# SWING LATTICE GAME AND A DIRECT PROOF OF THE SWING LEMMA FOR PLANAR SEMIMODULAR LATTICES

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*Dedicated to the eighty-fifth birthday of Béla Csákány*

Abstract. The Swing Lemma, due to G. Grätzer for slim semimodular lattices and extended by G. Czédli, G. Grätzer, and H. Lakser for all planar semimodular lattices, describes the congruence generated by a prime interval in an efficient way. Here we present a new, direct proof of this lemma, which is shorter than the earlier ones. Also, motivated by the Swing Lemma and mechanical pinball games with flippers, we construct an online game called Swing Lattice Game.

## 1. Introduction

The last decade has witnessed a rapid development of the theory of planar semimodular lattices; see the bibliographic section in the present paper and the many additional papers referenced in the book chapter Czédli and Grätzer [7]. Since every planar semimodular lattice can be obtained from a slim semimodular lattice, particular attention was paid to slim (hence necessarily planar) semimodular lattices; definitions will be given later.

**First target: the Swing Lemma.** A finite lattice $L$ is *planar* if it has a planar Hasse-diagram. We will assume that a planar diagram of our lattice is *fixed* somehow. The edges of a planar semimodular lattice divide its diagram into quadrangles, which we call 4-*cells*. For a prime interval $\mathfrak{p} = [a, b]$ of $L$, that is, for an edge of the diagram, we denote $a$ and $b$ by $0_\mathfrak{p}$ and $1_\mathfrak{p}$, respectively. The least congruence collapsing (the two elements of) a prime interval $\mathfrak{p}$ is denoted by $\mathrm{con}(\mathfrak{p})$ or $\mathrm{con}(0_\mathfrak{p}, 1_\mathfrak{p})$. In order to characterize whether $\mathrm{con}(\mathfrak{p})$ collapses another prime interval $\mathfrak{q}$ or not, we need the following definition.

**Definition 1.1.** Let $\mathfrak{r}$ and $\mathfrak{s}$ be distinct prime intervals of a 4-cell $S$ of a planar semimodular lattice.

(i) If $\mathfrak{r}$ and $\mathfrak{s}$ are opposite sides of $S$ then $\mathfrak{r}$ is *cell-perspective* to $\mathfrak{s}$.
(ii) If $1_\mathfrak{r}$ has at least three lover covers, $1_\mathfrak{r} = 1_\mathfrak{s}$, and $0_\mathfrak{s}$ is neither the leftmost, nor the rightmost lower cover of $1_\mathfrak{r}$, then $\mathfrak{r}$ *swings* to $\mathfrak{s}$.
(iii) If $0_\mathfrak{r}$ has at least three covers, $0_\mathfrak{r} = 0_\mathfrak{s}$, and $1_\mathfrak{s}$ is neither the leftmost, nor the rightmost cover of $0_\mathfrak{r}$, then $\mathfrak{r}$ *tilts* to $\mathfrak{s}$.

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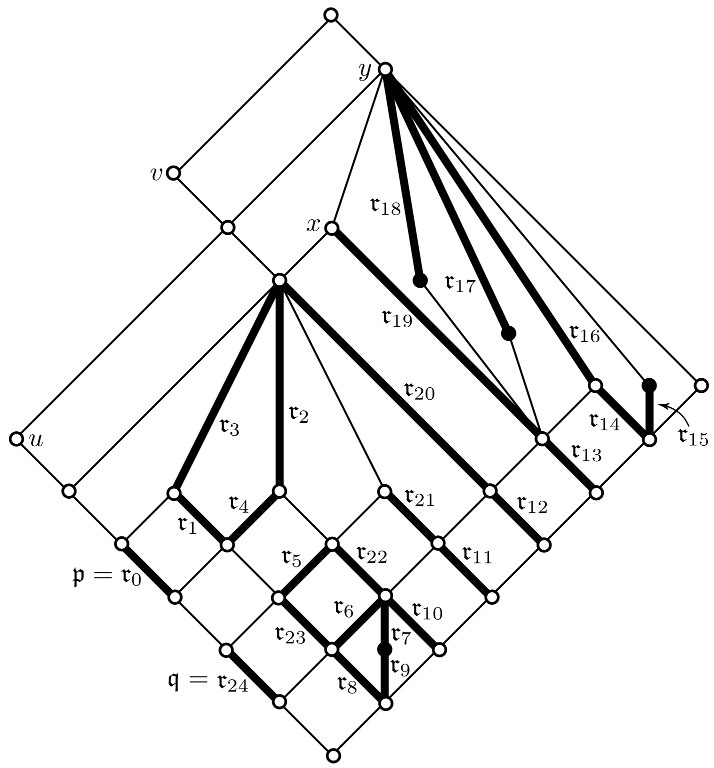


FIGURE 1. A GS-sequence from $\mathfrak{p}$ to $\mathfrak{q}$ in a planar semimodular lattice

For $n \in \{0, 1, 2, \dots\}$, a sequence

$$\vec{\mathfrak{r}} : \mathfrak{r}_0, \mathfrak{r}_1, \dots, \mathfrak{r}_n \tag{1.1}$$

of prime intervals is called a *GS-sequence* if for each $i \in \{1, \dots, n\}$, $\mathfrak{r}_{i-1}$ is cell-perspective to, swings to, or tilts to $\mathfrak{r}_i$. ("GS" is an acronym for "General Swing"; see Lemma 1.2 later.) In (1.1), $\mathfrak{r}_0$ and $\mathfrak{r}_n$ play a distinguished role, and we will say that $\vec{\mathfrak{r}}$ is a *GS-sequence from* $\mathfrak{r}_0$ *to* $\mathfrak{r}_n$. It is *cyclic* if $\mathfrak{r}_0 = \mathfrak{r}_n$.

