Acta Math. Hung.
53 (1—2) (1989), 169—171.

TWO NOTES ON INDEPENDENT SUBSETS IN LATTICES

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The present paper deals with two kinds of independent subsets in lattices. A subset H of a lattice L is called *weakly independent* (cf. [1]) iff for all $h, h_1, h_2, \ldots, h_n \in H$ satisfying $h \leqq h_1 \vee h_2 \vee \ldots \vee h_n$ there is an $i \in \{1, 2, \ldots, n\}$ such that $h \leqq h_i$. Analogously, let us call a subset H of L *$*$-independent* iff for all $h, h_1, h_2, \ldots, h_n \in H$ satisfying $h = h_1 \vee h_2 \vee \ldots \vee h_n$ there is an $i \in \{1, 2, \ldots, n\}$ such that $h = h_i$. A maximal weakly independent subset, resp. maximal $*$-independent subset, is called a *weak basis*, resp. *$*$-basis*, of L.

Denoting the set of join-irreducible elements of a lattice L by $J_0(L)$, it is well-known that

THEOREM A (cf., e.g., Grätzer [2]). *For every maximal chain C in a finite distributive lattice L, $|J_0(L)| = |C|$.*

As $J_0(L)$, provided L is distributive, and every maximal chain are always weak bases of finite lattices (cf. [1]), the following assertion is a generalization of Theorem A.

THEOREM B ([1]). *Any two weak bases of a finite distributive lattice have the same number of elements.*

Our first goal is to present another generalization of Theorem A. First we observe that, for every maximal chain $C = \{0 = c_0 < c_1 < c_2 < \ldots < c_{n-1} < c_n = 1\}$ in an arbitrary finite lattice L, $|J_0(L)| \geqq |C|$. Indeed, let $H_0 = \{0\}$ and, for $i \in \{1, 2, \ldots, n\}$, let H_i be the set of minimal elements of $(c_i] \setminus (c_{i-1}]$. Then the H_i are pairwise distinct and nonempty, and $H_0 \cup H_1 \cup \ldots \cup H_n \subseteq J_0(L)$. Therefore the equality in Theorem A is equivalent (modulo lattice theory) to the inequality $|J_0(L)| \leqq |C|$. As $J_0(L)$ and all maximal chains are $*$-bases of a finite lattice L, the following statement generalizes Theorem A, indeed.

THEOREM 1. *Every $*$-basis of a finite distributive lattice L has at least $|J_0(L)|$ elements.*

It was observed in [1] that Theorem B fails to hold for all finite modular lattices. However, modularity is still relevant as we have

THEOREM 2. *If any two weak bases of a finite lattice L have the same cardinality then L is modular.*

* The authors' work was partially supported by Hungarian National Foundation for Scientific Research Grant No. 1813.

Proof of Theorem 1. Let H be a $*$-basis of L. It suffices to give a map φ from $J_0(L)$ into $P(H)$, the power set of H, such that $x\varphi\neq\emptyset$ and $x\varphi\cap y\varphi=\emptyset$ for any two distinct $x, y\in J_0(L)$. Define φ as follows. For $x\in J_0(L)$ put $x\varphi=\{x\}$ if $x\in H$ and put

$x\varphi=\{h\in H$: there is a positive integer n and there are $h_1, h_2, \ldots, h_n\in H$ such that $h\notin\{x, h_1, h_2, \ldots, h_n\}$ and $h=x\vee h_1\vee h_2\vee\ldots\vee h_n\}$

if $x\notin H$. As H is a *maximal* $*$-independent subset, $x\varphi$ is never empty. To check $x\varphi\cap y\varphi=\emptyset$ for $x\neq y\in J_0(L)$, the following three cases have to be considered. If $x, y\in H$ then $x\varphi\cap y\varphi=\emptyset$ is evident. If $x\notin H$ and $y\in H$ but $x\varphi\cap y\varphi\neq\emptyset$ then $y\in x\varphi$. Thus $y=x\vee h_1\vee h_2\vee\ldots\vee h_n$ and $y\notin\{x, h_1, h_2, \ldots, h_n\}$ for some $h_1, h_2, \ldots, h_n\in H$, which contradicts $y\in J_0(L)$. If $x\notin H$ and $y\notin H$ but $x\varphi\cap y\varphi\neq\emptyset$, say $h\in x\varphi\cap y\varphi$, then there are $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n\in H$ such that $x\vee a_1\vee a_2\vee\ldots\vee a_m=h=$ $=y\vee b_1\vee b_2\vee\ldots\vee b_n$ and $h\notin\{x, a_1, a_2, \ldots, a_m\}\cup\{y, b_1, b_2, \ldots, b_n\}$. We may assume that $x\not\leqq y$. Then, by distributivity, $x=x\wedge h=x\wedge(y\vee b_1\vee b_2\vee\ldots\vee b_n)=(x\wedge y)\vee$ $\vee(x\wedge b_1)\vee(x\wedge b_2)\vee\ldots\vee(x\wedge b_n)$. As $x\in J_0(L)$ and $x\neq x\wedge y$, we infer $x=x\wedge b_i$, i.e. $x\leqq b_i$, for some $i\in\{1, 2, \ldots, n\}$. Hence $h=b_i\vee h=b_i\vee x\vee a_1\vee a_2\vee\ldots\vee a_m=b_i\vee a_1\vee$ $\vee a_2\vee\ldots\vee a_m$. The $*$-independence of H yields $h\in\{b_i, a_1, a_2, \ldots, a_m\}$, a contradiction. Q.e.d.

