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On classes of ordered algebras and quasiorder distributivity

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0. Introduction. Many kinds of partially ordered algebras have appeared in the literature so far, for example partially ordered groups, rings, fields, etc. In some cases all the fundamental operations were supposed to be monotonic, but in some others there are operations having only special monotonity domains; moreover, some operations may be order reversing (or „antitone") with respect to a (may be the whole) part of their variables. (See FUCHS [5], [6].) There is no doubt that one gets the most general concept, if one imposes no assumption on the monotonity or antitonity domains of the operations. But then it seems to be hopeless to develop such an elegant (or at least approximately so elegant) theory, as the theory of varieties, equational logic, Mal'cev conditions, and so on. The circumstances for obtaining such results become far more advantageous if we require all operations to be monotonic in all of their variables. So we accept the following definition (the exact origin of which is not known for us):

Definition 0.1. By a *partially ordered algebra* (in the sequel simply *ordered algebra*) we mean a triple $\mathfrak{A}=(A; F, \leqq)$, where $(A; F)$ is a universal algebra, $\leqq$ is a partial ordering on A, and all the operations $f\in F$ are monotone with respect to this ordering. (If there is no danger of confusion, we shall simply say „f is monotone".)

Note that, according to this definition, partially ordered algebras are essentially the same as the algebras in the category of partially ordered sets (see FREYD [4], PAREIGIS [9]).

In our work we make an attempt to give a unified theory for these algebras, using such concepts as subalgebras, direct products, homomorphic images, subdirect decompositions, congruences, inequalities, Mal'cev conditions.

1. Basic concepts and facts. In this section we remind the reader of the concepts of homomorphisms, subalgebras, direct and subdirect products, and then we define two kinds of congruences.

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The operations on *subalgebras* and *direct products* are given as usual, the ordering is *restricted* to the subset in question and is understood *componentwise*. It would be possible to define subalgebras such that the ordering on them is obtained by weakening the restricted ordering, but we need not use such subalgebras, so we do not allow them. This agreement will seem to be natural after investigating varieties and their Birkhoff-type characterization, due to S. L. Bloom [1].

By a *homomorphism* we mean a monotone, operation-preserving map from one algebra to another. A homomorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$ is said to be a *Q-homomorphism,* if the ordering of $\mathfrak{B}$ restricted to $\mathrm{Im}\,\varphi$ cannot be weakened so that φ remains still monotone and the operations on $\mathrm{Im}\,\varphi$ remain still isotone („isotone" is used as a synonym for „monotone"). (It would be possible to describe Q-homomorphisms constructively, but since it is straightforward from the proof of Theorem 1.1 below, it will be omitted.)

$\mathfrak{B}$ is a *homomorphic image* (resp. *Q-homomorphic image*) of $\mathfrak{A}$, if there exists a surjective homomorphism (Q-homomorphism) $\varphi: \mathfrak{A} \to \mathfrak{B}$.

Definition 1.1. A binary relation Θ over A will be called an *order-congruence* of $\mathfrak{A}$, if the following hold:

(i) Θ is a congruence on $(A; F)$;

(ii) whenever for some natural numbers n, m and elements $a, b, a_1, \dots, a_{n-1}, b_1, \dots, b_{m-1} \in A$ we have

$$a\,\Theta\,a_1 \leqq a_2\,\Theta\,a_3 \leqq \dots a_n = b\,\Theta\,b_1 \leqq b_2\,\Theta\,b_3 \leqq \dots b_m = a,$$

we always have also $a\Theta b$. (The sequence of elements of this form is a *Θ-circle* with distinguished elements a, b.)

It is clear, that finitely many Θ-circles (with fixed distinguished elements) can always be unified so that they have common n and common m, moreover, n and m can be required to be equal.

For a homomorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$ let $\mathrm{Ker}\,\varphi$ denote the kernel of φ, i.e. the relation $\{(a, b) \in A^2 | a\varphi = b\varphi\}$. The proof of the following theorem can also be found in [3], so here we give only the necessary construction. The theorem justifies Definition 1.1.

Theorem 1.1. *Θ is an order-congruence of $\mathfrak{A}$ iff $\Theta = \mathrm{Ker}\,\varphi$ for some homomorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$ ($\mathfrak{B}$ is an ordered algebra of the same similarity type), or equivalently, $\Theta = \mathrm{Ker}\,\varphi'$ for some surjective Q-homomorphism $\varphi': \mathfrak{A} \to \mathfrak{B}'$.*

Proof. The „if" part is obvious. Assume Θ is an order-congruence, and consider the ordered algebra $(A/\Theta; F, \preceq)$, where $(A/\Theta; F)$ is the corresponding quotient algebra, and

$$[a]\Theta \preceq [b]\Theta \quad \text{iff} \quad a\,\Theta\,a_1 \leqq a_2\,\Theta\,a_3 \leqq \dots a_n = b \quad \text{for some } n \text{ and } a_1, \dots, a_{n-1} \in A.$$

Then the natural map $a \mapsto [a]\Theta$ is a surjective Q-homomorphism onto the constructed ordered algebra (which will be usually denoted by $\mathfrak{A}/\Theta$).

However, the order-congruences or, what are the same, the kernels of homomorphisms are not sufficient to reproduce the corresponding homomorphic images in the case when the homomorphisms are surjective, unless they are Q-homomorphisms. But we need also homomorphic images, which are not Q-images. So it is desirable to introduce such relations on the ordered algebras, which enable us to describe all homomorphic images completely. The following definition can be found implicitly in BLOOM [1].

Definition 1.2. Let $\varphi\colon \mathfrak{A}\to\mathfrak{B}$ a homomorphism. By the *directed kernel of* φ we mean the relation

$$\overrightarrow{\mathrm{Ker}}\,\varphi = \{(a, b)\in A^2 \mid a\varphi \leqq b\varphi \text{ in } \mathfrak{B}\}.$$

The *isomorphisms* are those homomorphisms having a two-sided inverse map, which is also a homomorphism.

It is obvious that knowing $\overrightarrow{\mathrm{Ker}}\,\varphi$ for a surjective homomorphism φ, we can construct — up to isomorphism — the corresponding homomorphic image, thanks to the fact that $\overrightarrow{\mathrm{Ker}}\,\varphi$ determines on $\mathrm{Im}\,\varphi$ the equality, the ordering and the operations, as well.

