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ON CONGRUENCE n-DISTRIBUTIVITY OF ORDERED ALGEBRAS

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1. Introduction. The triple $(A, F, \leqq)$ is said to be an ordered algebra of type τ if (A, F) is a universal algebra of type τ, $(A, \leqq)$ is a partially ordered set, and all $f \in F$ are monotone with respect to $\leqq$. It is worth mentioning that τ-terms induce monotone term-functions on ordered τ-algebras. For τ-terms g, h the string $g \leqq h$ is called an order-identity or, shortly, identity. (Note that an identity $g=h$ is equivalent to the conjunction of $g \leqq h$ and $h \leqq g$). Let **H**, **S**, **P** be the operators of taking homomorphic images, subalgebras and direct products, respectively. (These concepts are defined in the natural way. I.e., a homomorphism is a *monotone* map preserving the operations, $u \leqq v$ in $\prod_{\gamma \in \Gamma} A_\gamma$ means $(\forall \gamma \in \Gamma)(u(\gamma) \leqq v(\gamma))$ and the original order is restricted in case of subalgebras.) The following result of Bloom [2] shows that the counterpart of the classical Birkhoff Theorem is valid for classes of ordered algebras: **HSP** is a closure operator on classes of similar ordered algebras, and a class of similar ordered algebras is closed under **HSP** iff it can be defined by a set of order-identities.

The concept of n-distributivity was introduced by Huhn [8, 10]. This concept has proved to be a very useful tool in several investigations (cf., e.g., Huhn [8, 9, 10] and Herrmann—Huhn [7]).

A lattice is called *n-distributive* if the n-distributive identity

$$x \wedge \bigvee_{i=0}^{n} y_i \leqq \bigvee_{j=0}^{n} \big(x \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i\big)$$

holds in it.

A variety of lattices is said to be a congruence variety (Jónsson [13]) if it is generated by the class of congruence lattices of all members of some variety of universal algebras. It is known (cf. Nation [16]) that n-distributive congruence varieties are distributive, and this fact plays an important rôle in the theory of congruence varieties. Our aim is to generalize this result for the case of ordered algebras.

2. Order-congruences. If congruence relations of an ordered algebra $(A, F, \leqq)$ were defined as congruences of (A, F), they would not depend on the ordering. Moreover, there would be no reasonable way to define orders on factor algebras so that factor algebras would be order-homomorphic images under the canonical map. That is why the concept of order-congruences is introduced. Since our motivation will be given only in Proposition 2.1, the following definition might seem astounding at the first sight.

DEFINITION. A binary relation Θ is called an order-congruence of the ordered algebra $(A, F, \leqq)$ if Θ is a congruence of the universal algebra (A, F) and

$$(*)\qquad \begin{cases} \text{whenever} \quad a, b, a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_k \in A \quad \text{such that} \\ a = a_0 \Theta a_1 \leqq a_2 \Theta a_3 \leqq a_4 \Theta a_5 \leqq \ldots a_m = b = b_0 \Theta b_1 \leqq b_2 \Theta b_3 \leqq \\ \leqq b_4 \Theta b_5 \leqq \ldots b_k = a \quad \text{then} \quad a \Theta b. \end{cases}$$

PROPOSITION 2.1. *Assume that Θ is a binary relation on an ordered algebra A. Then Θ is an order-congruence if and only if there exist an ordered algebra B and a homomorphism $\varphi\colon A \to B$ such that $\Theta = \operatorname{Ker} \varphi$.*

PROOF. Suppose Θ is an order-congruence of $(A, F, \leqq)$. Set $B = A/\Theta$ and, for $a, b \in A$, define $[a]\Theta \leqq [b]\Theta$ by "there exist $a_0, a_1, \ldots, a_t \in A$ such that $a = a_0 \Theta a_1 \leqq a_2 \Theta a_3 \leqq a_4 \Theta \ldots a_t = b$". The reflexivity of Θ and $\leqq$ (over A) together with $(*)$ yield that $\leqq$ is an ordering of B. If f is an m-ary operation of A and $[a^i]\Theta \leqq [b^i]\Theta$ $(i = 1, \ldots, m)$ then we have $a^i = a_0^i \Theta a_1^i \leqq a_2^i \Theta \ldots a_t^i = b^i$ where, without loss of generality, we assume that t does not depend on i. Since f preserves both Θ and $\leqq$ we obtain

$$f(a^1, \ldots, a^m) = f(a_0^1, \ldots, a_0^m) \Theta f(a_1^1, \ldots, a_1^m) \leqq f(a_2^1, \ldots, a_2^m) \Theta \ldots f(a_t^1, \ldots, a_t^m) = \\ = f(b^1, \ldots, b^m),$$

which shows that

$$f([a^1]\Theta, \ldots, [a^m]\Theta) = [f(a^1, \ldots, a^m)]\Theta \leqq [f(b^1, \ldots, b^m)]\Theta = f([b^1]\Theta, \ldots, [b^m]\Theta).$$

Hence, equipped with this ordering, B is an ordered algebra. Now the map $\varphi\colon A \to B$, $a\varphi = [a]\Theta$ is a homomorphism and $\Theta = \operatorname{Ker} \varphi$.