While (i) describes a symmetric relation, (ii) and (iii) do not. To see some examples, consider the planar semimodular lattice in Figure 1. Then $\mathfrak{r}_{11}$ and $\mathfrak{r}_{12}$ are cell-perspective to each other, $\mathfrak{r}_2$ and $\mathfrak{r}_3$ swing to each other, so do $\mathfrak{r}_{16}$ and $\mathfrak{r}_{17}$; $\mathfrak{r}_8$ tilts to $\mathfrak{r}_9$, and $\mathfrak{r}_6$ swings to $\mathfrak{r}_7$. However, $\mathfrak{r}_9$ does not tilt to $\mathfrak{r}_8$ and $\mathfrak{r}_7$ does not swing to $\mathfrak{r}_6$. The sequence $\mathfrak{r}_0$, $\mathfrak{r}_1$, ..., $\mathfrak{r}_{24}$ is a GS-sequence from $\mathfrak{p}$ to $\mathfrak{q}$, and it remains a GS-sequence if we omit $\mathfrak{r}_7$ and $\mathfrak{r}_8$. In Figure 2, the sequence $\mathfrak{r}_0$, $\mathfrak{r}_1$, ..., $\mathfrak{r}_{14} = \mathfrak{r}_0$ is a cyclic GS-sequence in $\mathsf{M}_6$.

The following result was proved in Czédli, Grätzer, and Lakser [8].

**General Swing Lemma 1.2** (Czédli, Grätzer, and Lakser [8]). *Let $L$ be a planar semimodular lattice, and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals of $L$. Then $\langle 0_{\mathfrak{q}}, 1_{\mathfrak{q}} \rangle \in \operatorname{con}(\mathfrak{p})$ if and only if there is a GS-sequence from $\mathfrak{p}$ to $\mathfrak{q}$.*

As a bit stronger but more technical variant of the General Swing Lemma, we will formulate and prove Theorem 2.2. Although the proof of the General Swing

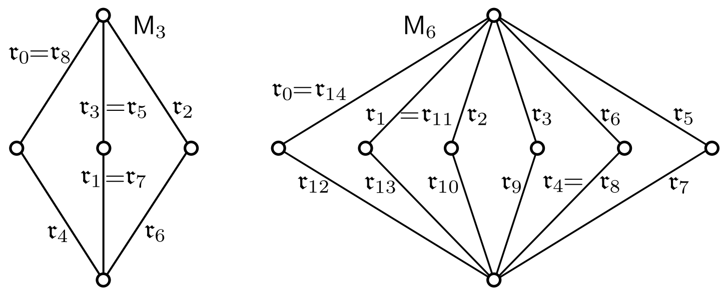


FIGURE 2. Cyclic SL-sequences in $\mathsf{M}_3$ and $\mathsf{M}_6$

Lemma in Grätzer, Czédli, and Lakser [8] is short, it relies on a particular case, which we will call *Swing Lemma* without the adjective "general"; see Section 3. The Swing Lemma is due to Grätzer [15] and there is another proof in Czédli [6], but both these papers give long and complicated proofs. Furthermore, the proof in [8] uses a nontrivial lemma from Czédli [3]. So, if [15] (or the relevant part of [6]) and the three pages from [3] are also counted, the proof of the General Swing Lemma is quite long. Our main goal is to give a direct and shorter proof.

**Second target: the Swing Lattice Game.** Section 5 describes our online game called *Swing Lattice Game*; it is available from the authors' websites. In addition to the General Swing Lemma, the game is also motivated by mechanical pinball games with flippers. This paper is dedicated to Professor Emeritus Béla Csákány, who is not only a highly appreciated algebraist and the scientific father or grandfather of almost all algebraists in Szeged, but has interest in mathematical games. This interest is witnessed by Csákány [2] and Csákány and Juhász [1].

## 2. Preliminaries and a short survey

This section gives a short survey of planar semimodular lattices; see Czédli and Grätzer [7] for a more extensive survey. For a recent general introduction to lattices, the reader can turn to the "A Brief Introduction to Lattices" part of Grätzer [16].

A lattice $L$ is *semimodular* if $x \preceq y \Rightarrow x \vee z \preceq y \vee z$ for all $x, y, z \in L$. A sublattice $S$ of $L$ is a *cover-preserving sublattice* if for any $a, b \in S$ such that $a \prec_S b$, we have that $a \prec_L b$. A *diamond* is an $\mathsf{M}_3$ (sub)lattice; see Figure 2. A lattice $L$ is *slim* if it contains no cover-preserving diamond and it is planar. This concept is due to Grätzer and Knapp [17]. It is motivated by Czédli and Schmidt [10] and Czédli and Grätzer [7, Lemma 3-4.1] that, as opposed to [17], we include planarity in the definition of slimness. For example, by Czédli and Grätzer [7, Theorem 3-4.3] or by Proposition 2.3, Figure 3 is a slim semimodular lattice. Also, if we omit the four black-filled elements from the planar semimodular lattice given in Figure 1, then we obtain a slim semimodular lattice. Note that planar lattices are *finite* by definition, and so are slim lattices.

In a planar semimodular lattice $L$, let $a < b$ but $a \not\prec b$. If $C_1$ and $C_2$ are maximal chains in the interval $[a, b]$ such that $C_1 \cap C_2 = \{a, b\}$ and every element of $C_2 \setminus \{a, b\}$ is on the right of $C_1$, then the elements of $[a, b]$ that are simultaneously on the right of $C_1$ and on the left of $C_2$ form a *region* of $L$. (Remember, the planar diagram of $L$ is always fixed.) Note that $C_1 \cup C_2$ is a subset of this region. For example, the