The directed kernels can be characterized as follows:

Theorem 1.2. *A binary relation Θ over A is the directed kernel for some homomorphism of $\mathfrak{A}$ into some ordered algebra if and only if Θ is a quasiorder compatible with the operations, which extends the ordering of $\mathfrak{A}$ (i.e. $a\leqq b$ implies $a\Theta b$).*

Proof. The „only if" part is trivial; for the converse let us consider the relation $\Phi=\Theta\cap\Theta^{-1}$. It is easily seen, that Φ is an order-congruence; let $[a]\Phi\leqq[b]\Phi$ iff $a\Theta b$. Then $\leqq$ is a (well-defined) partial ordering on A/Φ preserved by the operations of the quotient algebra. Now obviously $\Theta=\overrightarrow{\mathrm{Ker}}\,\eta$ with η the natural map $a\mapsto[a]\Phi$ onto $(A/\Phi;\, F,\, \leqq)$. (The latter need not be equal to $\mathfrak{A}/\Phi$!)

(Note that Bloom called such quasiorders „admissible preorders".)

Let us denote the ordered algebra constructed in the previous proof by $\mathfrak{A}/\Theta$. We essentially proved also

Theorem 1.3 (Homomorphism Theorem). *If $\varphi\colon\mathfrak{A}\to\mathfrak{B}$ is a surjective homomorphism, then $\mathfrak{A}/\overrightarrow{\mathrm{Ker}}\,\varphi\cong\mathfrak{B}$, an isomorphism is given by $[a]\Phi\mapsto a\varphi$, where Φ denotes the order-congruence $\overrightarrow{\mathrm{Ker}}\,\varphi\cap(\overrightarrow{\mathrm{Ker}}\,\varphi)^{-1}$.*

Next we investigate the connection between order-congruences and directed kernels (in the sequel we refer to the latter simply as *quasiorders,* as they are quasiorders compatible with the operations, which extend the partial order on the algebra).

Proposition 1.4. *The order-congruences are exactly the relations* $\Theta\cap\Theta^{-1}$, *where* Θ *is a quasiorder.*

Proof. We have already seen (proof of Theorem 1.2), that the relations $\Theta\cap\Theta^{-1}$ are all order-congruences. Now if Φ is an order-congruence, then let Θ be the directed kernel of the natural homomorphism of $\mathfrak{A}$ onto the quotient algebra $\mathfrak{A}/\Theta$ (see Theorem 1.1). It is clear that $\Phi=\Theta\cap\Theta^{-1}$.

If Θ is a quasiorder, then $\Theta\cap\Theta^{-1}$ is called the *order-congruence associated with* Θ; cf. Bloom [1], where it is shown that $\Theta\cap\Theta^{-1}$ is a congruence in the usual sense. The same order-congruence may be associated with *distinct* quasiorders, as trivial examples show. But always there is a smallest among the them: namely, for an order-congruence Φ the Θ in the proof of Proposition 1.4 is the least quasiorder such that $\Phi=\Theta\cap\Theta^{-1}$; call it the *quasiorder associated with* Φ. It can also be defined as the only quasiorder Θ for which the natural map of $\mathfrak{A}$ onto $\mathfrak{A}/\Phi$ is a Q-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\Theta$.

For every binary relation $H\subseteq A^2$ there is a smallest quasiorder Θ on $\mathfrak{A}$ such that $H\subseteq\Theta$; this is *the quasiorder generated by H* (denoted by $\vec{\Theta}(H)$), and is equal to the intersection of all quasiorders containing H. If H consists only of the pair (a, b), then we say that $\vec{\Theta}(H)$ is *the principal quasiorder generated by* (a, b), and denote it by $\vec{\Theta}(a, b)$.

Theorem 1.5. *The quasiorders of an ordered algebra* $\mathfrak{A}$ *form an algebraic lattice under set inclusion with the universal relation of A as the unit and the ordering of* $\mathfrak{A}$ *as the zero. The join* $\bigvee_{\gamma\in\Gamma}\Theta_\gamma$ *of the quasiorders* Θ_γ, $\gamma\in\Gamma$, *is given by*

$$a\Big(\bigvee_{\gamma\in\Gamma}\Theta_\gamma\Big)b \quad \text{iff} \quad a\Theta_{\gamma_1}a_1\Theta_{\gamma_2}a_2\ldots a_{n-1}\Theta_{\gamma_n}b \quad \text{for some elements}$$
$$a_1, \ldots, a_{n-1}\in A \quad \text{and} \quad \gamma_1, \ldots, \gamma_n\in\Gamma.$$

From now on, this lattice is denoted by $\mathrm{Cqu}(\mathfrak{A})$ ("compatible quasiorders"). The straightforward proof of the next theorem will be omitted.

Theorem 1.6 (Second Isomorphism Theorem). *Let* Θ_1, Θ_2 *be quasiorders on* $\mathfrak{A}$ *with* $\Theta_1\leqq\Theta_2$, *and let* Φ_i *denote the order-congruence associated with* Θ_i, $i=1, 2$. *Then the relation* $\bar{\Theta}_2$ *on* $\mathfrak{A}/\Theta_1$ *defined by*

$$[a]\Phi_1\bar{\Theta}_2[b]\Phi_1 \quad \text{iff} \quad a\Theta_2 b$$

is a quasiorder on $\mathfrak{A}/\Theta_1$ *and* $(\mathfrak{A}/\Theta_1)/\bar{\Theta}_2$ *is isomoprhic* $\mathfrak{A}/\Theta_2$ *via the map*

$$\big[[a]\Phi_1\big]\bar{\Phi}_2\mapsto[a]\Phi_2,$$

where Φ_2 *is the order-congruence associated with* $\bar{\Theta}_2$. *Hence, the quasiorder-lattice of* $\mathfrak{A}/\Theta$ *is isomorphic to the interval* $[\Theta)$ *of* Cqu $(\mathfrak{A})$.