Conversely, if $\Theta = \operatorname{Ker} \varphi$ for some homomorphism φ and $a = a_0 \Theta a_1 \leqq a_2 \Theta \ldots a_m = b = b_0 \Theta b_1 \leqq b_2 \Theta \ldots b_k = a$ then $a\varphi = a_0\varphi = a_1\varphi \leqq a_2\varphi = \ldots = a_m = b$, implying $a\varphi \leqq b\varphi$. Since $b\varphi \leqq a\varphi$ follows similarly, $a\varphi = b\varphi$, whence $a \Theta b$. Q.e.d.

Let us mention two examples. The additive group $Z = (Z, +, \leqq)$ of integers with the usual ordering has many congruences, but it has only the two trivial order-congruences. (Indeed, its proper factor groups do not admit nontrivial orderings.) In case of lattices equipped with the usual ordering congruences and order-congruences are the same.

For an ordered algebra A let $\operatorname{Con}(A)$ denote the set of *order*-congruences of A. Since the meet of arbitrary many order-congruences is an order-congruence again, $\operatorname{Con}(A)$ is a complete lattice with respect to the set-theoretic inclusion. The join in $\operatorname{Con}(A)$ is described in the following

PROPOSITION 2.2. *Let A be an ordered algebra and let $\Theta_0, \Theta_1, \ldots, \Theta_k$ be order-congruences of A. Set $\Phi = \{(a, b) \in A^2 \mid$ there exist $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_t \in A$ such that*

$$a = a_0 \Theta_0 a_1 \Theta_1 a_2 \Theta_2 \ldots a_k \Theta_k a_{k+1} \leqq a_{k+2} \Theta_0 a_{k+3} \Theta_1 \ldots a_{2k+2} \Theta_k a_{2k+3} \leqq \ldots \leqq a_m = b$$

and

$$b = b_0 \Theta_0 b_1 \Theta_1 \ldots b_k \Theta_k b_{k+1} \leqq b_{k+2} \Theta_0 b_{k+3} \Theta_1 \ldots b_{2k+2} \Theta_k b_{2k+3} \leqq \ldots b_t = a\}.$$

Then $\Phi = \bigvee_{i=0}^{n} \Theta_i$ in the lattice $\operatorname{Con}(A)$.

PROOF. It is straightforward to check that Φ is an order-congruence. The inclusion $\Theta_i \subseteq \Phi$ is trivial. On the other hand, if $\Psi \in \mathrm{Con}(A)$ and $\Theta_i \subseteq \Psi$ for all i then $(*)$ yields $\Phi \subseteq \Psi$. Q.e.d.

3. The main theorem. Now we can formulate

THEOREM 3.1. *For any class $\mathcal{U}$ of ordered algebras which is closed under taking subalgebras and direct products, and for any natural number n the following three conditions are equivalent:*

(i) $\mathrm{Con}(A)$ *is n-distributive for all* $A \in \mathcal{U}$;

(ii) $\mathrm{Con}(A)$ *is distributive for all* $A \in \mathcal{U}$;

(iii) *There exist a natural number k and ternary terms* $t_0(x, y, z), t_1(x, y, z), \ldots, t_k(x, y, z)$ *(corresponding to the type of $\mathcal{U}$) such that the identities*

$$t_0(x,y,z) = x, \quad t_k(x,y,z) = z, \quad t_i(x,y,x) = x \quad \textit{for} \quad i = 0, 1, \ldots, k,$$

$$t_i(x,x,y) = t_{i+1}(x,x,y) \quad \textit{for} \quad i \equiv 0\ (3),\ 0 \leqq i < k,$$

$$t_i(x,y,y) = t_{i+1}(x,y,y) \quad \textit{for} \quad i \equiv 1\ (3),\ 0 \leqq i < k,$$

$$t_i(x,y,z) \leqq t_{i+1}(x,y,z) \quad \textit{for} \quad i \equiv 2\ (3),\ 0 \leqq i < k$$

hold in $\mathcal{U}$.

Before proving this theorem let some consequences and examples be mentioned.

COROLLARY 3.2. (Jónsson [12]). *A variety $\mathcal{V}$ of universal algebras of type τ is congruence distributive iff there exist a natural number k and ternary τ-terms $t_0, t_1, \ldots, t_k$ such that the identities*

$$t_0(x,y,z) = x, \quad t_k(x,y,z) = z, \quad t_i(x,y,x) = x \quad \textit{for} \quad 0 \leqq i \leqq k,$$

$$t_i(x,x,y) = t_{i+1}(x,x,y) \quad \textit{for} \quad i \textit{ even},\ 0 \leqq i < k,$$

$$t_i(x,y,y) = t_{i+1}(x,y,y) \quad \textit{for} \quad i \textit{ odd},\ 0 \leqq i < k$$

hold in $\mathcal{V}$.

