Notes on coalition lattices†

Gábor Czédli, Benoit Larose and György Pollák

JATE Bolyai Institute, Szeged, Aradi vértanúk tere 1, H–6720 HUNGARY
E-mail: CZEDLI@MATH.U–SZEGED.HU
LACIM Département de mathématiques, Université du Québec a Montréal, C.P. 8888, succ. centre-ville, Montréal, Qc H3C 3P8, CANADA
E-mail: LAROSE@DISCRETE.CONCORDIA.CA
Mathematical Research Institute, Budapest, HUNGARY
E-mail: POLLAK@MATH.U-SZEGED.HU

Key words: Lattice, coalition lattice, partially ordered set, quasiorder, coalition, winning coalition

Mathematics Subject Classification: Primary 06B99, Secondary 06A99, 90D99

Abstract. Given a finite partially ordered set P, for subsets or, in other words, coalitions X, Y of P let $X \leq Y$ mean that there exists an injection φ: $X \to Y$ such that $x \leq \varphi(x)$ for all $x \in X$. The set $\mathcal{L}(P)$ of all subsets of P equipped with this relation is a partially ordered set. When $\mathcal{L}(P)$ is a lattice, it is called the coalition lattice of P. It is shown that P is determined by the coalition lattice $\mathcal{L}(P)$. Further, any coalition lattice satisfies the Jordan-Hölder chain condition. The so-called winning coalitions, i.e. coalitions X such that $P \setminus X \leq X$ in $\mathcal{L}(P)$, are shown to form a dual ideal in $\mathcal{L}(P)$. Finally, an inductive formula on $|P|$ is given to describe the lattice operations in $\mathcal{L}(P)$, and this result also works for certain quasiordered sets P.

The authors' work was partially supported by the NFSR of Hungary (OTKA) grant no. T17005, NSERC of Canada and PAFAQQ, and NFSR of Hungary (OTKA) grant no. T29525, respectively.

1. Introduction and results

Given a finite partially ordered set $P = \langle P, \leq \rangle$, the set of all subsets, alias coalitions, of P will be denoted by $\mathcal{L}(P)$. For $X, Y \in \mathcal{L}(P)$, a map φ: $X \to Y$ is said to be an *extensive map* if φ is injective and for every $x \in X$ we have $x \leq \varphi(x)$. Let $X \leq Y$ mean that there exists an extensive map $X \to Y$; this definition turns $\mathcal{L}(P)$ into a partially ordered set $\mathcal{L}(P) = \langle \mathcal{L}(P), \leq \rangle$. When $\mathcal{L}(P)$ is a lattice then it is called the *coalition lattice* of P. This concept, with roots in game theory and the mathematics of human decision making,

† This paper, when first submitted in 1995, was dedicated to Evgeniĭ Sergeyevich Lyapin on his 80th birthday.

was introduced in [3]. To present a natural example, let P be a voting committee each of whose members has a certain strength measured on a numerical scale. The strength of a coalition is the sum of strengths of its members. Putting $x \le y$ for "the strength of x is smaller than or equal to that of y" we make P into a quasiordered set, most frequently a chain which gives rise to a coalition lattice $\mathcal{L}(P)$. This example motivates the following definition: given a coalition lattice $\mathcal{L}(P)$, an $X \in \mathcal{L}(P)$ is called a *winning coalition* if $P \setminus X \le X$. For undefined terminology the reader is referred to Grätzer [5]. Even without explicit mentioning, all sets occurring in this paper are assumed to be finite.

A partially ordered set P is called *upper bound free*, in short UBF, if for any $a, b, c \in P$ we have

$$((a \le c) \;\&\; (b \le c)) \implies ((a \le b) \;\text{ or }\; (b \le a)).$$

The equivalence classes of the equivalence generated by $\le_P$ will be called the *components* of P. If P is an UBF partially ordered set and has only one component then P is called a *tree*. A partially ordered set is called a *forest* if its components are trees. Clearly, a finite partially ordered set is a forest iff it is UBF. For $a \in P$ we will use the notation $(a] = \{x \in P\colon\ x \le a\}$. A partially ordered set P is a forest iff $(a]$ is a chain for every $a \in P$.

If $P = \langle P, \le\rangle$ is a finite quasiordered set rather than a partially ordered set then the definition of $\mathcal{L}(P) = \langle \mathcal{L}(P), \le\rangle$, the UBF property and the above-mentioned motivating example still make sense; then $\mathcal{L}(P)$ is a quasiordered set, of course. A quasiordered set Q is called a *quasilattice* if each two-element subset of Q has an infimum and a supremum in Q. (The infimum and supremum is defined only up to equivalence!) Equivalently, Q is a quasilattice iff the partially ordered set $\widetilde{Q}$ (to be defined soon) induced by Q is a lattice. Note that there is an algebraic characterization of quasilattices in Chajda [1], cf. also Chajda and Kotrle [2].