The following statement is equally trivial:

Theoerem 1.7 (First Isomorphism Theorem). *Given an ordered algebra* $\mathfrak{A}$, *a subalgebra* $\mathfrak{B}$ *of* $\mathfrak{A}$ *and a quasiorder* $\Theta \in \mathrm{Cqu}(\mathfrak{A})$, *define* $[B]=\{a\in A | a\Phi b$ for some $b\in B\}$, *where* Φ *is the order-congruence associated with* Θ. *Let* $\bar{\mathfrak{B}}$ *be the subalgebra of* $\mathfrak{A}$ *determined by* $[B]$. *Then the mapping* $[b](\Phi \restriction B) \mapsto [b](\Phi \restriction [B])$ *is an isomorphism between* $\mathfrak{B}/(\Theta \restriction B)$ *and* $\bar{\mathfrak{B}}/(\Theta \restriction [B])$. (Here $\restriction$ stands for restriction.)

Now turn back to considering quasiorders generated by given set of pairs of elements.

Proposition 1.8. *For* $c, d, a, b \in A$ $(c, d) \in \vec{\Theta}(a, b)$ *if and only if there exists a natural number* n, *unary algebraic functions* $q_1(x), \ldots, q_n(x)$ *over* $\mathfrak{A}$ *and a sequence* $c=u_1, u_2, \ldots, u_{2n}=d$ *of elements of* A *such that*

(i) $u_{2i} \leqq u_{2i+1}$ for $i=1, \ldots, n-1$ *and*

(ii) $u_{2i-1}=q_i(a)$, $u_{2i}=q_i(b)$ *for* $i=1, \ldots, n$.

We omit the easy proof. Of course, if $a \leqq b$, then $\vec{\Theta}(a, b)$ is just the ordering of $\mathfrak{A}$, as it follows at once from the definition of $\vec{\Theta}(a, b)$, but it also follows from this proposition. Replacing (a, b) in (ii) by an arbitrary $(v_i, v_i') \in H$, we get the description of $\vec{\Theta}(H)$.

For every $H \subseteq A^2$ let $\Theta_0(H)$ denote the *congruence* on $(A; F)$ generated by H, and for any congruence Θ of $(A; F)$ let $\hat{\Theta}$ denote the smallest order-congruence of $\mathfrak{A}$ containing Θ. Then we can state:

Proposition 1.9. *Let* Θ *be a congruence of* $(A; F)$. *Then for any* $a, b \in A$, $a\hat{\Theta}b$ *if and only if there is a sequence of the form*

$$a \,\Theta\, a_1 \leqq a_2 \,\Theta\, a_3 \leqq \ldots a_n = b \,\Theta\, b_1 \leqq b_2 \,\Theta\, b_3 \leqq \ldots b_m = a.$$

Consequently, $\widehat{\Theta_0(H)}$ *is the order-congruence generated by* H.

By means of Proposition 1.9 and the well-known Mal'cev lemma concerning $\Theta_0(H)$ it would be possible to give an explicit description for $\widehat{\Theta_0(H)}$, but we omit this. Obviously, Proposition 1.9 defines also the *join of* (arbitrarily many) *order-congruences*. The formulation and the proof of the analogue of Theorem 1.5 is left to the reader (cf. also [3], Proposition 2.2). The *order-congruence lattice of* $\mathfrak{A}$ is denoted by Con $(\mathfrak{A})$, and the order-congruence generated by (a, b) is $\Theta(a, b)$.

Finally, note that the Second Isomorphism Theorem holds also for order-congruences, but in general the First does not, because the ordering of the congruence classes is defined by means of certain sequences of elements, and it can happen that there is no such sequence between two elements of B *inside* of B, but there is already in $[B]$ (see Theorems 1.1 and 1.7). The corresponding variant of the Homomorphism Theorem is true for Q-epimorphisms (of course, replacing $\overrightarrow{\mathrm{Ker}}$ by Ker).

2. Operators on classes of ordered algebras. Varieties. Classes will always consist of ordered algebras of the same similarity type. Let **I**, **H**, **Q**, **S**, **P** and $\mathbf{P}_S$ be the operators of forming all isomorphic, homomorphic, Q-homomorphic images, subalgebras, direct products and subdirect products, respectively (products of empty families — with the obvious meaning — are also allowed). A class $\mathcal{K}$ is a *variety* (resp. *Q-variety*) provided it is closed under **H**, **S** and **P** (resp. **Q**, **S** and **P**). It is easy to check (cf. [1]) that

Theorem 2.1. *For any class $\mathcal{K}$,* $\mathbf{HSP}(\mathcal{K})$ *is the smallest variety containing $\mathcal{K}$.*

One would expect an analoguous result for Q-varieties, but it does not hold in general, because the operator inequality $\mathbf{SQ} \leqq \mathbf{QS}$ may be false, as it is seen from very simple counterexamples (see also the remark at the end of the previous section on the First Isomorphism Theorem). The characterization of the Q-variety generated by a class in terms of operators is an open problem yet.

By an *inequality of type* τ we mean a sequence of symbols $f \leqq g$, where f and g are τ-terms. The expression "$f \leqq g$ holds in an algebra $\mathfrak{A}$" (or more generally, in a class $\mathcal{K}$) has the obvious meaning.

There is a Birkhoff-type characterization for varieties (Bloom [1]):

Theorem 2.2. *A class $\mathcal{K}$ is a variety if and only if $\mathcal{K}$ consists exactly of all the algebras satisfying a given set of inequalities.*

For any fixed type τ, the varieties of type τ are in one-to-one correspondence with the *fully invariant quasiorders* (i.e. invariant under all endomoprhisms) of the absolutely free τ-algebra of rank $\aleph_0$. From this fact one can easily conclude Bloom's four rules for the corresponding „inequational logic":

(i) $t \leqq t$;

(ii) $t_1 \leqq t_2$ and $t_2 \leqq t_3$ imply $t_1 \leqq t_3$;

(iii) $t_i \leqq t_i'$, $i=1, \ldots, n$, imply $f(t_1, \ldots, t_n) \leqq f(t_1', \ldots, t_n')$ for any n-ary operation symbol f;

(iv) $t(x_1, \ldots, x_n) \leqq t'(x_1, \ldots, x_n)$ implies $t(q_1, \ldots, q_n) \leqq t'(q_1, \ldots, q_n)$ for arbitrary terms $q_1, \ldots, q_n$. (Of course, we are inside of τ).

Now we will consider free algebras over arbitrary posets; they will play an important role in the investigation of Mal'cev-type conditions.