COROLLARY 3.3. (Nation [16]). *If a variety $\mathcal{V}$ of universal algebras is congruence n-distributive then it is congruence distributive.*

Both corollaries follow by the same consideration: Equip the members of $\mathcal{V}$ with the trivial order. Then congruences are the same as order-congruences and an order-identity $t_i(x,y,z) \leqq t_{i+1}(x,y,z)$ is equivalent to $t_i(x,y,z) = t_{i+1}(x,y,z)$.

If we call lattice varieties generated by the class $\{\mathrm{Con}(A) | A \in \mathcal{U}\}$ for some variety of ordered algebras $\mathcal{U}$ *order-congruence varieties* and denote by $\mathcal{M}(T)$ the variety of all vector spaces over a field T then we can describe the minimal modular order-congruence varieties:

COROLLARY 3.4. *For any modular but not distributive order-congruence variety $\mathcal{U}$ there exists a prime field T such that the (order-) congruence variety*

$$\mathbf{HSP}\,\{\mathrm{Con}(V) | V \in \mathcal{M}(T)\}$$

is a subvariety of $\mathcal{U}$. (Note that $\mathbf{HSP}\,\{\mathrm{Con}(V) | V \in \mathcal{M}(T_i)\}$, $i = 1, 2$, *are incomparable provided T_1 and T_2 are non-isomorphic prime fields.)*

For congruence varieties the same result was announced by Freese [4]. Herrmann and Freese [5] gave a very elegant proof for Freese's result. Their proof is based on, among others, Corollary 3.3. Replacing Corollary 3.3 by Theorem 3.1 their argument proves Corollary 3.4. (Since their work [5] had not appeared when the present paper was written, let us mention that their proof can be found in [3, Theorem 3.2], too.)

If a variety $\mathcal{V}$ of algebras is congruence distributive then $\mathcal{V}$, as an **S** and **P** closed class of ordered algebras (with the equality relations as orderings) is order-congruence distributive. (Indeed, Jónsson's condition from Corollary 3.2 is stronger than (iii) of Theorem 3.1) Therefore if we intend to present examples for classes of ordered algebras satisfying the conditions of Theorem 3.1, we can equip any congruence distributive variety of algebras with the trivial orderings. Another example is the class of all lattices with the usual orderings. In order to give a nontrivial example (which is far from lattice orderings) consider the ordered algebra $A=(\{a, b, c\}, f, \leqq)$ where the ordering is $\{(x, x)|x\in A\}\cup\{(a, b), (a, c)\}$, and f is a ternary majority function defined by

$$f(x_1, x_2, x_3) = \begin{cases} a & \text{if} \quad |\{x_1, x_2, x_3\}| = 3 \\ u\in A & \text{if} \quad |\{i|x_i = u\}| \geqq 2. \end{cases}$$

Now the class $\mathbf{SP}\{A\}$ satisfies condition (iii) of Theorem 3.1 since we can put $k=2$ and $t_1(x, y, z)=f(x, y, z)$.

Finally, it is worth mentioning that for a single ordered algebra A the n-distributivity of Con (A) does not imply the distributivity of Con (A). (Indeed, choose a finite ordered A such that Con (A) is not distributive. Then Con (A) is n-distributive for any n greater than $|\text{Con}\,(A)|$.)

4. Proof of the main theorem. Let us define three further conditions besides the conditions of Theorem 3.1:

(iv) The identity

$$\delta\colon\ x\wedge \bigvee_{i=0}^{n} y_i \leqq \Big(x\wedge \bigvee_{i=0}^{n-1} y_i\Big)\vee\Big(x\wedge \bigvee_{i=1}^{n} y_i\Big)$$

holds in Con (A) for any $A\in\mathcal{U}$;

(v) There exist $k\geqq 1$ and $(n+2)$-ary terms $t_0, t_1, \ldots, t_k$ such that the identities

$$t_0(x_0, x_1, \ldots, x_{n+1}) = x_0, \quad t_k(x_0, x_1, \ldots, x_{n+1}) = x_{n+1},$$
$$t_i(x, y_1, y_2, \ldots, y_n, x) = x \quad \text{for} \quad 0 \leqq i \leqq n,$$
$$t_i(x, x, \ldots, x, \underbrace{y, y \ldots, y}_{j+1}) = t_{i+1}(x, x, \ldots, x, \underbrace{y, y, \ldots, y}_{j+1})$$

where $0\leqq j\leqq n$, $0\leqq i<k$ and $i\equiv j\ (n+2)$,

$$t_i(x_0, x_1, \ldots, x_{n+1})\leqq t_{i+1}(x_0, x_1, \ldots, x_{n+1})$$

for $i\equiv n+1\ (n+2)$ and $0\leqq i<k$ hold in $\mathcal{U}$;

(vi) $$(x_0, x_{n+1})\in \bigvee_{j=0}^{n} \Big(\Theta_{x_0x_{n+1}}\wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} \Theta_{x_{n-i}x_{n+1-i}}\Big)$$

where $\Theta_{x_sx_t}$ denotes the smallest order-congruence of $F_{\mathcal{U}}(x_0, x_1, \ldots, x_{n+1})$, the free $\mathcal{U}$-algebra over $\{x_0, x_1, \ldots, x_{n+1}\}$, under which x_s and x_t collapse.