The main result of [3] asserts that, for a finite quasiordered set Q, $\mathcal{L}(Q)$ is a quasilattice iff Q is UBF. In particular, for a finite partially ordered set P, $\mathcal{L}(P)$ is a lattice iff P is a forest. Therefore, from now on, P and Q will always denote a finite forest and a finite quasiordered set with UBF, respectively. The description of lattice operations in $\mathcal{L}(P)$, cf. [3], is not so simple as generally in case of other lattices related with mathematical structures. The structure of coalition lattices is described in [4]. The easy part of this description is the following

Lemma A. *([3]) Let $T_1, T_2, \ldots, T_s$ be the components of P. Then $\mathcal{L}(P)$ is isomorphic to the direct product of the $\mathcal{L}(T_i)$, $1 \le i \le s$.*

We will also need

Lemma B. *([3]) The lattice $\mathcal{L}(P)$ is distributive iff every tree component of P is a chain.*

Our goal is to prove the following four theorems.

Theorem 1. *Every coalition lattice $\mathcal{L}(P)$ satisfies the Jordan-Hölder chain condition. I.e., any two maximal chains of $\mathcal{L}(P)$ have the same number of elements.*

Theorem 2. *The coalition lattice $\mathcal{L}(P)$ determines the forest P up to isomorphism. In other words, if $\mathcal{L}(P) \cong \mathcal{L}(P')$ then $P \cong P'$.*

Theorem 3. *Given a coalition lattice $\mathcal{L}(P)$, the winning coalitions form a dual ideal of $\mathcal{L}(P)$. Equivalently, there exists a winning coalition $W \in \mathcal{L}(P)$ such that, for any $X \in \mathcal{L}(P)$, X is a winning coalition iff $W \leq X$.*

While the meet in $\mathcal{L}(P)$ can be defined via a recursion on the size of P ([3, Proposition 1]), the description of join is much more complicated in [3]. Now we are going to give a recursive formula for the join in $\mathcal{L}(Q)$.

Let $\widetilde{Q} = \langle \widetilde{Q}, \leq \rangle$ denote the partially ordered set obtained from a quasiordered set Q with UBF in the canonical way, i.e., consider the intersection $\sim$ of $\leq_Q$ with its inverse, let $\widetilde{Q}$ consist of the classes of the equivalence relation $\sim$, and for $A, B \in \widetilde{Q}$ let $A \leq B$ mean that $a \leq b$ for some $a \in A$ and $b \in B$. For $x, y \in Q$, we write $x < y$ if $x \leq y$ but $y \not\leq x$. A subset F of Q is called an *order filter* if $f \in F$, $q \in Q$ and $f < q$ always imply $q \in F$. E.g., the empty subset is always an order filter. Let us choose an element $m \in Q$ such that $M = \widetilde{m}$, the $\sim$-class of m, is a maximal element in $\widetilde{Q}$.

Consider the subset $\mathcal{I} = \{X \in \mathcal{L}(Q)\colon\ X \cap M = \emptyset\}$ of $\mathcal{L}(Q)$. Then $\mathcal{I}$ is an order ideal (dual order filter). Observe that for $X \in \mathcal{I}$, $Y \in \mathcal{L}(Q)$ if $X \sim Y$ then $Y \in \mathcal{I}$. Since $X \cap Y \subseteq X \wedge Y$ and $X \vee Y \subseteq X \cup Y$ (at least for one possible choice of $X \wedge Y$ and $X \vee Y$) follow easily from [3] (and are explicitly stated in [4, Proof of Thm. 1]), $\mathcal{I}$ is closed with respect to (*arbitrary choice of*) infima and suprema, and $\mathcal{I}$ is (isomorphic to) $\mathcal{L}(Q \setminus M)$. The subset $\{x \in Q\colon\ x < m\}$, which is disjoint from M, will be denoted by $D(m)$. Now any element of $\mathcal{L}(Q)$ is of the (unique) form $X \cup Y$ where $X \in \mathcal{I}$ and $Y \subseteq M$. The join in $\mathcal{L}(Q)$ is described by the following

Theorem 4. *Given $X_i \in \mathcal{I}$ and $Y_i \subseteq M$, $i = 1, 2, \ldots, n$, put $t := \max(|Y_1|, \ldots, |Y_n|)$, let A_i be an order filter in $X_i \cap D(m)$ consisting of $\min(|X_i \cap D(m)|, t - |Y_i|)$ elements, put $B_i := X_i \setminus A_i$, and let C be a t-element subset of M. Then*

$$\bigvee_{i=1}^{n} (X_i \cup Y_i) = C \cup \bigvee_{i=1}^{n} B_i. \tag{1}$$

The proof of the above theorem will use the fact that $\mathcal{L}(Q)$ is a quasilattice. By the remarks preceding Theorem 4 the join on the right hand side of (1) can be understood both in $\mathcal{L}(Q)$ and in $\mathcal{L}(Q \setminus M)$.

Now let m be a maximal element in a forest P. For $X \in \mathcal{L}(P)$, let $\breve{X} = X \setminus \{m\}$ if $m \in X$, put $\breve{X} = X \setminus \{c\}$ if $m \notin X$ and c is the maximal element of $X \cap D(m)$, and let $\breve{X} = X$ if $m \notin X$ and $X \cap D(m) = \emptyset$. (Note that, by the UBF, $X \cap D(m)$ is a chain or empty, whence c is uniquely determined.) Then $\breve{X}$ belongs to the sublattice, in fact ideal, $\mathcal{I} = \{Y \in \mathcal{L}(P)\colon\ m \notin Y\} \cong \mathcal{L}(P \setminus \{m\})$. The following assertion is an obvious consequence of Theorem 4.