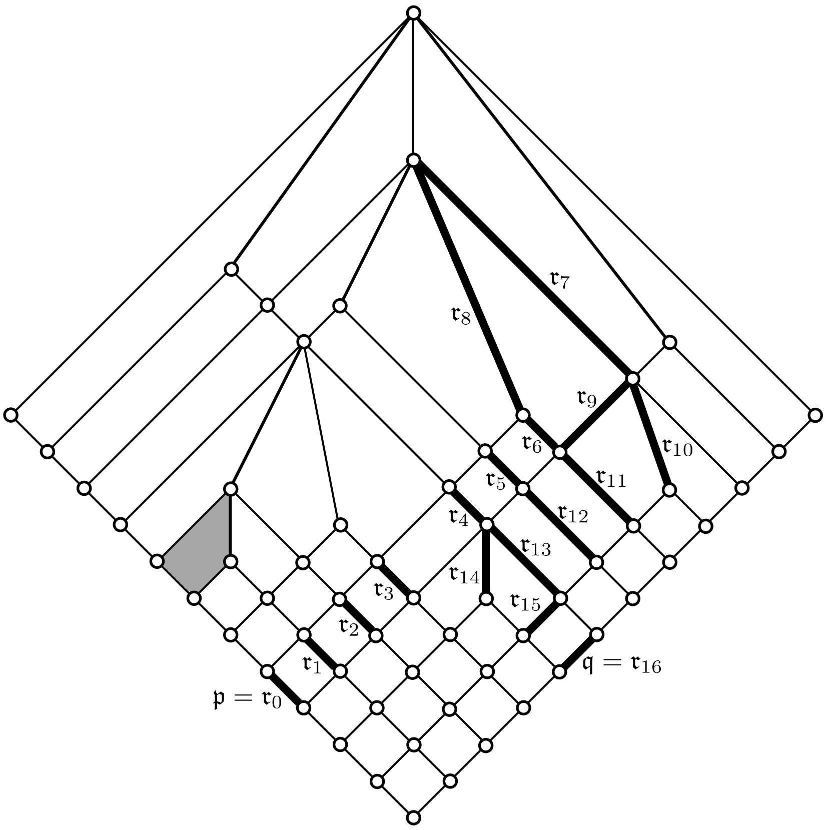


FIGURE 3. An (i&ii)-sequence from $\mathfrak{p}$ to $\mathfrak{q}$ in a slim semimodular (actually, a slim rectangular) lattice

elements belonging to the grey area in the second lattice of Figure 5 form a region denoted by $R$. We know from Kelly and Rival [20, Prop. 1.4 and Lemma 1.5] that, in a planar lattice,

(2.1) every interval is a region and every region is a cover-preserving sublattice.

If we drop the condition $C_1 \cap C_2 = \{a, b\}$ above, then we obtain a union (actually, a so-called glued sum) of regions, which also is a sublattice. More precisely, for elements $a < b$ in a planar lattice $L$,

(2.2) if $C_1$ and $C_2$ are maximal chains in $[a, b]$ such that every element of $C_2$ is on the right of $C_1$, then $\{x \in [a, b] : x$ is on the right of $C_1$ and on the left of $C_2\}$ is a cover-preserving sublattice of $L$.

For more about planar lattice diagrams (of planar semimodular lattices), see Kelly and Rival [20] (or Czédli and Grätzer [7]). Minimal regions are called *cells*. For example, the grey areas in Figure 3 and in the first lattice of Figure 5 are cells; actually, they are 4-*cells* since they are formed by four vertices and four edges. In a planar semimodular lattice, every cell is a 4-cell; see Grätzer and Knapp [17, Lemma 4]. Hence, by Czédli and Schmidt [11, Lemma 13],

(2.3) If $x$ and $y$ are neighboring lower covers of an element $z$ in a planar semimodular lattice, then $\{x \wedge y, x, y, z\}$ is a 4-cell.

A 4-cell can be turned into a diamond by adding a new element into its interior. The new element is called an *eye* and we refer to this step as *adding an eye*. Note that after adding an eye, one "old" 4-cell is replaced with two new 4-cells. We know from Czédli and Grätzer [7, Cor. 3-4.10] that

(2.4) every planar semimodular $L$ lattice is obtained from a slim semimodular lattice $L_0$ by adding eyes.

Note that $L_0$ is a sublattice of $L$. Although $L_0$ is not unique as a sublattice, it is unique up to isomorphism; see [7, Lemma 3-4.8]. We call $L_0$ the *full slimming* of $L$, while $L$ is an *antislimming* of $L_0$. Note that we can obtain the full slimming of $L$ by omitting all of its eyes. For example, we obtain the full slimming $L_0$ of the planar semimodular lattice $L$ given in Figure 1 by omitting the four black-filled elements. Conversely, we obtain $L$ from $L_0$ by adding an eye, four times. Based on, say, Grätzer and Knapp [17, Lemma 8], eyes are easy to recognize: an element $x$ of a planar semimodular lattice is an eye if and only if $x$ is doubly irreducible, its unique lower cover, denoted by $x_*$, has at least three covers, and $x$ is neither the leftmost, nor the rightmost cover of $x_*$.

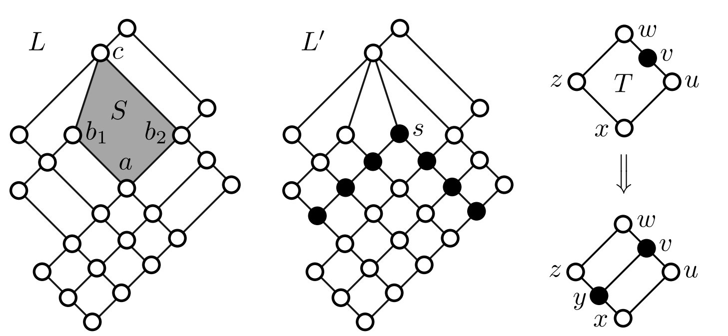


FIGURE 4. Inserting a fork

**Definition 2.1.** Let $\mathfrak{r}$ and $\mathfrak{s}$ be distinct edges of the *same* 4-cell in a planar semimodular lattice $L$, and let $\mathrm{Eyes}(L)$ denote the set of eyes of $L$.

(ii)′ $\mathfrak{r}$ *strongly swings* to $\mathfrak{s}$ if $\mathfrak{r}$ swings to $\mathfrak{s}$ and, in addition, the implication $0_{\mathfrak{r}} \in \mathrm{Eyes}(L) \Longrightarrow 0_{\mathfrak{s}} \in \mathrm{Eyes}(L)$ holds.

The sequence $\vec{\mathfrak{r}}$ in (1.1) will be called an SL-sequence if for each $i \in \{1, \dots, n\}$, the edge $\mathfrak{r}_{i-1}$ is cell-perspective to or tilts to or strongly swings to $\mathfrak{r}_i$. (The acronym "SL" comes from "Strong_Swing Lemma"; this is how Theorem 2.2 could be called.)

