset A of its domain will be denoted by $\alpha\rceil_A$. We claim that there is an automorphism ψ of $\mathrm{Aff}_F(F^n)$ such that

$$\psi\rceil_{\mathrm{rr}_{\vec{a}}(C)} = \varphi\rceil_{\mathrm{rr}_{\vec{a}}(C)} \quad \text{and} \quad \psi\big(\mathrm{rr}_{\vec{a}}(C)\big) = \mathrm{rr}_{\vec{a}'}(C'). \tag{7}$$

In order to prove this, extend $\{a_0, \dots, a_k\}$ and $\{a'_0, \dots, a'_k\}$ to maximal affine F-independent subsets $\{a_0, \dots, a_n\}$ and $\{a'_0, \dots, a'_n\}$ of $\mathrm{Aff}_F(F^n)$, respectively. Since $\{a_0, \dots, a_n\}$ and $\{a'_0, \dots, a'_n\}$ are free generating sets of $\mathrm{Aff}_F(F^n)$, there is a (unique) automorphism ψ of $\mathrm{Aff}_F(F^n)$ such that $\psi(a_i) = a'_i$ for $i = 0, \dots, n$.

Let $x \in \mathrm{rr}_{\vec{a}}(C)$ be arbitrary. Then there are $\xi_0, \dots, \xi_k \in \mathbb{Q}$ such that their sum equals 1 and

$$x = \xi_0 a_0 + \ldots + \xi_k a_k. \tag{8}$$

Observe that C and C' are $\mathbb{Q}$-convex and T-convex since they are F-convex. Hence we obtain from Lemma 4.2 and (8) that $\Phi^{\underline{I}^o(T)}_{\xi_0,\dots,\xi_k}(a_0, \dots, a_k; x)$ holds in $(C, \underline{I}^o(T))$. Since φ is an isomorphism, $\Phi^{\underline{I}^o(T)}_{\xi_0,\dots,\xi_k}(a'_0, \dots, a'_k; \varphi(x))$ holds in $(C', \underline{I}^o(T))$. Using Lemma 4.2 again, we conclude that $\varphi(x) = \xi_0 a'_0 + \ldots + \xi_k a'_k$. Therefore, (8) yields that $\psi(x) = \xi_0\psi(a_0) + \ldots + \xi_k\psi(a_k) = \xi_0 a'_0 + \ldots + \xi_k a'_k = \varphi(x) \in C'$. This gives that $\psi\rceil_{\mathrm{rr}_{\vec{a}}(C)} = \varphi\rceil_{\mathrm{rr}_{\vec{a}}(C)}$ and $\psi(x) \in \mathrm{rr}_{\vec{a}'}(C')$. Therefore, $\psi\big(\mathrm{rr}_{\vec{a}}(C)\big) \subseteq \mathrm{rr}_{\vec{a}'}(C')$. Working with (ψ^{-1},φ^{-1}) instead of (ψ,φ), we obtain $\psi^{-1}\big(\mathrm{rr}_{\vec a'}(C')\big) \subseteq \mathrm{rr}_{\vec a}(C)$ similarly. Thus, (7) holds.

If $F=\mathbb{Q}$, then (7) together with $C=\mathrm{rr}_{\vec a}(C)$ and $C'=\mathrm{rr}_{\vec a'}(C')$ implies the validity of the theorem. Thus we assume that at least one of C and C' is bounded. If, say, C is bounded, then so is $\mathrm{rr}_{\vec a}(C)$. The automorphisms of $\mathrm{Aff}_F(F^n)$ preserve this property, whence (7) implies that $\mathrm{rr}_{\vec a'}(C')$ is bounded. Since $\mathrm{rr}_{\vec a'}(C')$ is dense in C', we conclude that C' is bounded. Therefore, in the rest of the proof, we assume that both C and C' are bounded.