Corollary 1. *Let $X_1, \dots, X_n \in \mathcal{L}(P)$ and suppose that not all of them are in $\mathcal{I}$. Then*

$$\bigvee_{i=1}^{n} X_i = \{m\} \cup \bigvee_{i=1}^{n} \breve{X}_i.$$

The coalition lattice $\mathcal{L}(P)$ is the disjoint union of the above-defined ideal $\mathcal{I}$ and the filter $\mathcal{D} = \{Y \in \mathcal{L}(P)\colon\ m \in Y\}$, and both $\mathcal{I}$ and $\mathcal{D}$ are isomorphic to $\mathcal{L}(P \setminus \{m\})$ in a natural way. So, to compute meets via induction on $|P|$, it is sufficient to find the meet of E and $F \cup \{m\}$ for $E, F \in \mathcal{I}$. Let $\hat{F} = F$ if $D(m) \subseteq F$ and let $\hat{F} = F \cup \{u\}$ if u is the maximal element of $D(m) \setminus F$. Then $\hat{F} \in \mathcal{I}$ and we have

Corollary 2. $E \wedge (F \cup \{m\}) = E \wedge \hat{F}$.

The advantage of this Corollary over the analogous Proposition 1 in [3] is that $\mathcal{I} = \mathcal{L}(P \setminus \{m\}) \cong \mathcal{D}$ does not depend on the coalitions whose meet we intend to calculate.

Proofs

For $a \in P$ let $\mu(a)$ denote the cardinality of the chain $(a]$, i.e. $\mu(a) = |(a]|$. For $A \in \mathcal{L}(P)$ we define $\mu(A) = \sum_{a \in A} \mu(a)$. To avoid confusion, the elements of P resp. $\mathcal{L}(P)$ will be denoted by lower case resp. capital letters. The proof of Theorem 1 will rely on the following

Lemma 1. *Let $A, B \in \mathcal{L}(P)$. Then*

$$A < B \iff (A \le B \;\&\; \mu(A) < \mu(B)). \tag{2}$$

and

$$A \prec B \iff (A \le B \;\&\; \mu(A) + 1 = \mu(B)). \tag{3}$$

Proof. Suppose $A < B$ and choose an extensive map $\alpha\colon\ A \to B$. Then

$$\mu(A) = \sum_{a \in A} \mu(a) \le \sum_{a \in A} \mu(\alpha(a)) \le \sum_{b \in B} \mu(b) = \mu(B).$$

If both inequalities in the above formula were equations then $(\forall a)(a \le \alpha(a))$ and $\alpha(A) = B$ would imply $A = B$, a contradiction. Hence $\mu(A) < \mu(B)$. The converse direction of (2) is evident. The $\Longleftarrow$ direction of (3) follows from (2). To show the $\Longrightarrow$ direction of (3) let us assume that $A \prec B$. We have to distinguish two cases.

Case (i): $|A| < |B|$. Choose an extensive map $\varphi : A \to B$. Since $A \le \varphi(A) \le B$ but $\varphi(A)$, having less elements, is distinct from B, from $A \prec B$ we conclude that $A = \varphi(A)$.

Hence $A \subset B$. Let $\{b_1, b_2, \ldots, b_k\} = B \setminus A$. Since $A < A \cup \{b_1\} < A \cup \{b_1, b_2\} < \ldots < A \cup \{b_1, b_2, \ldots, b_k\} = B$, we conclude $k = 1$. Let z denote the smallest element in the chain $(b_1]$. If z belonged to A then $A < \big(A \setminus \{z\}\big) \cup \{b_1\} < A \cup \{b_1\} = B$ would contradict $A \prec B$. Hence $z \notin A$. The assumption $z < b_1$ would lead to $A < A \cup \{z\} < A \cup \{b_1\} = B$, another contradiction. Thus, $b_1 = z$ and $\mu(B) = \mu(A) + \mu(z) = \mu(A) + 1$, indeed.

Case (ii): $|A| = |B|$. Then we have an extensive bijection α: $A \to B$. The set $H = \{x \in A\colon\ x < \alpha(x)\}$ cannot be empty, for otherwise $A = \alpha(A) = B$ would follow. Let u be a minimal element of H and denote $\alpha(u)$ by v. We claim $u \notin B$. Indeed, otherwise $u = \alpha(y)$ would hold for some $y \in A$, the minimality of u would imply $y = u$, and $u = \alpha(y) = \alpha(u) = v$ would contradict $u < v$. Let $A_1 = A \setminus \{u\}$ and $B_1 = B \setminus \{v\}$. Since $u \notin B = \alpha(A)$, $(\alpha \setminus \{\langle u, v\rangle\}) \cup \{\langle u, u\rangle\}$: $A_1 \cup \{u\} \to B_1 \cup \{u\}$ is an extensive map. Hence $A = A_1 \cup \{u\} \le B_1 \cup \{u\} < B_1 \cup \{v\} = B$ yields $A_1 \cup \{u\} = B_1 \cup \{u\}$, whence $A_1 = B_1$ and the extensive map $\alpha_1 = \alpha \setminus \{\langle u, v\rangle\}$: $A_1 \to B_1$ must be the identical map. Since $\mu(B) - \mu(A) = \mu(v) - \mu(u)$, it suffices to show that $u \prec v$. Suppose this is not the case, i.e. $u < c < v$ holds for some $c \in P$. If $c \notin A_1$ then $A = A_1 \cup \{u\} < A_1 \cup \{c\} < A_1 \cup \{v\} = B_1 \cup \{v\} = B$ is a contradiction, so $c \in A_1$. Denoting $A_1 \setminus \{c\} = B_1 \setminus \{c\}$ by D we have $A = D \cup \{u, c\}$, $B = D \cup \{c, v\}$, and $A < D \cup \{u, v\} < B$ is a contradiction again. Hence $u \prec v$ and $\mu(B) = \mu(A) + 1$. ◇