Definition 2.1. Let $\mathcal{K}$ be a class of ordered algebras, $\mathfrak{X}=(X; \leqq)$ a poset, $\mathfrak{F}\in\mathcal{K}$, and let ϱ be a map $X\to F$. $\mathfrak{F}$ is the *free algebra over* $\mathfrak{X}$ *in* $\mathcal{K}$ with the canonical map ϱ, if the following hold:

(i) ϱ is monotone, and $X\varrho$ generates F;

(ii) given any monotone map $\varphi: \mathfrak{X}\to\mathfrak{A}$ into an algebra $\mathfrak{A}\in\mathcal{K}$, there exists a (unique) homomorphism ψ: $\mathfrak{F}\to\mathfrak{A}$ such that $\varrho\psi=\varphi$.

$\mathfrak{F}$ is denoted by $\mathfrak{F}^{\varrho}_{\mathcal{K}}(\mathfrak{X})$, or simply by $\mathfrak{F}_{\mathcal{K}}(\mathfrak{X})$, if we do not want to refer to ϱ explicitly. (Cf. [2] for topological algebras).

Proposition 2.3. *$\mathfrak{F}_{\mathcal{K}}(\mathfrak{X})$ is unique in the sense that always there is an isomorphism ξ between $\mathfrak{F}^{\varrho_1}_{\mathcal{K}}(\mathfrak{X})$ and $\mathfrak{F}^{\varrho_2}_{\mathcal{K}}(\mathfrak{X})$ such that $\varrho_1\xi=\varrho_2$.*

In what follows let us call the **ISP**-closed classes *prevarieties.*

Theorem 2.4. *If $\mathcal{K}$ is a prevariety, then for any poset $\mathfrak{X}$, $\mathfrak{F}^{\varrho}_{\mathcal{K}}(\mathfrak{X})$ exists with some ϱ. ϱ is an order-isomorphism onto a subset of F, provided $\mathcal{K}$ contains a nontrivially ordered member, or $\mathfrak{X}$ is trivially ordered and $\mathcal{K}$ contains an at least two-element member.*

Proof. The existence of $\mathfrak{F}^{\varrho}_{\mathcal{K}}(\mathfrak{X})$ can be seen in the usual way. For the second statement let $x, y\in X$, $x\not\leqq y$, and $a<b$, $a, b\in\mathfrak{A}$, $\mathfrak{A}\in\mathcal{K}$. Then there is a monotone map φ: $\mathfrak{X}\to\mathfrak{A}$ such that $y\varphi=a$, $x\varphi=b$. But then $x\varrho\not\leqq y\varrho$, otherwise with the ψ of (ii) in Definition 2.1 we would get $x\varrho\psi\leqq y\varrho\psi$, i.e. $b\leqq a$, a contradiction. The third statement is obvoius, since in that case we essentially work with usual universal algebras, and the statement simply expresses that ϱ is 1—1.

So, in the two cases mentioned above, we may think $\mathfrak{X}$ to be embedded in $\mathfrak{F}_{\mathcal{K}}(\mathfrak{X})$. If $\mathfrak{X}$ is trivially ordered, then $\mathfrak{F}_{\mathcal{K}}(\mathfrak{X})$ depends only on the cardinality of X. We will freely use such notations as $\mathfrak{F}_{\mathcal{K}}(a, b, c)$, $\mathfrak{F}_{\mathcal{K}}(n)$, etc. if this will result no confusion.

The structure of $\mathfrak{F}_{\mathcal{K}}(\mathfrak{X})$ is given very easily, when $\mathfrak{X}$ is trivially ordered: $p\leqq q$ in $\mathfrak{F}_{\mathcal{K}}(\mathfrak{X})$ (where p, q are terms applied to elements of X) iff the inequality $p\leqq q$ is identically true in $\mathcal{K}$. This remark will be frequently used later on. In the general case we have no satisfactory description yet.

3. Subdirect decompositions. For an ordered algebra $\mathfrak{A}=(A; F, \leqq)$, let Or $(\mathfrak{A})$ denote the ordering of $\mathfrak{A}$, i.e. the relation $\leqq$. If $\mathfrak{A}$ is a subdirect product of the algebras $\mathfrak{A}_i$, $i\in I$, then

$$\bigwedge_{i\in I} \overrightarrow{\operatorname{Ker}}\, \pi_i = \operatorname{Or}(\mathfrak{A}),$$

where π_i is the i^{th} natural projection. We show that this condition characterizes subdirect decompositions.

Theorem 3.1. *Let* $\mathfrak{A}$ *be an ordered algebra,* $\Theta_i \in \mathrm{Cqu}(\mathfrak{A})$, $i \in I$, *and* $\wedge\{\Theta_i | i \in I\} = \mathrm{Or}(\mathfrak{A})$. *Then* $\mathfrak{A}$ *is isomorphic to a subdirect product of the algebras* $\mathfrak{A}/\Theta_i$.

Proof. The map $\psi\colon a \mapsto ([a]\bar{\Theta}_i)_{i\in I}$, where $\bar{\Theta}_i$ is the order-congruence associated with Θ_i, gives the desired isomorphism (for the definition of $\mathfrak{A}/\Theta_i$ see Theorem 1.2 and the remark after it).

An ordered algebra is *subdirectly irreducible,* if in all its subdirect decompositions some of the projections is in fact an isomorphism, which by the preceding theorem is equivalent to saying that $\mathrm{Or}(\mathfrak{A})$ is completely meet-irreducible in $\mathrm{Cqu}(\mathfrak{A})$, or in other words, $\mathrm{Cqu}(\mathfrak{A})$ contains a smallest nonzero element. $\mathfrak{A}$ is called *simple* (resp. *weakly simple*), provided $\mathrm{Cqu}(\mathfrak{A})$ (resp. $\mathrm{Con}(\mathfrak{A})$) is the two-element chain. A simple algebra is always weakly simple, but not conversely.