REMARK. Since $\mathscr{U}$ is closed under **S** and **P**, the free algebra involved in (vi) (and that over an arbitrary generating set) exists. (The definition of free algebras and the proof of this remark are the same as in case of universal algebras, cf. Grätzer [6] or Birkhoff [1].) Note also that (v) is a generalization of Mederly's condition [15].

Via the implications (i)→(vi)→(v)→(iv)→(ii)→(i) we intend to show that all the six conditions, (i), ..., (vi), are equivalent. (No matter that (iii) is not involved above. For $n=1$ (v) and (iii) are the same, and 1-distributivity is the usual distributivity. Thus the equivalence of the other five conditions yields the equivalence of all the six ones.)

Since $(x_0, x_{n+1})\in\Theta_{x_0x_{n+1}}\wedge(\Theta_{x_0x_1}\circ\Theta_{x_1x_2}\circ\ldots\circ\Theta_{x_nx_{n+1}})\subseteq\Theta_{x_0x_{n+1}}\wedge\bigvee_{i=0}^{n}\Theta_{x_{n-i}x_{n+1-i}}$, the implication (i)→(vi) is trivial. Distributive lattices are n-distributive (cf. Huhn [8]), which settles (ii)→(i).

(vi) *implies* (v). Suppose (vi) and let $\bigvee_{\substack{i=0\\ i\neq j}}^{n}\Theta_{x_{n-i}x_{n+1-i}}$ be denoted by Φ_j ($j=0, 1, \ldots, n$). From the assumption $(x_0, x_{n+1})\in\bigvee_{i=0}^{n}(\Theta_{x_0x_{n+1}}\wedge\Phi_j)$ and Proposition 2.2 we obtain that there are elements $t_0(x_0, x_1, \ldots, x_{n+1})$, $t_1(x_0, x_1, \ldots, x_{n+1})$, ..., $t_k(x_0, x_1, \ldots, x_{n+1})$ in $F_{\mathscr{U}}(x_0, \ldots, x_{n+1})$ (where t_i is a term) such that

(1) $$x_0 = t_0(x_0, x_1, \ldots, x_{n+1}),\ t_k(x_0, x_1, \ldots, x_{n+1}) = x_{n+1},$$

(2) $$t_i(x_0, x_1, \ldots, x_{n+1}) \leqq t_{i+1}(x_0, x_1, \ldots, x_{n+1}) \quad\text{for}\quad i\equiv n+1 \quad (n+2),$$

(3) $$t_i(x_0, x_1, \ldots, x_{n+1})\,\Theta_{x_0x_{n+1}}\wedge\Phi_j\ t_{i+1}(x_0, x_1, \ldots, x_{n+1})$$

where $0\leqq j\leqq n$ and $i\equiv j\ (n+2)$.

Since $\Theta_{x_0x_{n+1}}\wedge\Phi_j=\Theta_{x_0x_{n+1}}\cap\Phi_j$, from (3) we obtain

(4) $$t_i(x_0, x_1, \ldots, x_{n+1})\,\Phi_j\, t_{i+1}(x_0, x_1, \ldots, x_{n+1})$$

where $0\leqq j\leqq n$ and $i\equiv j\ (n+2)$. Denoting $(x_0, x_1, \ldots, x_{n+1})$ by **x**, from (2) and (3) we obtain $x_0=t_0(\mathbf{x})\,\Theta_{x_0x_{n+1}}t_1(\mathbf{x})\,\Theta_{x_0x_{n+1}}\ldots\Theta_{x_0x_{n+1}}t_{n+1}(\mathbf{x})\leqq t_{n+2}(\mathbf{x})\,\Theta_{x_0x_{n+1}}\ldots\leqq\ldots t_i(\mathbf{x})\ldots\ldots\Theta_{x_0x_{n+1}}\ldots\leqq\ldots t_k(\mathbf{x})=x_{n+1}\,\Theta_{x_0x_{n+1}}x_0$, whence $(*)$ yields

(5) $$x_0\,\Theta_{x_0x_{n+1}}t_i(x_0, x_1, \ldots, x_{n+1}) \quad\text{for}\quad i=0, 1, \ldots, k.$$

Since $F_{\mathscr{U}}(x_0, \ldots, x_{n+1})$ is a *free* ordered algebra in $\mathscr{U}$, (1) and (2) show that all the identities of (v) which contain $n+2$ variables hold in $\mathscr{U}$. For the rest of identities (4) and (5) will be used. Consider indices i, j ($0\leqq j\leqq n$, $0\leqq i\leqq k$ and $i\equiv j\,(n+2)$) and the homomorphism $\varphi\colon F_{\mathscr{U}}(x_0, x_1, \ldots, x_{n+1})\to F_{\mathscr{U}}(x, y)$, $x_0\varphi=\ldots=x_{n-j}\varphi=x$, $x_{n+1-j}\varphi=\ldots=x_{n+1}\varphi=y$. Then $\operatorname{Ker}\varphi$ is an order-congruence by Proposition 2.1. Since, for $i\neq j$, $(x_{n-i}, x_{n+1-i})\in\operatorname{Ker}\varphi$, we have $\Phi_j=\bigvee_{i\neq j}\Theta_{x_{n-i}x_{n+1-i}}\subseteq\operatorname{Ker}\varphi$. Thus from (4) we obtain