Proof of Theorem 1. Let $\emptyset = C_0 \prec C_1 \prec C_2 \prec \ldots \prec C_t = P$ be a maximal chain in $\mathcal{L}(P)$. We infer from Lemma 1 that $\mu(P) = \mu(C_t) = \mu(C_{t-1}) + 1 = \mu(C_{t-2}) + 2 = \ldots = \mu(C_0) + t = t$, whence every maximal chain has $\mu(P) + 1$ elements. ◇

Proof of Theorem 2. Let $\mathcal{S} = \mathcal{S}(\mathcal{L}(P))$ denote the set of singleton coalitions in $\mathcal{L}(P)$, i.e., $\mathcal{S} = \{X \in \mathcal{L}(P)\colon\ |X| = 1\}$. For $a, b \in P$, $a \le b$ in P iff $\{a\} \le \{b\}$ in $\mathcal{L}(P)$. Therefore it suffices to describe $\mathcal{S}$ in a lattice theoretic language, i.e. in a way which is invariant under lattice isomorphisms; the theorem then will follow. Unfortunately, this description is not always possible. For example, if P is the three-element chain $\{0 < a < b\}$ then $\mathcal{L}(P)$ has an automorphism interchanging $\{a, 0\}$ and $\{b\}$, and the same can be said when one of the tree components of P is a three-element chain. That is why we deal with trees before settling the general case.

From now on let P be a tree. This property of P can be recognized from $\mathcal{L}(P)$ since it is easy to derive from Lemma A that P is a tree iff $\mathcal{L}(P)$ has exactly one atom. Note that the only atom of $\mathcal{L}(P)$ is $\{0\}$ where 0 is the smallest element of the tree P. A coalition $X \in \mathcal{L}(P)$ is called a *cycle* if the principal ideal $(X]$ is a chain in $\mathcal{L}(P)$. All singleton coalitions are cycles but not conversely. For a cycle X, distinct from the empty coalition, let X^- denote the unique coalition covered by X in $\mathcal{L}(P)$. Let $\mathcal{C}$ denote the set of cycles in $\mathcal{L}(P)$. For a coalition $X \in \mathcal{L}(P)$ let $h(X)$ denote the *height* of X, i.e. the length of any maximal chain from $\emptyset$ to X. Note that X is a cycle iff $|(X]| = h(X) + 1$. Now we define several subsets of $\mathcal{L}(P)$ as follows:

$$\mathcal{A} = \{X \in \mathcal{C}\colon\ h(X) = 2\},$$
$$\mathcal{B} = \{X \in \mathcal{C}\colon\ h(X) \ge 4\},$$

$$\begin{aligned}
\mathcal{T}_1 &= \{X \in \mathcal{C}\colon\ h(X) = 3 \ \text{ and }\ X < Y \ \text{ for some }\ Y \in \mathcal{B}\},\\
\mathcal{T}_2 &= \{X \in \mathcal{C}\colon\ h(X) = 3 \ \text{ and there is a }\ Z \in \mathcal{A}\\
&\qquad \text{such that }\ X^- \parallel Z \ \text{ and }\ |(X \vee Z]| \geq 8\}, \ \text{ and}\\
\mathcal{T}_3 &= \{X \in \mathcal{C}\colon\ h(X) = 3 \ \text{ and there is a }\ Y \in \mathcal{C}\\
&\qquad \text{such that }\ X \neq Y,\ X^- = Y^- \ \text{and}\ |(X \vee Y]| \geq 8\}.
\end{aligned}$$

Let

$$\mathcal{R} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \{\{0\}\}.$$

Here $\{0\}$ is, of course, the unique atom of $\mathcal{L}(P)$. We claim that

(4) $\qquad$ If $\mathcal{L}(P)$ is not distributive then $\mathcal{S} = \mathcal{R}$.

First we show $\mathcal{S} \subseteq \mathcal{R}$. Let g denote the height function on P. I.e., with μ defined in the previous proof, $g(a) = |(a]| - 1 = \mu(a) - 1$ for $a \in P$. Clearly, $h(\{a\}) = g(a) + 1$. Therefore $\{a\} \in \mathcal{R}$ for every $a \in P$ with $g(a) \neq 2$. Now assume that $g(a) = 2$. If a is not a maximal element in P then $\{a\} \in \mathcal{T}_1 \subseteq \mathcal{R}$. Therefore we can assume that a is a maximal element of P. Let b be the unique lower cover of a, i.e. $b \prec a$.