In a planar semimodular lattice,

(2.5) every SL-sequence is a GS-sequence,

but not conversely. For example, in Figure 1, the two-element sequence $\mathfrak{r}_{18}$, $[x, y]$ is a GS-sequence but not an SL-sequence. Now, we are in the position to formulate the following theorem, which is a stronger variant of Czédli, Grätzer, and Lakser [8]. By (2.5), this theorem implies Lemma 1.2, the General Swing Lemma.

**Theorem 2.2.** *If $L$ is a planar semimodular lattice and $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals of $L$, then the following two implications hold.*

(i) *If there exists a GS-sequence from $\mathfrak{p}$ to $\mathfrak{q}$ (in particular, if there is an SL-sequence from $\mathfrak{p}$ to $\mathfrak{q}$), then $\langle 0_{\mathfrak{q}}, 1_{\mathfrak{q}}\rangle \in \operatorname{con}(\mathfrak{p})$.*

(ii) *Conversely, if $\langle 0_{\mathfrak{q}}, 1_{\mathfrak{q}}\rangle \in \operatorname{con}(\mathfrak{p})$, then there exists an SL-sequence from $\mathfrak{p}$ to $\mathfrak{q}$.*

By (2.4), in order to have a satisfactory insight into planar semimodular lattices, it suffices to describe the slim ones. In order to do so, we need the following concepts.

Based on Czédli and Schmidt [11], Figure 4 shows how we *insert a fork* into a 4-cell $S$ of a slim semimodular lattice $L$ in order to obtain a new slim semimodular lattice $L'$. First, we add a new element $s$ into the interior of $S$. Next, we add two lower covers of $s$ that will be on the lower boundary of $S$ as indicated in the figure. Finally, we do a series of steps: as long as there is a chain $u \prec v \prec w$ such that $T = \{x = z \wedge u, z, u, w = z \vee u\}$ is a 4-cell in the original $L$ and $x \prec z$ at the present stage, then we insert a new element $y$ such that $x \prec y \prec z$ and $y \prec v$; see on the right of the figure. The new elements of $L'$, that is, the elements of $L' \setminus L$, are the black-filled ones in Figure 4.

A doubly irreducible element $x$ on the boundary of a slim semimodular lattice is called a *corner* if it has a unique upper cover $x^*$ and a unique lower cover $x_*$, $x^*$ covers exactly two elements, and $x_*$ is covered by exactly two elements. For example, after omitting the black-filled elements from Figure 1, there are exactly two corners, $u$ and $v$. Note that there is no corner in the slim semimodular lattice given by Figure 3. A *grid* is the (usual diagram of the) direct product of two finite non-singleton chains.

**Proposition 2.3** (Czédli and Schmidt [11]). *Every slim semimodular lattice with at least three elements can be obtained from a grid such that*

(i) *first we add finitely many forks,*

(ii) *and then we remove some corners, possibly no corner.*

*Furthermore, all lattices obtained in this way are slim and semimodular.*

Note that by Czédli and Schmidt [12, Prop. 2.3], the lattices we obtain by (i) but without (ii) are exactly the *slim rectangular lattices* introduced by Grätzer and Knapp [18]; see Figure 3 for an example. We can add eyes to these lattices to obtain the so-called *rectangular lattices*; see [12, Prop. 2.3] and Grätzer and Knapp [18].

## 3. Swing Lemma

**Definition 3.1.** The sequence $\mathfrak{r}$ from (1.1) is an (i&ii)-*sequence* if for each $i \in \{1, \dots, n\}$, the edge $\mathfrak{r}_{i-1}$ is cell-perspective to or swings to $\mathfrak{r}_i$.

For example, the edges $\mathfrak{r}_0$, $\mathfrak{r}_1$, …, $\mathfrak{r}_{16}$ in Figure 3 form an (i&ii)-sequence. In a planar semimodular lattice, every (i&ii)-sequence is a GS-sequence but, in general, not conversely. Since every element of a slim semimodular lattice has at most two covers by Grätzer and Knapp [17, Lemma 8], there are no tilts in *slim* semimodular lattices. That is,

(3.1) In a *slim* semimodular lattice, GS-sequences, SL-sequences, and (i&ii)-sequences are the same.

Therefore, the following statement is a particular case of Lemma 1.2.

**Swing Lemma 3.2** (Grätzer [15]). *Let $L$ be a slim semimodular lattice, and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals of $L$. Then $\langle 0_{\mathfrak{q}}, 1_{\mathfrak{q}}\rangle \in \mathrm{con}(\mathfrak{p})$ if and only if there is an* (i&ii)*-sequence from $\mathfrak{p}$ to $\mathfrak{q}$.*

Note that Grätzer [15] states this lemma in another way. In order to see that our version implies his version, we will make two easy observations. For prime intervals $\mathfrak{p}$ and $\mathfrak{q}$, if $1_{\mathfrak{p}} \vee 0_{\mathfrak{q}} = 1_{\mathfrak{q}}$ and $1_{\mathfrak{p}} \wedge 0_{\mathfrak{q}} = 0_{\mathfrak{p}}$, then $\mathfrak{p}$ is *up-perspective* to $\mathfrak{q}$ and $\mathfrak{q}$ is *down-perspective* to $\mathfrak{p}$. *Perspectivity* is the disjunction of up-perspectivity and down-perspectivity. As an important property of (i&ii)-sequences, we claim that, for prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ in a finite semimodular lattice $L$,

(3.2) If $\mathfrak{p}$ is up-perspective to $\mathfrak{q}$, then there is an (i&ii)-sequence $\vec{\mathfrak{r}} = \langle \mathfrak{r}_0, \dots, \mathfrak{r}_n\rangle$ from $\mathfrak{p}$ to $\mathfrak{q}$ such that $\mathfrak{r}_{i-1}$ is upward cell-perspective to $\mathfrak{r}_i$ for all $i \in \{1, \dots, n\}$. Conversely, if there is such an $\vec{\mathfrak{r}}$, then $\mathfrak{p}$ is up-perspective to $\mathfrak{q}$.