Proof of Theorem 2. In order to recall a result from Jakubík [3], let us call a sublattice S of a lattice L a *c-sublattice* of L iff for any $u, v\in S$ whenever v covers u in S then v covers u in L. What we need here is only the following weakened version of Jakubík [3, Theorem 1; note the misprint, "sublattice" should be "c-sublattice"]: If a finite lattice L is not modular then L includes a c-sublattice S such that S has a four element chain and either 1_S, the greatest element of S, is the join of two atoms of S or 0_S is the meet of two coatoms of S.

Now suppose L is a finite non-modular lattice and consider an above-mentioned c-sublattice S. Let

$$\{0_L = c_0 < c_1 < c_2 < \ldots < c_{m-1} < c_m = 0_S\}$$

and

$$\{1_S = d_0 < d_1 < d_2 < \ldots < d_{n-1} < d_n = 1_L\}$$

be maximal chains in the intervals $[0_L, 0_S]$ and $[1_S, 1_L]$, respectively ($0\leqq m, n$), and let D be an at least four element maximal chain in S. Then $C=\{c_0, c_1, \ldots, c_{m-1}\}\cup$ $\cup D\cup\{d_1, d_2, \ldots, d_n\}$ is a maximal chain in L, whence it is a weak basis of L. Now we distinguish two cases. (If both conditions hold, we can choose the first one.)

Case A. If $1_S=a_0\vee a_1$ for some atoms a_0 and a_1 of S then put $G=\{0_S, a_0, a_1\}$.

Case B. If $0_S=b_0\wedge b_1$ for some coatoms b_0 and b_1 of S then put $G=\{0_S, b_0, b_1\}$.

We claim that $H=\{c_0, c_1, \ldots, c_{m-1}\}\cup G\cup\{d_1, d_2, \ldots, d_n\}$ is a weak basis of L. It is easy to see that H is weakly independent, and we have to show that for every $x\in L\setminus H$ the subset $H\cup\{x\}$ is not weakly independent. Suppose the contrary, i.e., let $H\cup\{x\}$ be weakly independent for some $x\in L\setminus H$. We have $x\leqq d_0=1_S$ as otherwise $i=\min\{j: x\leqq d_j\}$ would be positive, and all the three possibilities $i=1$ in Case A, $i=1$ in Case B, $i>1$ would contradict the weak independence of $H\cup\{x\}$ via $d_1=a_0\vee a_1\vee x$, $d_1=b_0\vee b_1\vee x$, $d_i=d_{i-1}\vee x$, respectively. Similarly, $x\not\leqq c_m=0_S$ as otherwise $c_k=c_{k-1}\vee x$, where $k=\min\{j: x\leqq c_j\}$, would be a contradiction. In Case A, $x \leqq 1_S = a_0 \vee a_1$ implies $x \leqq a_i$ for some $i \in \{0, 1\}$. Since $x \not\leqq 0_S$ and a_i covers 0_S in L, $0_S < 0_S \vee x \leqq a_i$ yields $a_i \leqq 0_S \vee x$, which contradicts the weak independence of H. In Case B, $x \not\geqq 0_S = b_0 \wedge b_1$ yields $x \not\geqq b_i$ for some $i \in \{0, 1\}$. As 1_S covers b_i and $x \leqq 1_S$, $1_S = b_i \vee x$. But then $b_{1-i} \leqq b_i \vee x$ implies $b_{1-i} \leqq x$. Hence $x \leqq 1_S = b_i \vee b_{1-i}$ contradicts the weak independence of $H \cup \{x\}$. Now we have seen that H is a weak basis, and the proof is complete by $|H| < |C|$.

Concluding remarks. Theorem 1 is sharp in the sense that the distributivity of L cannot be omitted and a $*$-basis may have more element than $J_0(L)$. The five element non-distributive modular lattice and the eight element Boolean lattice are appropriate counterexamples. However, we have the following open

PROBLEM. Does every $*$-basis of an arbitrary finite modular lattice L have at least $|C|$ elements where C is a maximal chain in L?

An affirmative answer would be a generalization of Theorem 1. Modularity or some stronger assumption seems to be essential as we have the following example. Let $C = \{0 < a < 1\}$ be a three element chain, and insert a new element into each of the intervals $[(0, 0), (0, a)]$ and $[(0, 0), (a, 0)]$ of C^2. Then $\{(0, 0), (0, 1), (a, 1), (1, 0), (1, a)\}$ is a five element $*$-basis of the eleven element lattice L we obtained, albeit all maximal chains in L have exactly six elements.

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(Received August 26, 1986)

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