The analogue of Birkhoff's subdirect decomposition theorem holds:

Theorem 3.2. *Every ordered algebra is isomorphic to a subdirect product of its subdirectly irreducible quotient algebras.*

Proof. The claim follows from the fact that, $\mathrm{Cqu}(\mathfrak{A})$ being an algebraic lattice, every quasiorder of $\mathfrak{A}$ is the meet of completely meet-irreducible quasiorders, from the definition of the orderings on the quotient algebras, and from the preceding theorem. For a more direct proof, let us consider for every $a, b \in \mathfrak{A}$ with a $a \nleqq b$ a maximal quasiorder $\psi(a, b)$ not containing (a, b). Then $\wedge\{\psi(a, b) | a \nleqq b\} = \mathrm{Or}(\mathfrak{A})$, and $\psi(a, b) \vee \vec{\Theta}(a, b)$ is the least nonzero element of $\mathrm{Cqu}(\mathfrak{A}/\psi(a, b))$, from which the assertion follows.

Of course, there are several necessary and sufficient conditions on families of quasiorders to determine a *finite direct decomposition.* We formulate only the simplest of them:

Theorem 3.3. *Let* Θ_1, Θ_2 *be quasiorders on* $\mathfrak{A}$, *and let* Φ_1, Φ_2 *be the associated order-congruences. The correspondence*

$$a \mapsto ([a]\Phi_1, [a]\Phi_2)$$

defines an isomorphism between $\mathfrak{A}$ *and the direct product* $\mathfrak{A}/\Theta_1 \times \mathfrak{A}/\Theta_2$ *if and only if the following are satisfied:*

(i) $\Theta_1 \wedge \Theta_2 = \mathrm{Or}(\mathfrak{A})$;

(ii) $\Phi_1 \circ \Phi_2 = \Phi_2 \circ \Phi_1 = \iota$ *(the universal relation).*

Obviously, (i) implies that $\Phi_1 \wedge \Phi_2 = \omega$ (the identity relation), but it is easily seen, that the latter is not sufficient for (i).

4. Conditions in quasiorder-lattices. The analogue of Jónsson's lemma. In this section we investigate the analogues of such properties, as (n-) permutability and distributivity of all congruences on every algebra from a class, having so great importance in the theory of universal algebras.

Proposition 4.1. *Let $\mathcal{K}$ be a prevariety of ordered algebras with a nontrivially ordered member. Then there exist non-permutable order-congruences on an algebra from $\mathcal{K}$.*

Proof. See [3].

We shall not deal with the n-permutability of order-congruences for $n>2$, because the idea of the proof of the next statement carries over easily. (Cf. [7].)

Proposition 4.2. *Under the assumption of the above proposition, the n-permutability of quasiorders does not hold in $\mathcal{K}$.*

Proof. For technical reasons, let $n=2m$. Assume that the quasiorders $\Theta = \bigvee_{i<m} \vec{\Theta}(a_{2i}, a_{2i+1})$ and $\Phi = \bigvee_{i<m} \vec{\Theta}(a_{2i+1}, a_{2i+2})$ of the free algebra $\mathfrak{F}=\mathfrak{F}_{\mathcal{K}}(a_0, \ldots, \ldots, a_n)$ are n-permutable. Then $(a_0, a_n) \in \underbrace{\Theta \circ \Phi \circ \Theta \circ \ldots \circ \Phi}_{n \text{ times}}$ implies the existence of a sequence $a_0 = b_0 \Phi b_1 \Theta b_2 \Phi b_3 \ldots b_{n-1} \Theta b_n = a_n$. Here $b_i = q_i(a_0, \ldots, a_n)$ for a term q_i. If i is even, then

$$q_i(a_0, a_1, a_1, a_3, a_3, \ldots) \Phi q_i(a_0, a_1, a_2, a_3, a_4, \ldots) \Phi$$
$$\Phi q_{i+1}(a_0, a_1, a_2, a_3, a_4, \ldots) \Phi q_{i+1}(a_0, a_2, a_2, a_4, a_4, \ldots).$$

Consider the endomorphism ξ of $\mathfrak{F}$, which leaves a_0 fixed, and sends a_{2i+1} and a_{2i+2} to a_{2i+2} for every $i<m$. Then $\Phi \leqq \overrightarrow{\operatorname{Ker}}\, \xi$, so

$$q_i(a_0, a_2, a_2, a_4, a_4, \ldots) = q_i(a_0, a_1, a_1, a_3, a_3, \ldots)\xi \leqq$$
$$\leqq q_{i+1}(a_0, a_2, a_2, a_4, a_4, \ldots)\xi = q_{i+1}(a_0, a_2, a_2, a_4, a_4, \ldots).$$

Similarly, for i odd we have

$$q_i(a_1, a_1, a_3, a_3, \ldots) \leqq q_{i+1}(a_1, a_1, a_3, a_3, \ldots).$$

Now let $p_i(x, y, z) = q_i(x, \ldots, x, y, z, \ldots, z)$ with x occurring i times for $1 \leqq i < n$, $p_0(x, y, z) = x$ and $p_n(x, y, z) = z$. Then in $\mathcal{K}$ there hold the following inequalities:

$$(*) \qquad p_i(x, x, z) \leqq p_{i+1}(x, z, z), \quad i = 1, \ldots, n-1.$$

But take elements $a<b$ in some member of $\mathcal{K}$ and compute:

$$b = p_0(b, b, a) \leqq p_1(b, a, a) \leqq p_1(b, b, a) \leqq p_2(b, a, a) \leqq p_2(b, b, a) \leqq \ldots$$
$$\ldots \leqq p_n(b, a, a) = a;$$

this is a contradiction. (Note, that $p_i(b, a, a) \leqq p_i(b, b, a)$ is true by monotonity.)

This means that in nontrivially ordered prevarieties the description of the join of quasiorders cannot be reduced so that we take sequences of elements of a fixed length.

Corollary 4.3. *Let $\mathcal{K}$ be a prevariety of universal algebras, $n\geqq 2$ a natural number, and suppose that all compatible quasiorders on algebras in $\mathcal{K}$ are n-permutable. Then all these quasiorders are congruences (i.e. are also symmetric).*

Proof. Endow all $\mathcal{K}$-algebras with trivial order. Considering a pair $(a, b)\in \in\varrho\in \mathrm{Cqu}(\mathfrak{A})$, $\mathfrak{A}\in\mathcal{K}$, and defining the $p_i(x, y, z)$ as above, we can compute by $(*)$ (which gives now equations!) and the compatibility of ϱ:

$$b = p_0(b, b, a) = p_1(b, a, a)\,\varrho\, p_1(b, b, a) = p_2(b, a, a)\,\varrho\, p_2(b, b, a) =$$
$$\ldots = p_n(b, a, a) = a, \quad \text{from which } (b, a)\in\varrho \text{ follows.}$$

Fortunately, besides the negative phenomena mentioned so far, there are positive facts, too. The concept of *quasiorder distributivity* of all algebras in a prevariety is already useful. The significance of quasiorder distributivity is seen from the next two statements. All algebras are ordered algebras of a fixed type. We follow JÓNSSON's [8] original proofs mutatis mutandis, keeping also his notations.