For $X\subseteq\mathbb{R}^n$, the topological closure of X, that is the set of cluster points of X, will be denoted by $[X]^{\mathrm{top}}_{\mathbb{R}}$. Let $C^*=\psi^{-1}(C')$. It is an F-convex subset of F^n since the automorphisms of $\mathrm{Aff}_F(F^n)$ are also automorphisms of $(F^n,\underline{I}^o(F))$. By the same reason, the restriction $\psi^{-1}\rceil_{C'}$ is an isomorphism $(C',\underline{I}^o(T))\to(C^*,\underline{I}^o(T))$. Let $\gamma:=\psi^{-1}\rceil_{C'}\circ\varphi$ (we compose maps from right to left). Then, by (7), by $\gamma(a_i)=a_i$ for $0\le i\le n$, and by Lemma 4.3, we know that

$$\begin{aligned}&\gamma\colon (C,\underline{I}^o(T))\to(C^*,\underline{I}^o(T))\quad\text{is an isomorphism,}\\ &\mathrm{rr}_{\vec a}(C)=\mathrm{rr}_{\vec a}(C^*),\quad\text{and}\quad \gamma\rceil_{\mathrm{rr}_{\vec a}(C)}\text{ is the identical map,}\\ &C\subseteq V:=\mathrm{Span}^{\mathrm{aff}}_F(a_0,\dots,a_k)\quad\text{and}\quad C^*\subseteq V.\end{aligned}\tag{9}$$

It suffices to show that γ is the identical map; really, then the desired $\varphi=\psi\rceil_C$ would follow by the definition of γ. For $y\in C$, the element $\gamma(y)$ will often be denoted by y^*. We have to show that $y^*=y$ for all $y\in C$. Since this is clear by (9) if $y\in\mathrm{rr}_{\vec a}(C)$, we assume that

$$y\in C\setminus\mathrm{rr}_{\vec a}(C).$$

Next, we deal with C and C^* simultaneously. Since they play a symmetric role, we usually give the details only for C.

If $\vec b=(b_1,b_2,b_3,\dots)\in\mathrm{rr}_{\vec a}(C)^\omega=\mathrm{rr}_{\vec a}(C^*)^\omega$, then $\vec b$ is called an $\mathrm{rr}_{\vec a}(C)$-sequence. Convergence (without adjective) is understood in the usual sense in $\mathbb{R}^n$. We use the notation $\lim_{j\to\infty}b_j=y$ to denote that $\vec b$ converges to y. We say that $\vec b$ $(C,\underline{I}^o(T))$-*converges* to y, in notation $\vec b\to_{(C,\underline{I}^o(T))}y$, if for each $j\in\mathbb{N}$,

$$\text{there exist an } x_j\in C \text{ and a } q_j\in I^o(T) \text{ such that } q_j\le 1/j \text{ and } b_j=yx_j\,\underline{q_j}.\tag{10}$$

In virtue of Lemma 2.2, $\vec b\to_{(C,\underline{I}^o(T))}y$ iff

$$\text{for each } j\in\mathbb{N},\ \text{there is a } q_j\in I^o(T) \text{ such that } q_j\le 1/j \text{ and } yb_j\,\underline{1/q_j}\in C\ .\tag{11}$$

It follows from (9) and (10) that for all $\vec b\in\mathrm{rr}_{\vec a}(C)^\omega$,

$$\vec b\to_{(C,\underline{I}^o(T))}y\quad\text{iff}\quad \vec b\to_{(C^*,\underline{I}^o(T))}y^*.\tag{12}$$

For $X\subseteq\mathbb{R}^n$, let $\mathrm{diam}(X)$ denote the *diameter* $\sup\{\mathrm{dist}(u,v):u,v\in X\}$ of X. We know that $\mathrm{diam}(C)<\infty$ and $\mathrm{diam}(C^*)<\infty$. Hence if $q_j\le 1/j$, then Lemma 2.2 yields that $\mathrm{dist}(y,b_j)=q_j\cdot\mathrm{dist}(y,yb_j\,\underline{1/q_j})\le\mathrm{diam}(C)/j$. Hence (11) gives that for any $\mathrm{rr}_{\vec a}(C)$-sequence $\vec b$,

$$\begin{aligned}&\text{if}\quad \vec b\to_{(C,\underline{I}^o(T))}y,\quad\text{then}\quad \lim_{j\to\infty}b_j=y.\quad\text{Similarly,}\\ &\text{if}\quad \vec b\to_{(C^*,\underline{I}^o(T))}y^*,\quad\text{then}\quad \lim_{j\to\infty}b_j=y^*.\end{aligned}\tag{13}$$

Next, we intend to show that

(14) there exists a $\mathrm{rr}_{\vec{a}}(C)$-sequence $\vec{b}$ such that $\vec{b} \to_{(C,\underline{I}^o(T))} y$.