$$t_i(x, x, \ldots, x, \underbrace{y, y, \ldots, y}_{j+1}) = t_i(x_0\varphi, x_1\varphi, \ldots, x_{n+1}\varphi) = t_i(x_0, \ldots, x_{n+1})\varphi =$$

$$= t_{i+1}(x_0, \ldots, x_{n+1})\varphi = t_{i+1}(x_0\varphi, \ldots, x_{n+1}\varphi) = t_{i+1}(x, \ldots, x, \underbrace{y, \ldots, y}_{j+1}).$$

Hence the identity $t_i(x, \ldots, x, \underbrace{y, \ldots, y}_{j+1})=t_{i+1}(x, \ldots, x, \underbrace{y, \ldots, y}_{j+1})$ holds in $\mathscr{U}$. Similarly, considering the homomorphism $\psi\colon F_{\mathscr{U}}(x_0, \dots, x_{n+1}) \to F_{\mathscr{U}}(x, y_1, y_2, \dots, y_n)$, $x_0\psi = x_{n+1}\psi = x$, $x_s\psi = y_s$ for $1 \leqq s \leqq n$, and making use of (5) together with $\Theta_{x_0 x_{n+1}} \subseteqq \operatorname{Ker}\psi$, we obtain $t_i(x, y_1, \dots, y_n, x) = t_i(x_0\psi, x_1\psi, \dots, x_{n+1}\psi) = t_i(x_0, x_1, \dots, x_{n+1})\psi = x_0\psi = x$, whence $t_i(x, y_1, \dots, y_n, x) = x$ holds in $\mathscr{U}$.

(v) *implies* (iv). Suppose (v) and let $A \in \mathscr{U}$, $\alpha, \beta_0, \dots, \beta_n \in \operatorname{Con}(A)$. Considering a pair (a, b) of elements in $\alpha \wedge \bigvee_{i=0}^{n} \beta_i$ and denoting $\bigvee_{i=0}^{n-1} \beta_i$, $\bigvee_{i=1}^{n} \beta_i$ by γ_n and γ_0, respectively, $(a, b) \in (\alpha \wedge \gamma_n) \vee (\alpha \wedge \gamma_0)$ should be shown. Since the rôle of a and b can be interchanged, by Proposition 2.2 it suffices to find a sequence of elements $a = d_0, d_1, \dots, d_r = b$ such that for all $i(<r)$ we have either $d_i \alpha \wedge \gamma_j d_{i+1}$ for some $j \in \{0, n\}$ or $d_i \leqq d_{i+1}$. First of all $(a, b) \in \alpha$ and, by Proposition 2.2, there are elements $c_{ij}, c^{ij}, c_{s+1,0}, c^{s+1,0} \in A$ $(i = 0, 1, \dots, s, j = 0, 1, \dots, n+1)$ such that

$$a = c_{00}\beta_0 c_{01}\beta_1 c_{02}\beta_2 c_{03} \dots \beta_n c_{0,n+1} \leqq c_{10}\beta_0 c_{11}\beta_1 c_{12}\beta_2 c_{13} \dots \beta_n c_{1,n+1} \leqq$$
$$\leqq c_{20}\beta_0 c_{21}\beta_1 c_{22}\beta_2 c_{23} \dots \beta_n c_{2,n+1} \leqq \dots \leqq c_{s0}\beta_0 c_{s1}\beta_1 c_{s2}\beta_2 c_{s3} \dots \beta_n c_{s,n+1} \leqq$$
$$\leqq c_{s+1,0} = b$$

and

$$b = c^{00}\beta_0 c^{01}\beta_1 c^{02} \dots \beta_n c^{0,n+1} \leqq c^{10}\beta_0 c^{11}\beta_1 c^{12} \dots \beta_n c^{1,n+1} \leqq \dots \leqq c^{s0}\beta_0 c^{s1}\beta_1 c^{s2} \dots$$
$$\dots \beta_n c^{s,n+1} \leqq c^{s+1,0} = a.$$

Let us compute by the identities of (v) and keeping in mind that all term functions are monotone:

$$a = t_0(a, \dots, a, a, b) =$$
$$= t_1(a, \dots, a, a, b)\gamma_n t_1(a, \dots, a, c_{0n}, b)\gamma_0 t_1(a, \dots, a, c_{0,n+1}, b) \leqq$$
$$\leqq t_1(a, \dots, a, c_{10}, b)\gamma_n t_1(a, \dots, a, c_{1n}, b)\gamma_0 t_1(a, \dots, a, c_{1,n+1}, b) \leqq \dots \leqq$$
$$\leqq t_1(a, \dots, a, c_{s0}, b)\gamma_n t_1(a, \dots, a, c_{sn}, b)\gamma_0 t_1(a, \dots, a, c_{s,n+1}, b) \leqq$$
$$\leqq t_1(a, \dots, a, c_{s+1,0}, b) = t_1(a, \dots, a, b, b) =$$
$$= t_2(a, \dots, a, a, b, b)\gamma_n t_2(a, \dots, a, c_{0n}, b, b)\gamma_0 t_2(a, \dots, a, c_{0,n+1}, b, b) \leqq$$
$$\leqq t_2(a, \dots, a, c_{10}, b, b)\gamma_n t_2(a, \dots, a, c_{1n}, b, b)\gamma_0 t_2(a, \dots a, c_{1,n+1}, b, b) \leqq \dots \leqq$$
$$\leqq t_2(a, \dots, a, c_{s+1,0}, b, b) = t_2(a, \dots a, b, b, b) = t_3(a, \dots, a, b, b, b) = \dots =$$
$$= t_{n+1}(a, b, b, \dots, b) \leqq t_{n+2}(a, b, \dots, b) =$$
$$= t_{n+2}(a, c^{00}, \dots, c^{00}, b)\gamma_n t_{n+2}(a, c^{0n}, \dots, c^{0n}, b)\gamma_0 t_{n+2}(a, c^{0,n+1}, \dots, c^{0,n+1}, b) \leqq$$
$$\leqq t_{n+2}(a, c^{10}, \dots, c^{10}, b)\gamma_n t_{n+2}(a, c^{1n}, \dots, c^{1n}, b)\gamma_0 t_{n+2}(a, c^{1,n+1}, \dots, c^{1,n+1}, b) \leqq \dots \leqq$$
$$\leqq t_{n+2}(a, c^{s0}, \dots, c^{s0}, b)\gamma_n t_{n+2}(a, c^{sn}, \dots, c^{sn}, b)\gamma_0 t_{n+2}(a, c^{s,n+1}, \dots, c^{s,n+1}, b) \leqq$$
$$\leqq t_{n+2}(a, c^{s+1,0}, \dots, c^{s+1,0}, b) = t_{n+2}(a, \dots, a, a, b) =$$
$$= t_{n+3}(a, \dots, a, a, b)\gamma_n t_{n+3}(a, \dots, a, c_{0n}, b)\gamma_0 t_{n+3}(a, \dots, a, c_{0,n+1}, b) \leqq$$
$$\leqq t_{n+3}(a, \dots, a, c_{10}, b)\gamma_n t_{n+3}(a, \dots, a, c_{1n}, b)\gamma_0 t_{n+3}(a, \dots, a, c_{1,n+1}, b) \leqq \dots \leqq \dots =$$
$$= t_k(a, \text{ some elements of } A, b) = b.$$

Now if we replaced γ_0 and γ_n by $\alpha\wedge\gamma_0$ and $\alpha\wedge\gamma_n$, respectively, we would obtain a required sequence $a=d_0, d_1, \ldots, d_r=b$. But this is possible since for any $u_1, \ldots, u_n\in A$ we have $t_i(a, u_1, \ldots, u_n, b)\alpha t_i(a, u_1, \ldots, u_n, a)=a$, whence the elements of the above sequence are pairwise congruent modulo α.

(iv) *implies* (ii). Suppose the identity δ holds in a lattice L and let x, y, z be arbitrary elements of L. Let $x\wedge y\wedge z$ be denoted by w. Then

$$x\wedge(y\vee z) = x\wedge(y\vee w\vee w\vee\ldots\vee w\vee z) = (x\wedge(y\vee w\vee\ldots\vee w))\vee(x\wedge(w\vee\ldots\vee w\vee z) = $$
$$= (x\wedge y)\vee(x\wedge z),$$

i.e., L is distributive.

The proof of Theorem 3.1 is complete.

5. Some Mal'cev conditions. Roughly saying, a Mal'cev condition is a condition on classes of algebras (ordered algebras, resp.) of the form "there are certain terms which satisfy certain prescribed identities (order-identities, resp.)." (For a precise definition and classification of Mal'cev conditions cf., e.g., Jónsson [13].) For example, (iii) of Theorem 3.1, Jónsson's condition in Corollary 3.2, and (v) in the previous section are Mal'cev conditions. These conditions are named after A. I. Mal'cev, who has proved in [14] that a variety $\mathcal{U}$ of universal algebras is congruence permutable if and only if there exists a ternary term t, corresponding to the type of $\mathcal{U}$, such that the identities $t(x, z, z)=x$ and $t(x, x, z)=z$ are satisfied in $\mathcal{U}$. An analogous result is true for **SP** closed classes of ordered algebras with the surprising consequence that these classes allow only trivial orderings whenever they are order-congruence permutable. (Therefore the permutability of *order*-congruences seems to have not much importance. However, to claim its unimportance we need the following generalization of Mal'cev's result.)