Lemma 4.4. *If $\mathfrak{A}$ is a subalgebra of $\prod(\mathfrak{C}_i|i\in I)$, $\mathrm{Cqu}(\mathfrak{A})$ is distributive and $\mathfrak{A}/\varphi$ is subdirectly irreducible, where $\varphi\in\mathrm{Cqu}(\mathfrak{A})$, then there exists an ultrafilter U over I such that $U^\wedge\restriction A\leqq\varphi$.* (For any filter V over I, $V^\wedge$ denotes the relation defined by $xV^\wedge y$ iff $\{i|x(i)\leqq y(i)\}\in V$.)

Proof. Obviously, the $V^\wedge$ are always quasiorders on $\mathfrak{C}=\prod(\mathfrak{C}_i|i\in I)$. Write $J^\wedge$ instead of $V^\wedge$, if V is the principal filter generated by a subset J of I. Let $D=\{J|J\subseteqq I$ and $J^\wedge\restriction A\leqq\varphi\}$, and let U be a maximal filter contained in D (Zorn's lemma applies since $I\in D$). Then $U^\wedge=\bigcup(J^\wedge|J\in U)$, so $U^\wedge\restriction A\leqq\varphi$. We show, that U is an ultrafilter. For every $J, K\subseteqq I$

(1) $$I\supseteqq J\supseteqq K \text{ and } K\in D \text{ implies } J\in D,$$

and $(J\cup K)^\wedge\restriction A=(J^\wedge\restriction A)\cap(K^\wedge\restriction A)$, so by distributivity

(2) $$\varphi=\varphi\vee((J\cup K)^\wedge\restriction A)=(\varphi\vee(J^\wedge\restriction A))\cap(\varphi\vee(K^\wedge\restriction A)) \text{ if } J\cup K\in D.$$

But φ is meet-irreducible, so $\varphi\vee(J^\wedge\restriction A)=\varphi$ or $\varphi\vee(K^\wedge\restriction A)=\varphi$, i.e.

(3) $$J\cup K\in D \text{ implies } J\in D \text{ or } K\in D.$$

If U were not an ultrafilter, then we would have $J\notin U$ and $I\setminus J\notin U$ for some $J\subseteqq I$. Then by (1) and the maximality of U there exist sets K', $K''\in U$ such that $J\cap K'\notin D$ and $(I\setminus J)\cap K''\notin D$. However, $K=K'\cap K''\in U$, so $K\in D$, and $K=(J\cap K)\cup$

$\bigcup((I\setminus J)\cap K)$. But this contradicts (3), since the members of the latter union do not belong to D by (1).

Lemma 4.5. (Jónsson-lemma). *If $\mathcal{K}$ is a class of ordered algebras, $\mathcal{V}$ is the variety generated by $\mathcal{K}$, and all the* $\mathrm{Cqu}(\mathfrak{A})$, $\mathfrak{A}\in\mathcal{V}$, *are distributive, then all subdirectly irreducible members of $\mathcal{V}$ belong to* $\mathbf{HSP}_U(\mathcal{K})$, *where* $\mathbf{P}_U$ *denotes the model-theoretic operator of forming ultraproducts. Consequently,* $\mathcal{V}=\mathbf{IP}_S\mathbf{HSP}_U(\mathcal{K})$.

Proof. Every algebra in $\mathcal{V}$ is of the form $\mathfrak{A}/\varphi$, where $\mathfrak{A}$ is a subalgebra of a direct product $\prod\{\mathfrak{C}_i|i\in I\}$, $\mathfrak{C}_i\in\mathcal{K}$, and $\varphi\in\mathrm{Cqu}(\mathfrak{A})$. If $\mathfrak{A}/\varphi$ is subdirectly irreducible, then $U^\wedge\restriction A\leqq\varphi$ for a suitable ultrafilter U over I by the preceding lemma. Therefore, $\mathfrak{A}/\varphi$ is a homomorphic image of $\mathfrak{A}(U^\wedge\restriction A)$, and the latter is obviously a subalgebra of $(\prod\{\mathfrak{C}_i|i\in I\})/U^\wedge$.

It remains to show, that $(\prod\{\mathfrak{C}_i|i\in I\})/U^\wedge$ is an ultraproduct of members of $\mathcal{K}$. We point out, that this is just the ultraproduct of the $\mathfrak{C}_i$ over the ultrafilter U. Indeed, let $[f]U$ denote the equivalence class modulo U of any function $f\in\prod C_i$ according to the definition of ultraproduct, and let Θ be the order-congruence associated with $U^\wedge$, i.e. $\Theta=U^\wedge\cap(U^\wedge)^{-1}$. Now $[f]\Theta=[g]\Theta$ means that $\{i|f(i)\leqq g(i)\}\in U$ and $\{i|g(i)\leqq f(i)\}\in U$, which is equivalent to $\{i|f(i)=g(i)\}\in U$, i.e. $[f]U=[g]U$. From this it follows at once, that the operations are also the same. Let $[f]\Theta\leqq[g]\Theta$, then $fU^\wedge g$ (see the proof of Theorem 1.2), which means $\{i|f(i)\leqq g(i)\}\in U$. But this expresses just the fact that $[f]U\leqq[g]U$ in the ultraproduct.

Let us mention, that many results of Jónsson's fundamental paper [8] on congruence distributivity can be reformulated and proved for ordered varieties, using quasiorders instead of congruences. To work with order-congruences is generally more difficult, although not always: for example, the authors succeeded in characterizing order-congruence distributivity of prevarieties in [3] by Mal'cev-type conditions, while for quasiorder distributivity there is no such result yet; there is only a criterion in terms of weak Mal'cev conditions (see below).