The second part of (3.2) is trivial. In order to see its first part, assume that $\mathfrak{p}$ is up-perspective to $\mathfrak{q}$, and pick a maximal chain $0_{\mathfrak{p}} = x_0 \prec x_1 \prec \cdots \prec x_n = 0_{\mathfrak{q}}$. For $i \in \{1, \dots, n\}$, the set $\{x_{i-1}, x_i, 1_{\mathfrak{p}} \vee x_{i-1}, 1_{\mathfrak{p}} \vee x_i\}$ is a covering square by semimodularity. (For more details, if necessary, see the argument justifying Figure 1 in Czédli and Schmidt [9].) Covering squares are 4-cells by Czédli and Grätzer [7, Thm. 3-4.3(v)], whence there is an (i&ii)-sequence $\vec{\mathfrak{r}}$ from $\mathfrak{p}$ to $\mathfrak{q}$ with the required property. This proves (3.2).

It is clear from Czédli and Schmidt [10, Lemma 2.8], and we can also derive it from Proposition 2.3 by induction, that in a slim semimodular lattice,

(3.3) For a *repetition-free* (i&ii)-sequence $\vec{\mathfrak{r}}$ from (1.1) in a *slim* semimodular lattice, if $\mathfrak{r}_{i-1}$ is up-perspective to $\mathfrak{r}_i$, then $\mathfrak{r}_{j-1}$ is up-perspective to $\mathfrak{r}_j$ for all $j \in \{1, 2, \dots, i\}$.

Now it is clear that, by (3.2) and (3.3), Lemma 3.2 and its original version in Grätzer [15] imply each other easily.

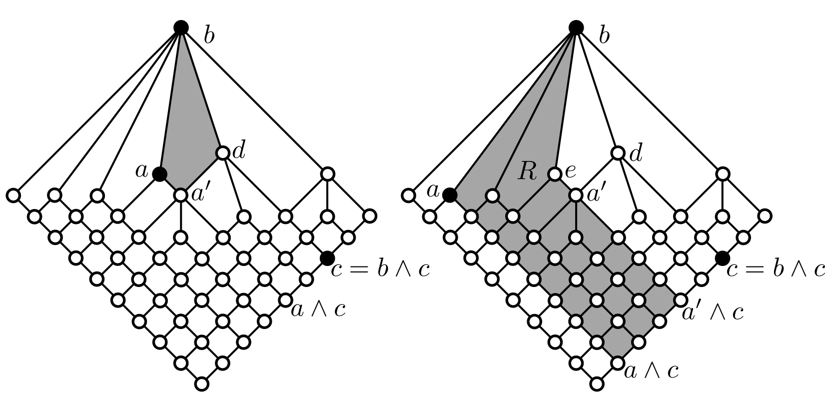


FIGURE 5. Illustration for (4.2)

## 4. A SHORTER, DIRECT PROOF

*Proof of Theorem 2.2.* Part (i) follows easily from known results and (2.5). For example, it follows from Czédli [5, Theorems 3.7 and 5.5 (or 7.3)] and Czédli [6, Thm. 2.2, Cor. 2.3, and Prop. 5.10]; however, the reader will certainly find it more convenient to observe that both $\mathrm{con}(w_\ell, t)$ and $\mathrm{con}(w_r, t)$ collapse the pairs $\langle s_i, t\rangle$ of $\mathsf{S}_7^{(\mathsf{n})}$ in [5, Fig. 1].

Before proving part (ii), some preparation is needed. For $n \in \{3, 4, 5, \dots\}$, let $\mathsf{M}_n$ be the $(n+2)$-element modular lattice of length 2. For example, $\mathsf{M}_3$ and $\mathsf{M}_6$ are given in Figure 2. As this figure suggests, it is easy to see that, for $n \in \{3, 4, 5, \dots\}$,

(4.1) $\mathsf{M}_n$ has a cyclic SL-sequence that contains all edges.

For a prime interval $\mathfrak{r}$ and elements $u \le v$ of a planar semimodular lattice $L$, we will say that $\mathfrak{r}$ *SSL-spans* (respectively, $\mathfrak{r}$ (i&ii)*-spans*) the interval $[u, v]$ if there is an $n \in \{0, 1, 2, \dots\}$ and there exists a maximal chain $u = w_0 \prec w_1 \prec \cdots \prec w_n = v$ in $[u, v]$ such that, for each $i \in \{1, \dots, n\}$, there is an SL-sequence (respectively, an (i&ii)-sequence) from $\mathfrak{r}$ to $[w_{i-1}, w_i]$. First, we focus on (i&ii)-spanning. We claim the following; see Figure 5.

(4.2) If $a, b, c$ are elements of a *slim* semimodular lattice $K$ such that $a \prec b$, then $[a, b]$ (i&ii)-spans $[a \wedge c, b \wedge c]$.

We prove (4.2) by induction on $|K|$. The base of the induction, $|K| \le 4$, is obvious. We can assume that $c \le b$, because otherwise we can replace $c$ with $b \wedge c$. Actually, we assume that $c < b$ but $c \nleq a$, since otherwise (4.2) is trivial. Pick an element $d$ such that $c \le d \prec b$; see Figure 5. Since $c \nleq a$ and $a \prec b$, the elements $a$ and $d$ are distinct lower covers of $b$. By left-right symmetry, we assume that $a$ is to the left of $d$. There are two cases to consider.

First, assume that among the lower covers of $b$, the element $a$ is immediately to the left of $d$; see the first lattice of Figure 5. Let $a' = a \wedge d$. By (2.3), $\{a', a, d, b\}$ is a 4-cell. Hence, there is a "one-step" (i&ii)-sequence from $[a, b]$ to $[a', d]$, which consists of a downwards cell-perspectivity. Observe that

$$a \wedge c = a \wedge (d \wedge c) = (a \wedge d) \wedge c = a' \wedge c$$

and the principal ideal $\downarrow d$ does not contain $a$. Hence, $|\downarrow d| < |K|$. Thus, the induction hypotheses yields that $[a', d]$ (i&ii)-spans $[a' \wedge c, c] = [a \wedge c, b \wedge c]$. This is witnessed by some (i&ii)-sequences; combining them with the one-step (i&ii)-sequence mentioned above, we conclude that $[a, b]$ (i&ii)-spans $[a \wedge c, b \wedge c]$, as required.