Extend $\{y\}$ to a maximal affine F-independent subset $\{y, z_1, \dots, z_k\}$ of C. It follows from Lemma 4.3 that this set consists of $1+k$ elements, and V equals $\mathrm{Span}_F^{\mathrm{aff}}(y, z_1, \dots, z_k)$. For a given $j \in \mathbb{N}$, choose a $q_j \in I^o(T)$ such that $q_j \leq 1/j$. For $i = 1, \dots, k$, let $u_i := yz_i\,\underline{q_j}$. By the F-convexity of C, $u_i \in C$. Since $z_i = yu_i\,\underline{1/q_j}$ by Lemma 2.2, $\{y, u_1, \dots, u_k\}$ also F-spans V, whence it is affine F-independent by Lemma 4.3(iii). Hence $\mathrm{Cnv}_F(y, u_1, \dots, u_k) \subseteq C$ is a (non-degenerate) k-dimensional simplex of V, so its interior (understood in V) is nonempty. Since $\mathrm{rr}_{\vec{a}}(C)$ is dense in C and $\mathrm{rr}_{\vec{a}}(C) \subseteq C \subseteq V$, we can choose a point $b_j \in \mathrm{Cnv}_F(y, u_1, \dots, u_k)$. By (1), b_j is of the form $yu_1 \dots u_k \boldsymbol{\tau}$. Let $x_j := yz_1 \dots z_k\,\boldsymbol{\tau} \in C$. Using that $\underline{q_j}$ commutes with $\boldsymbol{\tau}$ and the terms are idempotent, we have that

$$\begin{aligned} yx_j\,\underline{q_j} &= y(yz_1 \dots z_k\,\boldsymbol{\tau})\,\underline{q_j} = (yy \dots y\,\boldsymbol{\tau})(yz_1 \dots z_k\,\boldsymbol{\tau})\,\underline{q_j} \\ &= (yy\,\underline{q_j})(yz_1\,\underline{q_j}) \dots (yz_k\,\underline{q_j})\boldsymbol{\tau} = yu_1 \dots u_k\boldsymbol{\tau} = b_j. \end{aligned}$$

(Notice that the parentheses above can be omitted.) Therefore, the sequence $\vec{b} = (b_1, b_2, \dots)$ proves (14).

Finally, it follows from (14), (12) and (13) that $y^* = y$. Therefore, γ is the identical map. ☐

Proof of Corollary 2.5. As we have already mentioned, with reference to [12], $(F^n, \underline{h})$ is term equivalent to $(F^n, \underline{I}^o(\mathbb{D}))$. Hence the first part of the statement is clear.

To prove the second part, assume that D and D' are isomorphic subalgebras of $(\mathbb{Q}^n, \underline{h})$ such that D' is a geometric subset of $\mathbb{Q}^n$. By Proposition 2.3(ii), D' is $\mathbb{Q}$-convex. Let $\varphi\colon (D, \underline{h}) \to (D', \underline{h})$ be an isomorphism. Let $a, b \in D$. Their φ-images are denoted by a' and b', respectively. If $y' \in D'$, then y will stand for $\varphi^{-1}(y') \in D$. Assume that $r/q \in I^o(\mathbb{Q})$ such that $r < q \in \mathbb{N}_0$; we have to show that $ab\,\underline{r/q} \in D$. Since D' is $\mathbb{Q}$-convex, $u_i' = a'b'\,\underline{i/q} \in D'$ for $i \in \{0, \dots, q\}$. Clearly, $u_j' = u_{j-1}'u_{j+1}'\,\underline{h}$ for $j \in \{1, \dots, q-1\}$. Hence, $u_j = u_{j-1}u_{+1}\,\underline{h}$ for all these j, and we conclude that $ab\,\underline{r/q} = u_r \in D$. This proves that D is $\mathbb{Q}$-convex. ☐

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