5. Characterization of quasiorder-distributivity. Some examples. Now we intend to characterize the distributivity of quasiorders in a prevariety by a (weak) Mal'cev condition. This characterization will enable us to present some nontrivial examples, too.

Theorem 5.1. *Let $\mathcal{K}$ be a class of ordered algebras closed under* **I**, **S** *and* **P** *(i.e. a prevariety). Then the following two conditions are equivalent:*

(i) $\mathrm{Cqu}(\mathfrak{A})$, *the lattice of quasiorders of* $\mathfrak{A}$, *is distributive for any member* $\mathfrak{A}$ *of* $\mathcal{K}$;

(ii) *For any even integer $n\geqq2$ there exists a positive multiple k of $n/2$ such that $U(n,k)$ holds in $\mathcal{K}$, where $U(n,k)$ is a (strong) Mal'cev condition defined as follows ($(x_0, x_1, \ldots, x_n)$ is denoted by* $\mathbf{x}$ *and $n/2$ by m):*

"There exist ((n+1)-ary and (n+2)-ary) terms

$$p_0(\mathbf{x}),\ p_1(\mathbf{x}),\ \dots,\ p_k(\mathbf{x}),$$
$$q^i_j(t, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i,\ j \leqq k,$$
$$r^i_j(t, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i \leqq k,\ i \textit{ odd, and } 0 \leqq j \leqq k-1, \textit{ and}$$
$$s^i_j(t, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i,\ j \leqq k,\ i \textit{ even},$$

such that the following inequalities and identities hold:

$$p_0(\mathbf{x}) = x_0, \quad p_k(\mathbf{x}) = x_n,$$
$$p_{i-1}(\mathbf{x}) = q^i_1(x_0, \mathbf{x}), \quad p_i(\mathbf{x}) = q^i_k(x_n, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i \leqq k,$$
$$q^i_l(x_n, \mathbf{x}) \leqq q^i_{l+1}(x_0, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i \leqq k, \quad 1 \leqq l < k,$$
$$p_{i-1}(\mathbf{x}) = r^i_0(x_0, \mathbf{x}), \quad p_i(\mathbf{x}) = r^i_{k-1}(x_{n-1}, \mathbf{x}) \quad \textit{for } i \textit{ odd}, \quad 1 \leqq i \leqq k,$$
$$r^i_l(x_{2j+1}, \mathbf{x}) \leqq r^i_{l+1}(x_{2j+2}, \mathbf{x}) \quad \textit{for } i \textit{ odd}, \quad 1 \leqq i \leqq k,\ 0 \leqq j < m,\ 0 \leqq l < k-1,$$

$j \equiv l(m)$, *where* $+$ *is understood modulo n so that* $0 \leqq 2j+2 < n$,

$$p_{i-1}(\mathbf{x}) = s^i_1(x_1, \mathbf{x}), \quad p_i(\mathbf{x}) = s^i_k(x_n, \mathbf{x}) \quad \textit{for } i \textit{ even}, \quad 1 < i \leqq k,$$
$$s^i(x_{2j}, \mathbf{x}) \leqq s^i_{l+1}(x_{2j+1}, \mathbf{x}) \quad \textit{for } i \textit{ even}, \quad 1 < i \leqq k,\ 0 < j \leqq m,\ 1 \leqq l < k,\ j \equiv l(m),$$

where $+$ *is understood modulo n so that* $0 < 2j+1 \leqq n$."

Proof. Suppose (i) holds, n is an even positive integer, and consider the quasi-orders $\alpha = \vec{\Theta}(x_0, x_n)$, $\beta = \vec{\Theta}(\{(x_0, x_1), (x_2, x_3), \dots, (x_{n-2}, x_{n-1})\})$, $\gamma = \vec{\Theta}(\{(x_1, x_2), (x_3, x_4), \dots, (x_{n-1}, x_n)\})$ on the free algebra $\mathfrak{F} = \mathfrak{F}_{\mathscr{K}}(n+1)$ freely generated by $\{x_0, x_1, \dots, x_n\}$. Since $(x_0, x_n) \in \alpha \wedge (\beta \vee \gamma)$, we have $(x_0, x_n) \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ as well. Therefore, $x_0 = p_0 \alpha \wedge \beta p_1 \alpha \wedge \gamma p_2 \alpha \wedge \beta p_3 \alpha \wedge \gamma \dots p_k = x_n$ holds for some multiple k of m and elements $p_i = p_i(\mathbf{x})$ of $\mathfrak{F}$. Since $(p_{i-1}, p_i) \in \alpha = \vec{\Theta}(x_0, x_n)$, by Proposition 1.8 there are unary algebraic functions $\hat{q}^i_l(t)$ on $\mathfrak{F}$, which can be considered as $(n+2)$-ary terms $q^i_l(t, \mathbf{x})$ $1 \leqq l \leqq k_i$, such that $q^i_l(x_0, \mathbf{x}) = p_{i-1}(\mathbf{x})$, $q^i_k(x_n, \mathbf{x}) = p_i(\mathbf{x})$ and $q^i_l(x_n, \mathbf{x}) \leqq q^i_{l+1}(x_0, \mathbf{x})$ for $1 \leqq l < k_i$. Both k and k_i can be enlarged by repeating the last terms, whence they can be assumed to be equal. Now all the identities and inequalities involving some q^i_l hold for the generators of $\mathfrak{F}$, therefore they hold throughout $\mathscr{K}$. The case of the r^i_l and s^i_l is a little bit more complicated from technical point of view, but can be handled similarly, while $p_0(\mathbf{x}) = x_0$ and $p_k(\mathbf{x}) = x_n$ are evidently true in $\mathscr{K}$.