Firstly, assume that a is the only element of P which covers b. Since $\mathcal{L}(P)$ is not distributive, P is not a chain by Lemma B. Hence $P \setminus (a] \neq \emptyset$. Let c be a minimal element of $P \setminus (a]$. Denoting $\{a\}$, $\{b\}$ and $\{c\}$ by X, X^- and Z, respectively, we obtain $\{a\} \in \mathcal{T}_2$, for $(X \vee Z] = (\{a, c\}]$ contains $\emptyset$, $\{0\}$, $\{b\}$, $\{c\}$, $\{a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, a\}$, $\{b, c\}$, $\{a, c\}$, i.e. more than eight distinct coalitions.

Secondly, assume that $\{a = a_1, a_2, \ldots, a_k\}$ is the set of elements covering b, $k \geq 2$. Putting $X = \{a\}$ and $Y = \{a_2\}$ we see that $\{a\} \in \mathcal{T}_3$, for the coalitions $\emptyset$, $\{0\}$, $\{b\}$, $\{a\}$, $\{a_2\}$, $\{0, b\}$, $\{0, a\}$, $\{0, a_2\}$ all belong to $(X \vee Y] = \{a, a_2\}$. We have shown $\mathcal{S} \subseteq \mathcal{R}$.

As a first step towards the converse inclusion in (4) we claim

(5) $$X \in \Big(\mathcal{L}(P) \setminus \big(\mathcal{S} \cup \{\emptyset\}\big)\Big) \cap \mathcal{C} \Longrightarrow X = \{0, b\} \ \text{ for some }\ 0 \prec b.$$

Let $X \in \mathcal{L}(P) \setminus \big(\mathcal{S} \cup \{\emptyset\}\big)$ be a cycle. If $|X| \geq 3$ then, for any maximal element u of X, $\{u\} \parallel X \setminus \{u\}$, contradicting the fact that $(X]$ is a chain. Therefore $|X| = 2$. Let $X = \{a, b\}$. From $\{a\}, \{b\} \in (X]$ we infer that a and b are comparable, so we assume $0 \leq a < b$. If $0 < a < b$ then $\{0, a\} \parallel \{b\}$ in $(X]$, a contradiction. Hence $X = \{0, b\}$. If $0 < c < b$ for some $c \in P$ then $\{0, c\} \parallel \{b\}$ in $(X]$, a contradiction again. Therefore $0 \prec b$, proving (5).

For $0 \prec b$ we have $h(\{0, b\}) = 3$. This fact and (5) clearly yield $\mathcal{A} \cup \mathcal{B} \cup \{\{0\}\} \subseteq \mathcal{S}$. Hence, by $\mathcal{B} \subseteq \mathcal{S}$, $\mathcal{T}_1 \subseteq \mathcal{S}$ follows immediately. Suppose $X \in \mathcal{T}_2 \setminus \mathcal{S}$. By (5), $X = \{0, b\}$ for some $0 \prec b$. We have $X^- = \{b\}$, $Z = \{a\}$ from $\mathcal{A} \subseteq \mathcal{S}$, $a \parallel b$ and, by $h(Z) = 2$, $0 \prec a$. Since $X \vee Z = \{a, b\}$, $(X \vee Z] \cong \mathcal{L}(Q) \setminus \{Q\}$ where Q is $\{0, a, b\}$, as a sub-poset of P. Hence $|(X \vee Z]| = 2^3 - 1 = 7$, contradicting $X \in \mathcal{T}_2$. Thus, $\mathcal{T}_2 \subseteq \mathcal{S}$.

Suppose $X \in \mathcal{T}_3 \setminus \mathcal{S}$. As previously, $X = \{0, b\}$ and $X^- = \{b\} = Y^-$ for some $0 \prec b$. Now Y is a singleton, for otherwise $h(Y) = h(Y^-) + 1 = 3$ and (5) would imply $Y = \{0, b\} = X$, a contradiction. Therefore $Y = \{a\}$ for some $b \prec a$. We have $X \vee Y = \{0, a\}$. Using $Q = \{0, a, b\}$ as before we can derive $|(X \vee Y]| = 2^3 - 1 = 7$. This contradiction shows $\mathcal{T}_3 \subseteq \mathcal{S}$. This proves $\mathcal{R} \subseteq \mathcal{S}$ and (4).

Now let us assume first that $\mathcal{L}(P)$ has only one atom, i.e. P is a tree. If $\mathcal{L}(P)$ is distributive then P is a chain by Lemma B. Since the chain P is determined by $|P|$ and $|P|$ uniquely comes from $2^{|P|} = |\mathcal{L}(P)|$, this case is settled. If $\mathcal{L}(P)$ is not distributive then $P \cong \mathcal{S}$ is determined up to isomorphism by (4).