PROPOSITION 5.1. *For any* **S** *and* **P** *closed class* $\mathcal{U}$ *of ordered algebras the following three conditions are equivalent:*

(i) $\mathcal{U}$ *is order-congruence permutable, i.e., if* Φ *and* Ψ *are order-congruences of any member of* $\mathcal{U}$ *then* $\Phi\circ\Psi=\Psi\circ\Phi$;

(ii) $\mathcal{U}$ *is congruence permutable (i.e., congruences in the usual sense of its members commute) and its members have trivial (i.e., equality) orderings*;

(iii) *There exists a ternary term t (corresponding to the type of* $\mathcal{U}$*) such that the (order-) identities* $t(x, z, z)=x$, $t(x, x, z)=z$ *hold in* $\mathcal{U}$.

PROOF. (i) *implies* (iii). Suppose (i) and consider Θ_{xy}, Θ_{yz}, the order-congruences of the free algebra $F_{\mathcal{U}}(x, y, z)$, generated by (x, y) and (y, z), respectively. Now $(x, z)\in\Theta_{xy}\circ\Theta_{yz}$ implies $(x, z)\in\Theta_{yz}\circ\Theta_{xy}$, whence $(x, t)\in\Theta_{yx}$ and $(t, z)\in\Theta_{xy}$ for some $t=t(x, y, z)\in F_{\mathcal{U}}(x, y, z)$. Defining a homomorphism $\varphi\colon F_{\mathcal{U}}(x, y, z)\to F_{\mathcal{U}}(x, z)$ by $x\mapsto x, y\mapsto z, z\mapsto z$, we have $\Theta_{yz}\subseteq\operatorname{Ker}\varphi$. Thus $x=x\varphi=t(x, y, z)\varphi=t(x\varphi, y\varphi, z\varphi)=t(x, z, z)$, while the satisfaction of the other identity follows similarly.

(iii) *implies* (ii). It suffices to show that the members of $\mathcal{U}$ do not allow nontrivial orderings, because then congruences and order-congruences are the same and Mal'cev's above mentioned theorem applies. (No matter that $\mathcal{U}$ is not necessarily a variety, consider the variety generated (in the usual sense) by it.) Assume that $a, b\in A\in\mathcal{U}$, $a\neq b$ and $a\leqq b$. Then $b=t(a, a, b)\leqq t(a, b, b)=a$ is a contradiction.

Finally, (ii) trivially implies (i).

Now we intend to present an algorithm which associates a strong (i.e., containing a finite number of prescibed formulae) Mal'cev type condition $M(p^{(m)} \leqq q^{(n)})$ with an arbitrary lattice identity $p \leqq q$ and integers $m, n \geqq 2$ such that the following result can be stated. (Note that $M(p^{(m)} \leqq q^{(n)})$ is not a Mal'cev condition in the sense of Jónsson [13].)

THEOREM 5.2. *For any class* $\mathcal{U}$ *of ordered algebras closed under* **S** *and* **P** *and for any lattice identity* $p \leqq q$ *the following three conditions are equivalent:*

(i) *The lattice identity* $p \leqq q$ *holds in the lattice of order-congruences of any member of* $\mathcal{U}$;

(ii) *For any integer* $m \geqq 2$ *there exists an integer* $n \geqq 2$ *such that the Mal'cev type condition* $M(p^{(m)} \leqq q^{(n)})$ *is satisfied in* $\mathcal{U}$;

(iii) $p \leqq q$ *holds in the order-congruence lattices of finitely generated members of* $\mathcal{U}$.

Before defining the Mal'cev type conditions involved in Theorem 5.2 some remarks will be made. This theorem can be considered as a generalization of Wille's one [18]. (Really, if $\mathcal{U}$ happens to consist of trivially ordered algebras then any universal Horn sentence of $M(p^{(m)} \leqq q^{(n)})$ is equivalent to an identity and $M(p^{(m)} \leqq q^{(n)})$ turns into a strong Mal'cev condition, which only slightly differs from Wille's one.) Even their proofs are similar, the only essential difference is the use of Proposition 2.2 instead of the well-known description of join of congruences. (For the proof of Wille's theorem see, beside [18], Pixley [17], but the proof cited in [11] is also recommended since its form is near to our approach.) Hence the proof of Theorem 5.2 would not be surprising for those who are acquainted with that of Wille's theorem and Theorem 3.1. Thus the proof will be omitted because of its length.

To make our Mal'cev type conditions visible we shall use a pictorial approach. Finally note that if $p \leqq q$ is the distributive law then (ii) of Theorem 5.2 is much less handlable than condition (iii) of Theorem 3.1.

The definition of $M(p^{(m)} \leqq q^{(n)})$ starts with the recursive definition of $G_m(p)$, the graph of the lattice term p of order m. The graph $G_m(p)$ has coloured edges (the colours are the sign $\leqq$ and the variables of p) and two of its vertices, the so-called left and right endpoints, have special rôle. In the figures the left endpoint will be placed on the left-hand side, and dually.

If p is a variable then $G_m(p)$ has only a single edge coloured by p, which connects the two endpoints.

To obtain $G_m(p_1 \wedge p_2)$ take disjoint copies of $G_m(p_1)$ and $G_m(p_2)$ and glue their left (right, resp.) endpoints together (Figure 1).