Second, assume that there is a lower cover of $b$ strictly to the right of $a$ and to the left of $d$. Let $e$ denote the rightmost one of these lower covers and let $a' := e \wedge d$; see the second lattice in Figure 5. Since $\{a', e, d, b\}$ is a 4-cell by (2.3), there is a one-step (i&ii)-sequence from $[e, b]$ to $[a', d]$. Combining it with a sequence of swings from $[a, b]$ to $[e, b]$, we obtain an (i&ii)-sequence from $[a, b]$ to $[a', d]$. Applying the induction hypothesis to $\downarrow d$, we obtain that $[a', d]$ (i&ii)-spans $[a' \wedge c, d \wedge c] = [a' \wedge c, b \wedge c]$. Taking the above-mentioned (i&ii)-sequence into account, it follows that $[a, b]$ (i&ii)-spans $[a' \wedge c, c] = [a' \wedge c, b \wedge c]$. We know from Czédli and Grätzer [7, Exercise 3.4] and it also follows from (2.1) that $a \wedge d \le e \wedge d = a'$. Hence, $a \wedge c = a \wedge d \wedge c \le a' \wedge c$. In the interval $[a \wedge c, b]$, let $C_2$ be a maximal chain such that $\{a' \wedge c, a', e\} \subseteq C_2$.

The elements of $[a \wedge c, b]$ on the left of $C_2$ form a cover-preserving sublattice $L_1$, because (2.2) applies for the leftmost maximal chain of $[a \wedge c, b]$ and $C_2$. Since $a$ is on the left of $e$, the element $a$ is in $L_1$ by Kelly and Rival [20, Prop. 1.6]. Pick a maximal chain $C_1$ in $L_1$ such that $a \in C_1$, and let $R$ denote the cover-preserving sublattice of $L_1$ determined by $C_1$ and $C_2$ in the sense of (2.2). Since $d$ is strictly on the right of $e \in C_2$, Kelly and Rival [20, Prop. 1.6] gives that $d \notin R$. Thus, $|R| < |K|$. Hence, the induction hypothesis applies for $\langle R, a, b, a' \wedge c\rangle$ as $\langle K, a, b, c\rangle$, and we obtain that $[a, b]$ (i&ii)-spans $[a \wedge c, a' \wedge c]$ in $R$. Since $R$ is a cover-preserving sublattice and also a region, the same holds in $K$. Thus, since $[a, b]$ (i&ii)-spans both $[a \wedge c, a' \wedge c]$ and $[a' \wedge c, b \wedge c]$, it (i&ii)-spans $[a \wedge c, b \wedge c]$. This proves (4.2).

Next, we claim that

(4.3) If $a, b, c$ are elements of a *planar* semimodular lattice $L$ such that $a \prec b$, then $[a, b]$ SSL-spans $[a \wedge c, b \wedge c]$.

It follows from (3.1) that (4.3) generalizes (4.2). In order to prove (4.3), let $K$ denote the full slimming of $L$. Its elements and edges will be called *old*, while the rest of elements and edges are *new*; this terminology is explained by (2.4) and the paragraph following it. The new elements are exactly the eyes. As in the proof of (4.2), we can assume that $c < b$ but $c \nleq a$. First, we deal only with the case where $[a, b]$ is an *old edge*. Since (the segments of) (i&ii)-sequences are also SL-sequences by (3.1), we conclude from (4.1) that

(4.4) if $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are old edges and there is an (i&ii)-sequence from $\mathfrak{s}_1$ to $\mathfrak{s}_2$ in $K$, then there is an SL-sequence from $\mathfrak{s}_1$ to $\mathfrak{s}_2$ in $L$.

Hence, for an old prime interval $\mathfrak{s}$ and old elements $u \leq v$,

(4.5) if $\mathfrak{s}$ (i&ii)-spans $[u, v]$ in $K$, then $\mathfrak{s}$ SSL-spans $[u, v]$ in $L$.

If $c$ is also an old element, then $\{a \wedge c, b \wedge c\} \subseteq K$, so the validity of (4.3) follows from (4.2) and (4.5). Hence, we can assume that $c$ is an eye. Let $c^*$ and $c_*$ stand for its (unique) cover and lower cover, respectively; they are old elements. Since $c < b$ and $c$ is meet-irreducible, $c^* \leq b$. (4.2) yields that $[a, b]$ (i&ii)-spans $[a \wedge c_*, b \wedge c_*] = [a \wedge c_*, c_*]$ in $K$. Since $c \nleq a$, we have that $a \wedge c < c$. Using that $c$ is join-irreducible, we obtain that $a \wedge c = a \wedge c_*$. Hence, by (4.5),

(4.6) $[a, b]$ SSL-spans $[a \wedge c, c_*] = [a \wedge c_*, c_*]$ in $L$.

On the other hand, $a \wedge c^* < c^*$, since otherwise $c < c^* \leq a$ would contradict $c \nleq a$. (4.2) yields that $[a, b]$ (i&ii)-spans $[a \wedge c^*, b \wedge c^*] = [a \wedge c^*, c^*]$. Thus, we can pick an old element $w$ such that $a \wedge c^* \leq w \prec c^*$ and there is an (i&ii)-sequence from $[a, b]$ to $[w, c^*]$ in $K$. By (4.4), we have an SL-sequence from $[a, b]$ to $[w, c^*]$ in $L$. By left-right symmetry, we can assume that $w$ is to the left of $c$. Listing them from left to right, let $w = w_0, w_1, \dots, w_t$ be the old lower covers of $c^*$ that are neither strictly to the left of $w$, nor strictly to the right of $c$; see Figure 6 for $t = 3$. Note that the old elements are empty-filled while the new ones are black-filled, and the elements in the figure do not form a sublattice. Let $w_{t+1}$ be the neighboring old lower cover of $c^*$ to the right of $w_t$ in $K$; it is also to the right of $c$. By (2.3), $\{w_{i-1} \wedge w_i, w_{i-1}, w_i, c^*\}$ is a 4-cell of $K$ for $i \in \{1, \dots, t\}$; these 4-cells are colored by alternating shades of grey in the figure. Clearly, $[w_{i-1}, c^*]$ strongly swings to $[w_i, c^*]$ in $K$, for $i \in \{1, \dots, t\}$. Hence, there is an (i&ii)-sequence in $K$ from $[w, c^*] = [w_0, c^*]$ to $[w_t, c^*]$. By (4.4), we have an SL-sequence from $[w, c^*]$, and thus also from $[a, b]$, to $[w_t, c^*]$. Also, since $c_*$, $c^*$, $w_t$, $w_{t+1}$, and the lower covers of $c^*$ between $w_t$ and $w_{t+1}$ form a region in $L$ and a cover-preserving sublattice $\mathsf{M}_n$ for some $n$, (4.1) allows us to continue the above-mentioned SL-sequence to $[c_*, c]$. Hence, $[a, b]$ SSL-spans $[c_*, c] = [c_*, b \wedge c]$ in $L$. This fact and (4.6) yield that $[a, b]$ SSL-spans $[a \wedge c, b \wedge c]$ in $L$, proving (4.3) for *old edges* $[a, b]$.