Secondly let us assume that $\mathcal{L}(P)$ has more than one atom. Then, by Lemma A,

$$\mathcal{L}(P) \cong \prod_{i=1}^{k} \mathcal{L}(T_i), \tag{6}$$

where the T_i are the tree components of P. But, as we mentioned before, the $\mathcal{L}(T_i)$ are directly indecomposable. It is known, cf. Grätzer [3, p. 153, Cor. III.4.4] that if we decompose a finite lattice as a direct product of directly indecomposable factors then these factors are uniquely determined up to isomorphism. Applying this to (6) we infer that the $\mathcal{L}(T_i)$ are determined up to isomorphism. But any one of them has only one atom. Consequently, by the previous part of the proof, they determine the T_i, i.e. the tree components, and therefore the whole P, up to isomorphism. ◇

The proof of Theorem 3 requires three lemmas. For $a \in P$ we set $U_a = \{x \in P\colon\ x > a\} = [a) \setminus \{a\}$ and $D_a = \{x \in P\colon\ x < a\} = (a] \setminus \{a\}$. We define the i-th layer P_i of P via induction as follows. Let P_1 consist of the maximal elements of P. If $P_1 \cup P_2 \cup \ldots \cup P_{i-1} \neq P$ then let P_i be the set of maximal elements of $P \setminus (P_1 \cup P_2 \cup \ldots \cup P_{i-1})$. There are finitely many layers, say $P_1, P_2, \ldots, P_r$, they are disjoint and their union is P. The subset $P_1 \cup P_2 \cup \ldots \cup P_i$ will be denoted by Q_i. For a coalition $X \in \mathcal{L}(P)$, $P \setminus X$ will be denoted by $\overline{X}$.

Lemma 2. *A coalition $C \in \mathcal{L}(P)$ is winning iff*

$$|C \cap [x)| \geq |\overline{C} \cap [x)| \quad \textit{for every} \quad x \in P. \tag{7}$$

Proof. Let C be a winning coalition and let $\varphi\colon\ \overline{C} \to C$ be an extensive map. Then φ maps $\overline{C} \cap [x)$ into $C \cap [x)$ and (7) follows from injectivity.

Conversely, suppose that (7) holds. We will define extensive maps $\varphi_i\colon\ \overline{C} \cap Q_i \to C \cap Q_i$ via induction. This is sufficient, for $\varphi_r\colon\ \overline{C} \to C$ will imply that C is winning. In virtue of (7) we have $\overline{C} \cap Q_1 = \emptyset$, so we let φ_1 be the empty map, which is clearly extensive. Suppose that φ_{i-1} is already defined and consider an arbitrary $x \in \overline{C} \cap P_i$. Since φ_{i-1} maps $\overline{C} \cap U_x$ into $C \cap U_x$ and from (7) we obtain $|\overline{C} \cap U_x| = |\overline{C} \cap [x)| - 1 \leq |C \cap [x)| - 1 < |C \cap [x)| = |C \cap U_x|$, we can fix an element $y_x \in C \cap U_x$ such that $y_x \notin \varphi_{i-1}(\overline{C} \cap U_x)$. It follows from the UBF property and $x < y_x$ that for distinct $x_1, x_2 \in \overline{C} \cap P_i$ we have $y_{x_1} \neq y_{x_2}$. Therefore

$$\varphi_i = \varphi_{i-1} \cup \{\langle x, y_x\rangle\colon\ x \in \overline{C} \cap P_i\}\colon\ \overline{C} \cap Q_i \to C \cap Q_i$$

is an extensive map, proving the assertion. ◇

Lemma 3. *Let C be a winning coalition and suppose that*

$$|C \cap U_a| > |\overline{C} \cap U_a|. \tag{8}$$

holds for some $a \in C$. Then there exists a winning coalition B such that $B < C$.

Proof. Let us fix an extensive map φ: $\overline{C} \to C$. Since φ maps $\overline{C} \cap U_a$ into $C \cap U_a$, by (8) we can fix an element $b \in C \cap U_a$ such that $b \notin \varphi(\overline{C} \cap U_a)$. Firstly, we consider the case $\overline{C} \cap D_a = \emptyset$. Then let $B = C \setminus \{a\}$. Clearly, $B < C$ and the map $\varphi \cup \{\langle a, b\rangle\}$: $\overline{B} \to B$ is extensive, whence B is winning.

Secondly, suppose that $\overline{C} \cap D_a$ is nonempty, and let c be the greatest element of the chain $\overline{C} \cap D_a$. Now we set $B = (C \setminus \{a\}) \cup \{c\}$. The relation $B < C$ is clear. We can assume that $\varphi(c) = a$. Indeed, if $a \notin \varphi(\overline{C})$ then we can take $(\varphi \setminus \{\langle c, \varphi(c)\rangle\}) \cup \{\langle c, a\rangle\}$ instead of φ. If $\varphi(t) = a \neq \varphi(c)$ then, by the choice of c, $t < c$ and φ can be replaced by $(\varphi \setminus \{\langle c, \varphi(c)\rangle, \langle t, a\rangle\}) \cup \{\langle c, a\rangle, \langle t, \varphi(c)\rangle\}$. Thus, $\varphi(c) = a$. Define a map

$$\psi\colon \overline{B} \to B, \quad x \mapsto \begin{cases} b, & \text{if } x = a, \\ c, & \text{if } \varphi(x) = b \\ \varphi(x), & \text{otherwise.} \end{cases}$$

Note that if $\varphi(x) = b$ then $x < c$ by the choice of b, c and the fact that $(b]$ is a chain. Hence ψ is an extensive map and B is a winning coalition. ◇

Lemma 4. *There is exactly one minimal winning coalition in $\mathcal{L}(P)$. If W denotes this coalition then, for any $x \in P$, we have*

$$x \in W \iff |W \cap U_x| = |\overline{W} \cap U_x|. \tag{9}$$

Proof. By finiteness, there is at least one minimal winning coalition $W \in \mathcal{L}(P)$. After showing that W satisfies (9) and at most one coalition can satisfy (9) the lemma will follow.