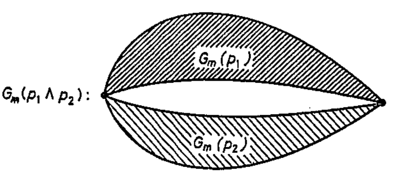

Fig. 1.

To define $G_m(p_1 \vee p_2)$ consider $2m$ disjoint graphs $H_1, H_2, \ldots, H_m, H^1, H^2, \ldots, \ldots, H^m$ where H_i and H^i are copies of $G_m(p_j)$ for $i \equiv j$ (3) and $j \in \{1, 2\}$, while for

$i \equiv 0$ (3) let H_i and H^i be copies of the graph consisting of a single oriented edge coloured by $\leqq$:

$$\circ \xrightarrow{\leqq} \circ$$

Now glue together:

the right endpoint of H_i and the left one of H_{i+1} for $i=1, ..., n-1$,
the right endpoint of H^i and the left one of H^{i+1} for $i=1, ..., n-1$,
the left endpoint of H_1 and the right endpoint of H^m,
the left endpoint of H^1 and the right endpoint of H_m.

The obtained graph is $G_m(p_1 \vee p_2)$, its left (right, resp.) endpoint is the left endpoint of H_1 (H^1, resp.). (Note that, exceptionally, the left endpoint of H^i is placed on the right-hand side on Figure 2, and conversely.)

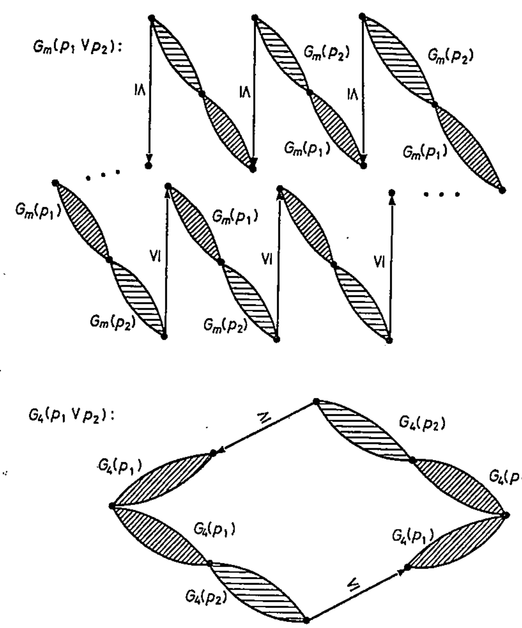

Fig. 2.

The graph $G_n(q)$ is defined in the same way. Let $X=\{x_0, x_1, ..., x_k\}$ and $T=\{t_0, t_1, ..., t_s\}$ be the vertex set of $G_m(p)$ and $G_n(q)$, respectively, such that x_0 and t_0 are the left endpoints while x_1 and t_1 are the right ones. For each variable α occurring in $p \leqq q$ let Θ_α be the smallest equivalence relation of the set $\{0, 1, ..., k\}$ under which i and j collapse whenever x_i and x_j are connected with an α-coloured edge in $G_m(p)$. Now $G(p^{(m)} \leqq q^{(n)})$ is obtained from $G_n(q)$ via replacing the colour α, for all variables α of $p \leqq q$, by Θ_α on each α-coloured edge of $G_n(q)$.

For an equivalence Θ of $\{0, 1, ..., k\}$ and $i \in \{0, 1, ..., k\}$ let $i\Theta = \min \{j \mid j\Theta i\}$. With a Θ-coloured edge of $G(p^{(m)} \leqq q^{(n)})$ connecting the vertices t_u and t_v we associate the universally quantified Horn sentence "if $x_i \leqq x_j$ for all edges

$x_i \circ\!\xrightarrow{\leqq}\!\circ\, x_j$ of $G_m(p)$ then $t_u(x_{0\Theta}, x_{1\Theta}, \dots, x_{k\Theta}) = t_v(x_{0\Theta}, x_{1\Theta}, \dots, x_{k\Theta})$" while with an edge $t_u \circ\!\xrightarrow{\leqq}\!\circ\, t_v$ of $G(p^{(m)} \leqq q^{(n)})$ the universal Horn sentence "if $x_i \leqq x_j$ for all edges $x_i \circ\!\xrightarrow{\leqq}\!\circ\, x_j$ of $G_m(p)$ then $t_u(x_0, x_1, \dots, x_k) \leqq t_v(x_0, x_1, \dots, x_k)$" will be associated.

Finally, $M(p^{(m)} \leqq q^{(n)})$ is defined to be the following condition:

"There exist (k+1)-ary terms $t_0(x_0, x_1, \dots, x_k)$, $t_1(x_0, x_1, \dots, x_k), \dots, \dots, t_s(x_0, x_1, \dots, x_k)$ such that the two endpoint Horn sentences "$x_i \leqq x_j$ for all edges $x_i \circ\!\xrightarrow{\leqq}\!\circ\, x_j$ of $G_m(p)$ imply $t_l(x_0, x_1, \dots, x_k) = x_l$" ($l=0, 1$) and the Horn sentences associated with the edges of $G(p^{(m)} \leqq q^{(n)})$ are satisfied".

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