Conversely, let (ii) be satisfied. Assume $\mathfrak{A} \in \mathscr{K}$, $\alpha, \beta, \gamma \in \mathrm{Cqu}(\mathfrak{A})$ and $(a, b) \in \alpha \wedge (\beta \vee \gamma)$; then $(a, b) \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ has to be shown. From the assumption we obtain a sequence of the form $a = a_0 \beta a_1 \gamma a_2 \beta a_3 \gamma \dots \beta a_{n-1} \gamma a_n = b$ for some even n; moreover $a_0 \alpha a_n$. Let k be such a multiple of $n/2$ for which $U(n, k)$ holds in $\mathscr{K}$. It is sufficient to show that for $p_0(a_0, \ldots, a_n)=p_0(\mathbf{a})$ (notation!), $p_1(\mathbf{a}), \ldots, p_k(\mathbf{a})$ we have

$$a_0 = p_0(\mathbf{a})\ \alpha\wedge\beta\ p_1(\mathbf{a})\ \alpha\wedge\gamma\ p_2(\mathbf{a})\ \alpha\wedge\beta\ p_3(\mathbf{a})\ \alpha\wedge\gamma \ldots p_k(\mathbf{a}) = a_n.$$

Indeed, $p_{i-1}(\mathbf{a}) = q_1^i(a_0, \mathbf{a})\alpha q_1^i(a_n, \mathbf{a}) \leqq q_2^i(a_0, \mathbf{a})\alpha q_2^i(a_n, \mathbf{a}) \leqq q_3^i(a_0, \mathbf{a})\alpha \ldots \leqq$ $\leqq q_k^i(a_0, \mathbf{a})\alpha q_k^i(a_n, \mathbf{a}) = p_i(\mathbf{a})$ yields $(p_{i-1}(\mathbf{a}), p_i(\mathbf{a}))\in\alpha$, for i odd $p_{i-1}(\mathbf{a}) =$ $= r_0^i(a_0, \mathbf{a})\beta r_0^i(a_1, \mathbf{a}) \leqq r_1^i(a_2, \mathbf{a})\beta r_1^i(a_3, \mathbf{a}) \leqq \ldots \leqq r_{k-1}^i(a_{n-2}, \mathbf{a})\beta r_{k-1}^i(a_{n-1}, \mathbf{a}) = p_i(\mathbf{a})$ implies $(p_{i-1}(\mathbf{a}), p_i(\mathbf{a}))\in\beta$, while $(p_{i-1}(\mathbf{a}), p_i(\mathbf{a}))\in\gamma$ for i even follows similarly.

Before formulating a corollary to this theorem, two relevant remarks will be made. Firstly, the theorem is obviously applicable for any class $\mathcal{K}$ of ordered algebras, containing all free algebras $\mathfrak{F}_{\mathcal{K}}(X)$ for finite unordered X. Secondly, any universal algebra can be considered as a trivially ordered algebra. Thus the theorem also holds for certain classes (including varieties and prevarieties) of universal algebras. In this case Cqu $(\mathfrak{A})$ is the lattice of all compatible, reflexive and transitive binary relations of $\mathfrak{A}$, and the inequalities in $U(n, k)$ simply turn into identities.

Corollary 5.2. *Let $\mathcal{K}$ be a class as in Theorem* 5.1, *and let there exist a ternary term $u(x, y, z)$ for which the identities $u(x, x, y)=u(x, y, x)=u(y, x, x)=x$ hold throughout $\mathcal{K}$ (i.e. u induces a majority function on the members of $\mathcal{K}$). Then* Cqu$(\mathfrak{A})$ *is distributive for any $\mathfrak{A}$ in $\mathcal{K}$.*

Proof. It is sufficient to show that $U(n, n)$ holds in $\mathcal{K}$ for any even n. Let us agree that all the terms p, q, r, s, h, g (with indices) contain at least the variables $x_0, x_1, \ldots, x_n$, but, for the sake of brevity, these common variables will not be indicated. First we define $p_0, \ldots, p_n$ and $h_0(t), \ldots, h_n(t)$ by induction:

$$h_0(t) = t, \quad p_0 = h_0(x_0),$$
$$h_i(t) = u(p_{i-1}, x_n, h_{i-1}(t)), \quad p_i = h_i(x_i).$$

The terms $g_1(t), \ldots, g_n(t)$ are determined by

$$g_1(t) = h_1(t), \quad g_{i+1}(t) = u(g_i(t), x_n, h_{i-1}(x_i)).$$

For $1\leqq i\leqq n$ set $q_1^i(t)=q_2^i(t)=\ldots=q_{n-1}^i(t)=p_{i-1}$ (so in fact these terms do not depend on t) and $q_n^i(t)=u(p_{i-1}, g_i(t), h_{i-1}(x_i))$. For i odd, $1\leqq i<n$, let $j=(i-1)/2$,

$$r_0^i(t) = \ldots = r_{j-1}^i(t) = p_{i-1},$$
$$r_j^i(t) = u(p_{i-1}, x_n, h_{i-1}(t)), \quad \text{and} \quad r_{j+1}^i(t) = \ldots = r_{n-1}^i(t) = p_i.$$

For i even, $1<i\leqq n$, set $j=i/2$,

$$s_1^i(t) = \ldots = s_{j-1}^i(t) = p_{i-1},$$
$$s_j^i(t) = u(p_{i-1}, x_n, h_{i-1}(t)), \quad \textit{and} \quad s_{j+1}^i(t) = \ldots = s_n^i(t) = p_i.$$

A trivial induction shows that $h_i(x_n)=x_n$ $(0\leqq i\leqq n)$, $g_i(x_n)=x_n$ and $g_i(x_0)=p_{i-1}$ $(1\leqq i\leqq n)$. Thus it is not difficult to check that the terms p, q, r, s (with the corresponding indices) satisfy the identities and inequalities required in $U(n, n)$.

We note that it is possible to state and prove an analogous general theorem which „translates" every lattice identity holding in all quasiorder-lattices of members in a prevariety, similarly as it was done in [3] for order-congruence lattices. This is straighforward enough, so we omit it.

To conclude our paper, we present some examples. Since lattices are ordered algebras with their natural orderings and $u(x, y, z)=(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$ induces a majority function on any lattice, Cqu $(\mathcal{L})$ is distributive for any lattice $\mathcal{L}$. To give another example which is far from lattice orders, set $\mathfrak{A}=(A; u, \leqq)$ where $A=\{a, b, c\}$, u is a ternary majority function such that $u(x, y, z)=c$ provided $\{x, y, z\}=\{a, b, c\}$, and $a<c$, $b<c$ are the only comparable pairs of distinct elements in $(A, \leqq)$. Then $\mathfrak{A}$ is an ordered algebra, and any member of $\mathbf{HSP}(\mathfrak{A})$ is quasiorder distributive by corollary 5.2.

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