Let W be a minimal winning coalition and suppose that (9) is violated by some $x \in P$. First let $x \in W$ but $|W \cap U_x| \neq |\overline{W} \cap U_x|$. Since any extensive mapping $\overline{W} \to W$ must map $\overline{W} \cap U_x$ into $W \cap U_x$, $|\overline{W} \cap U_x| \leq |W \cap U_x|$. Hence $|\overline{W} \cap U_x| < |W \cap U_x|$ and Lemma 3 yields that W is not a *minimal* winning coalition, a contradiction. Therefore $x \notin W$ but $|W \cap U_x| = |\overline{W} \cap U_x|$. Then $|W \cap [x)| = |W \cap U_x| = |\overline{W} \cap U_x| = |\overline{W} \cap [x)| - 1 < |\overline{W} \cap [x)|$, contradicting Lemma 2. Thus, any minimal winning coalition satisfies (9).

Suppose that both W_1 and W_2 satisfy (9) for every $x \in P$ but $W_1 \neq W_2$. Take a maximal element x in $(W_2 \setminus W_1) \cup (W_1 \setminus W_2)$. By the maximality of x, $W_2 \cap U_x = W_1 \cap U_x$ and $\overline{W_2} \cap U_x = \overline{W_1} \cap U_x$. Hence, by (9), we conclude $x \in W_2 \iff |W_2 \cap U_x| = |\overline{W_2} \cap U_x| \iff |W_1 \cap U_x| = |\overline{W_1} \cap U_x| \iff x \in W_1$, which contradicts the choice of x. This proves the uniqueness, and the assertion follows. ◇

Proof of Theorem 3. Let us denote the set of winning coalitions by $\mathcal{W}$. Then $\mathcal{W}$ has a unique minimal element by Lemma 4 and clearly has the property

$$(\forall X, Y \in \mathcal{L}(P))\ (X \leq Y \ \&\ X \in \mathcal{W} \Longrightarrow Y \in \mathcal{W}).$$

By finiteness, $\mathcal{W}$ is a dual ideal. $\diamond$

It is worth noting that Lemma 4 gives a straightforward algorithm to construct the minimal winning coalition.

Proof of Theorem 4. Denoting the right hand side of (1) by R first we show that R is an upper bound of the $X_j \cup Y_j$, $1 \leq j \leq n$. Since $|A_j| \leq t - |Y_j|$ and $|C| = t$, any injective map $A_j \to C$ can be extended to an injective map α: $A_j \cup Y_j \to C$. There is an extensive map β: $B_j \to \bigvee_{i=1}^n B_i$. Clearly, $\alpha \cup \beta$: $X_j \cup Y_j \to R$ is an extensive map. Hence R is an upper bound of the $X_j \cup Y_j$, $1 \leq j \leq n$.

Now let $U \cup T \in \mathcal{L}(Q)$, where $U \in \mathcal{I}$ and $T \subseteq M$, be an arbitrary upper bound of the $X_j \cup Y_j$, $1 \leq j \leq n$. Since any extensive map $X_j \cup Y_j \to U \cup T$ maps Y_j to T, we infer $|C| \leq |T|$. We may assume that $|Y_1| \leq |Y_2| \leq \ldots \leq |Y_n|$. Notice that for any $X \in \mathcal{I}$ if $Y', Y'' \subseteq M$ and $|Y'| = |Y''|$ then the coalitions $X \cup Y'$ and $X \cup Y''$ are equivalent, i.e. $X \cup Y' \leq X \cup Y''$ and $X \cup Y'' \leq X \cup Y'$. Therefore we may assume, without loss of generality, that $Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y_n = C \subseteq T$. All we have to show is

$$B_j \leq U \cup (T \setminus C) \tag{10}$$

for $1 \leq j \leq n$; indeed, then $\bigvee_{i=1}^n B_i \leq U \cup (T \setminus C)$ and $R = C \cup \bigvee_{i=1}^n B_i \leq C \cup U \cup (T \setminus C) = U \cup T$ will already follow.

Assume first that $|X_j \cap D(m)| \leq t - |Y_j|$ and let φ: $X_j \cup Y_j \to U \cup T$ be an extensive map. Then $A_j = X_j \cap D(m)$, and $B_j \cap D(m) = \emptyset$ yields $\varphi(B_j) \cap T = \emptyset$. Hence $B_j \leq \varphi(B_j) \subseteq U \subseteq U \cup (T \setminus C)$ and (10) follows.