Second, we assume that $[a,b]$ is a new edge. If $b$ is an eye, which has only one lower cover, then $c<b$ gives that $c\le a$, whence $[a\wedge c, b\wedge c]$ is a singleton, which is clearly SSL-spanned. So we can assume that $a$ is an eye with upper and lover covers $a^*=b$ and $a_*$, respectively. Let $S=\{a_*, w_\ell, w_r, b\}$ denote the 4-cell of $K$ into which $a$ has been added. Here this is understood so that several eyes could have been added to this 4-cell simultaneously, whence $[a_*,b]_L$ is isomorphic to $\mathsf{M}_n$ for some $n\in\{3,4,\dots\}$. Applying (4.1) to $[a_*,b]_L$ and using (2.1), we obtain that

$$[a,b] \text{ SSL-spans both } [a_*,w_r] \text{ and } [w_r,b] \text{ in } L. \tag{4.7}$$

Since (4.3) has already been proved for "old" edges,

$$[a_*,w_r] \text{ SSL-spans } [a_*\wedge c, w_r\wedge c] \text{ and } [w_r,b] \text{ SSL-spans } [w_r\wedge c, b\wedge c]. \tag{4.8}$$

In (4.7), prime intervals are SSL-spanned, whence (4.7) yields SL-sequences. Combining these SL-sequences with those provided by (4.8) and using transitivity, we obtain that $[a,b]$ SSL-spans $[a_*\wedge c, b\wedge c]$. Hence, we need to show only that $a_*\wedge c = a\wedge c$. If we had that $a\le c$, then $a\prec b$ and $b<c$ would give that $a=c$, contradicting $c\nleq a$. Thus, $a\nleq c$ and $a\wedge c<a$. Since $a_*$ is the only lower cover of $a$, we have that $a\wedge c\le a_*$ and so $a\wedge c\le a_*\wedge c$. Since the converse inequality is obvious, $a\wedge c = a_*\wedge c$, as required. This completes the proof of (4.3).

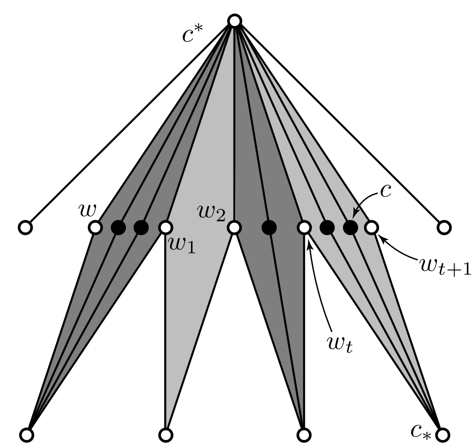


FIGURE 6. From $[w,c^*]$ to $[c_*,c]$

Next, let $\boldsymbol{\alpha}=\{\langle x,y\rangle\in L^2 : \mathfrak{p} \text{ SSL-spans } [x\wedge y, x\vee y]\}$, where $\mathfrak{p}$ is the prime interval from Theorem 2.2(ii). We are going to show that $\boldsymbol{\alpha}$ is a congruence. Obviously, $\langle x,y\rangle\in\boldsymbol{\alpha} \iff \langle x\wedge y, x\vee y\rangle\in\boldsymbol{\alpha}$ and

$$\bigl(x\le y\le z,\ \langle x,y\rangle\in\boldsymbol{\alpha},\ \text{and } \langle y,z\rangle\in\boldsymbol{\alpha}\bigr)\Longrightarrow\langle x,z\rangle\in\boldsymbol{\alpha}. \tag{4.9}$$

Hence, by Grätzer [14, Lemma 11], it suffices to show that whenever $x\le y$, $\langle x,y\rangle\in\boldsymbol{\alpha}$, and $z\in L$, then $\langle x\vee z, y\vee z\rangle\in\boldsymbol{\alpha}$ and $\langle x\wedge z, y\wedge z\rangle\in\boldsymbol{\alpha}$. To do so, pick a maximal chain $x=u_0\prec u_1\prec\cdots\prec u_n=y$ that witnesses $\langle x,y\rangle\in\boldsymbol{\alpha}$. Then, for each $i\in\{1,\dots,n\}$, there is an SL-sequence from $\mathfrak{p}$ to $[u_{i-1},u_i]$. By (4.3), $\langle u_{i-1}\wedge z, u_i\wedge z\rangle\in\boldsymbol{\alpha}$ for $i\in\{1,\dots,n\}$, and (4.9) yields that $\langle x\wedge z, y\wedge z\rangle = \langle u_0\wedge z, u_n\wedge z\rangle\in\boldsymbol{\alpha}$. By semimodularity, either $[u_{i-1},u_i]$ is up-perspective to $[u_{i-1}\vee z, u_i\vee z]$, or $u_{i-1}\vee z = u_i\vee z$. Hence, either by (3.2) or trivially, $\langle u_{i-1}\vee$

$z, u_i \vee z\rangle \in \boldsymbol{\alpha}$. Thus, (4.9) implies that $\langle x \vee z, y \vee z\rangle = \langle u_0 \vee z, u_n \vee z\rangle \in \boldsymbol{\alpha}$, and we have shown that $\boldsymbol{\alpha}$ is a congruence.