In the rest of the proof we assume that $|X_j \cap D(m)| > t - |Y_j|$. Since $X_j \cup Y_j \leq U \cup T$, there exists an extensive map τ: $X_j \to U \cup (T \setminus Y_j)$ such that $|\tau(B_j) \cap T|$ is minimal. Since τ maps B_j into $U \cup (\tau(B_j) \cap T)$, (10) clearly follows from

$$|\tau(B_j) \cap T| \leq |T \setminus C|, \tag{11}$$

which we are going to show. We may suppose $\tau(B_j) \cap T \neq \emptyset$, for otherwise (11) is evident.

Suppose first that $\tau(A_j) \not\subseteq T$, and choose $a \in A_j$, $b \in B_j$ such that $\tau(a) \notin T$ and $\tau(b) \in T$. $D(m)$ is a chain by the UBF property, whence a and b are comparable elements. Since A_j is a filter in $X_j \cap D(m)$, we conclude $b \leq a$, whence $b \leq \tau(a)$. From $a \in D(m)$ we infer $a \leq \tau(b)$. Therefore

$$\psi\colon\ X_j \to U \cup (T \setminus Y_j), \quad x \mapsto \begin{cases} \tau(b), & \text{if } x = a, \\ \tau(a), & \text{if } x = b \\ \tau(x), & \text{otherwise.} \end{cases}$$

is also an extensive map, and $|\psi(B_j) \cap T| < |\tau(B_j) \cap T|$ contradicts the choice of τ.

Thus $\tau(A_j) \subseteq T$, and we obtain

$$|\tau(B_j) \cap T| = |\tau(B_j) \cap (T \setminus Y_j)| =$$
$$|\big(\tau(X_j) \cap (T \setminus Y_j)\big) \setminus \big(\tau(A_j) \cap (T \setminus Y_j)\big)| =$$
$$|\tau(X_j) \cap (T \setminus Y_j)| - |\tau(A_j)| \leq$$
$$|T \setminus Y_j| - (t - |Y_j|) = |T| - |Y_j| - (|C| - |Y_j|) =$$
$$|T| - |C| = |T \setminus C|,$$

proving (11). ◇

Proof of Corollary 2. By [3, Prop. 2] we have

$$Z_1 \wedge Z_2 = \overline{\overline{Z}_1 \vee \overline{Z}_2} \tag{12}$$

for any $Z_1, Z_2 \in \mathcal{L}(P)$. Put $X_1 := \overline{E \cup \{m\}}$, $Y_1 := \{m\}$, $X_2 := \overline{F \cup \{m\}}$ and $Y_2 := \emptyset$. With the notations of Theorem 4 we have $t = 1$, $C = \{m\}$, $A_1 = \emptyset$ and $B_1 = X_1 = \overline{E \cup \{m\}}$. If $D(m) \subseteq F$ then $A_2 = \emptyset$ and $B_2 = X_2$, otherwise $A_2 = \{u\}$ and $B_2 = X_2 \setminus \{u\}$, so $B_2 = \overline{\hat{F} \cup \{m\}}$ in both cases. Let us compute based on (12), Theorem 4 and $\mathcal{D} \cong \mathcal{I}$:

$$E \wedge (F \cup \{m\}) = \overline{\overline{E} \vee \overline{F \cup \{m\}}} = \overline{(X_1 \cup Y_1) \vee (X_2 \cup Y_2)} =$$
$$\overline{\{m\} \cup (B_1 \vee B_2)} = \overline{B_1 \vee B_2} \setminus \{m\} = (\overline{B_1} \wedge \overline{B_2}) \setminus \{m\} =$$
$$\big((E \cup \{m\}) \wedge (\hat{F} \cup \{m\})\big) \setminus \{m\} =$$
$$\big((E \wedge \hat{F}) \cup \{m\}\big) \setminus \{m\} = E \wedge \hat{F},$$

indeed. ◇

Remark. While revising the present paper, we were notified that Michelle Davidson and George Grätzer found a new proof of the fact that $\mathcal{L}(Q)$ is a quasilattice iff Q is UBF, cf. [6]. Their approach also offers a recursive construction of joins in $\mathcal{L}(Q)$, which is entirely different from our Theorem 4.

References

[1] I. Chajda , Lattices in quasiordered sets , Acta Palack. Univ. Olomouc , 31 , 1992 , 6–12 .

[2] I. Chajda and M. Kotrle , Subdirectly irreducible and congruence distributive q-lattices , Czechoslovak Math. Journal , 43 (118) , 1993 , 635–642 .

[3] G. Czédli and Gy. Pollák , When do coalitions form a lattice? , Acta Sci. Math. (Szeged) , 60 , 1995 , 197–206 .

[4] G. Czédli , A Horn sentence for coalition lattices , Acta Math. Hungar. , 72 , 1996 , 99–104 .

[5] G. Grätzer , General Lattice Theory , Akademie-Verlag (Berlin), Birkhäuser Verlag (Basel), Academic Press (New York, N. Y.) , 1978 .

[6] M. Davidson and G. Grätzer , A note on coalitions , Acta Sci. Math. (Szeged) , 61 , 1995 , 34–34 .