Finally, since $\boldsymbol{\alpha}$ collapses $\mathfrak{p}$, we have that $\operatorname{con}(\mathfrak{p}) \subseteq \boldsymbol{\alpha}$. So if $\langle 0_{\mathfrak{q}}, 1_{\mathfrak{q}}\rangle \in \operatorname{con}(\mathfrak{p})$, then the containment $\langle 0_{\mathfrak{q}}, 1_{\mathfrak{q}}\rangle \in \boldsymbol{\alpha}$ and the definition of $\boldsymbol{\alpha}$ yield an SL-sequence from $\mathfrak{p}$ to $\mathfrak{q}$. This completes the proof of Theorem 2.2. □

**Remark 4.1.** For a *slim* semimodular lattice $L$, (4.3) is equivalent to (4.2) by (3.1). Actually, (4.3) is not needed in this case. In this way, we obtain a proof for the Swing Lemma (Lemma 3.2) that is much shorter than the proof above.

## 5. Swing Lattice Game

The Swing Lattice Game is available at

`http://www.math.u-szeged.hu/~czedli/swinglattice/` or
`http://www.math.u-szeged.hu/~makay/swinglattice/`

as a JavaScript program. The program works in the latest versions of most modern browsers, including Mozilla Firefox, Google Chrome, Internet Explorer, and Microsoft Edge.

After describing the Game, we will point out its connections to Theorem 2.2 and, more generally, to planar semimodular lattices. In order to demonstrate that the player need not be a mathematician and does not have to know what a lattice is, we describe the game in a plane language that avoids the terminology of lattices as much as possible. The description below is close to what the program gives to its users; the main difference is that the program displays the diagrams instead of describing them as "slim semimodular".

5.1. **Description of the Game.** The program displays a diagram $D$ of a random slim semimodular lattice $L$ of length in the interval $[6, 13]$ (default) or $[4, 29]$ (upon request). An invisible *monkey* keeps moving from edge to edge such that the two edges in question have to belong to the same cell. The monkey's recent position is always indicated by a red thick edge. The monkey moves at a constant speed at the beginning; later, this speed slowly increases. The monkey can

(i) either *jump* to the opposite edge of the same cell,
(ii) or *swing* to the other upper edge of the same cell provided that the destination edge hangs between two other edges from the same vertex,
(iii) or *tilt* to the other lower edge of the same cell provided that the destination edge stands between two other edges from the same vertex.

However, the monkey cannot move back to the edge it came from in the very next step. If the monkey can make several moves, then the program chooses the actual move randomly. The monkey looses a life when no move is possible; this can happen only at a boundary edge of the lattice. If a life is lost but the monkey still has at least one life, then the game continues on a new random diagram. When the monkey has no more lives left, the game terminates.

The purpose is to keep the monkey alive as long as the player's luck and, much more significantly, his skill allow. The player's tools are the following.

(5.1) When a new diagram $D$ appears, the player has three seconds to choose the initial edge.

If the player is late, the program chooses the initial edge randomly. At any time,

(5.2) the player is allowed to add an eye to one of the cells by a mouse click;

this eye will be indicated by a black-filled vertex. Whenever he adds a new eye, the old one disappears. Therefore, except for the beginning when there is no eye, the diagram contains exactly one black-filled vertex. If the player clicks on a cell while the monkey is moving from or to an edge ending at the old eye, then the action of adding a new eye is delayed till the monkey arrives at an edge not adjacent to the old eye. From time to time in a random way, the program turns a cell into a *grey bonus cell*;

(5.3) if the monkey can jump or swing between two edges of the grey cell within ten moves, then it earns an extra life.

Similarly, the program offers *blue candidate cells* from time to time randomly. If the player accepts the candidate cell by clicking on it within three moves, then this cell becomes a *purple adventure cell* and

(5.4) the monkey earns two extra lives if it jumps or swings between two edges of the adventure cell within 20 moves but it looses a life otherwise.

When the monkey has no more lives left, the game terminates. Using (5.2) appropriately, the player can increase the probability that the monkey will go in a desired direction. In order to make a good decision how to use (5.1), when to use (5.4), and when and how to apply (5.2), the player should have some experience and insight into the process. Hence, the Swing Lattice Game is not only a reflex game.

5.2. **Mathematics beyond the Game.** The Swing Lattice Game grew out from Lattice Theory; our motivation was to make lattices more popular. Besides the general development of the theory of planar semimodular lattices as surveyed in Czédli and Grätzer [7], three milestones toward the Game are worth separate mentioning.

First, in Czédli [6], a class $\mathcal{C}_2$ of aesthetic planar semimodular lattice diagrams has been introduced. Instead of repeating the long definition of $\mathcal{C}_2$ here, we only mention that the diagrams in Figures 1, 3, and 5 and $L'$ in Figure 4 belong to $\mathcal{C}_2$, but the diagrams in Figure 2 and $L$ in Figure 4 do not. Whenever the program displays a new diagram $D$, the diagram belongs to $\mathcal{C}_2$. After adding an eye, $D$ turns into the diagram $D' \in \mathcal{C}_2$ of a larger planar semimodular lattice.

Second, the diagrams of length $n$ in $\mathcal{C}_2$ are given by their Jordan-Hölder permutations belonging to the symmetric group $S_n$; see Czédli and Schmidt [13] for details. Since not every diagram in $\mathcal{C}_2$ of a given length is appropriate for the game, the program defines the concept of "good diagrams". For example, neither a distributive diagram, nor a glued sum decomposable diagram is good. We have characterized goodness in terms of permutations. Whenever a new diagram is needed, the program generates a random good permutation $\pi \in S_n$, and the diagram is derived from $\pi$. The lattice theoretical background of this algorithm is not quite trivial. However, instead of going into details in the *present* paper, we only mention that several tools given by Czédli [4] and [6] and Czédli and Schmidt [13] have extensively been used.

Third, the main link between the Game and Lattice Theory is Theorem 2.2. Let us call a GS-sequence $\vec{\mathfrak{r}}$ from (1.1) an *SLG-sequence* if, for $i \in \{1, 2, \dots, n\}$, $\mathfrak{r}_{i-1} \neq \mathfrak{r}_i$. (The acronym comes from "Swing Lattice Game".) As long as the player does not change the lattice by adding an eye or repositioning the eye, the edges visited by the monkey in the Game form an *SLG-sequence*.

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