LARGE RIGID SETS OF ALGEBRAS WITH RESPECT TO EMBEDDABILITY

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Dedicated to Anatolij Dvurečenskij on his 65th birthday

Abstract. Let τ be a nonempty similarity type of algebras. A set H of τ-algebras is called rigid with respect to embeddability, if whenever $A, B \in H$ and $\varphi : A \to B$ is an embedding, then $A = B$ and φ is the identity map. We prove that if τ is a nonempty similarity type and $\mathfrak{m}$ is a cardinal such that no inaccessible cardinal is smaller than or equal to $\mathfrak{m}$, then there exists a set H of τ-algebras such that H is rigid with respect to embeddability and $|H| = \mathfrak{m}$. This result strengthens a result proved by the second author in 1980.

1. Introduction and our results

An infinite cardinal $\mathfrak{m}$ is called *inaccessible* (cf. any standard textbook, e.g., Levy [9]) if the following three conditions hold:

- $\aleph_0 < \mathfrak{m}$;
- for all cardinals $\mathfrak{n}$, if $\mathfrak{n} < \mathfrak{m}$, then $2^{\mathfrak{n}} < \mathfrak{m}$;
- $\mathfrak{m}$ is a *regular cardinal*, that is, for every set I of cardinals, if $|I| < \mathfrak{m}$ and all members of I are smaller than $\mathfrak{m}$, then $\sum_{\mathfrak{n} \in I} \mathfrak{n} < \mathfrak{m}$.

Note that there exists a model of set theory in which there exists no inaccessible cardinal; see Kuratowski [8]. In this model, our theorems hold for all cardinals $\mathfrak{m}$.

Before formulating our first theorem, we also need the following concept. Let τ be a similarity type of algebras. (For example, if τ consists of a single binary operation, then we speak about the similarity type of groupoids.)

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Definition 1.1. Let H be a set or a class of algebras of the similarity type τ. Then H is said to be *rigid with respect to embeddability* (*e-rigid*, for short), if whenever $A, B \in H$, $\varphi : A \to B$ is an embedding (that is, injective homomorphism), then $A = B$ and φ is the identity map $\mathrm{id}_A : A \to A$, $x \mapsto x$. We say that A is an *e-rigid algebra* if it is an algebra and $\{A\}$ is an e-rigid set.

Let us remark that this is a very strong condition even if H is a singleton set. Note also that category theorists would call an e-rigid H as a discrete full subcategory of the category of τ-algebras with embeddings. Further, note that the concept of e-rigidity is interesting not only for algebras; see Primavesi and Thompson [10] for an example.

Our first goal is to prove the following statement.

Theorem 1.2. *Let τ be a similarity type of algebras containing an at least unary operation, and let $\mathfrak{m}$ be a cardinal number. If there is no inaccessible cardinal $\mathfrak{k}$ such that $\mathfrak{k} \leq \mathfrak{m}$, then there exists an e-rigid set H of τ-algebras such that $|H| = \mathfrak{m}$.*

If τ consists of a single unary operation f, then τ-algebras are called *monounary algebras* (for basic notions see [6]). A monounary algebra $A = \langle A, f \rangle$ is *connected*, if for each $\langle x, y \rangle \in A^2$ there exists a pair $\langle i, j \rangle$ of nonnegative integers such that $f^i(x) = f^j(y)$. If a connected monounary algebra has an idempotent element, that is, an element $x \in A$ with $f(x) = x$, then this element is uniquely determined and it is called the *top* of A. The top of A will be denoted $\mathrm{top}(A)$. A monounary algebra of this form is said to be a *root algebra* (cf. [6] and Jónsson [7]). For $x \in A$ we denote $f^{-1}(x) = \{y \in A : f(y) = x\}$.

We assert that

Theorem 1.3. *Let $\mathfrak{m}$ be a cardinal such that there is no inaccessible cardinal $\mathfrak{k}$ with $\mathfrak{k} \leq \mathfrak{m}$. Then there exists an e-rigid set H of monounary algebras such that $|H| = \mathfrak{m}$.*

Actually, we prove slightly more in the sense that H will consist of root monounary algebras. Note that both theorems strengthen [5] by the second author, where smaller (but still very large) cardinals are considered.

2. Proofs and auxiliary statements

First of all, observe that Theorem 1.2 is a straightforward consequence of Theorem 1.3 according to the following argument. Let $\mathfrak{m}$ be as in Theorem 1.2, and let g be an operation symbol in τ. Take an e-rigid set H provided by Theorem 1.3. We can turn the monounary algebras $\langle A, f \rangle \in H$ to τ-algebras by defining $g(x_1, x_2, \dots) = f(x_1)$ and letting the rest of τ-operations act as the first projection. This way H turns into an e-rigid set of τ-algebras.

Armed with the observation above, in the rest of paper, it suffices to deal with Theorem 1.3 and monounary algebras.

Replacing "embedding" by "isomorphism" in Definition 1.1, we obtain the concept of *rigidity* (without the prefix "e-"). In connection with our results, we mention that, by Comer and Le Tourneau [1], for every nonempty similarity type τ and each cardinal $\mathfrak{m}$, there exists a rigid set of τ-algebras consisting of $\mathfrak{m}$ members. Equivalently, they proved that for each $\mathfrak{m}$, there is a τ-algebra A such that $|A| \geq \mathfrak{m}$ and A has no nontrivial automorphism. Note, however, that it is more difficult to find e-rigid sets than rigid ones, and that the theory of monounary algebras in itself brings new interesting achievements, like [2], Halušková [3], and Horváth, Kátai-Urbán, Pach, Pongrácz, Pluhár and Szabó [4].

The proof of Theorem 1.3 requires two constructions; they are illustrated in Figures 1 and 2. Note that we depict our monounary algebras as Hasse diagrams; the action of the operation to an element x is the unique upper cover of x if this cover exists, and it is x otherwise.

Construction 2.1. Let $X = \langle X, f_X \rangle$ and $Y = \langle Y, f_Y \rangle$ be disjoint root algebras, and let $t, u_1, u_2, w_1, w_2, w_3$ be fixed elements. (Now we assume that distinct symbols represent distinct elements and that they do not belong to $X \cup Y$.) We denote by $U(X, Y) = \langle U(X, Y), g \rangle$ the unique root monounary algebra that satisfies the following four requirements:

(1) $U(X, Y) = X \cup Y \cup \{t, u_1, u_2, w_1, w_2, w_3\}$;
(2) $g(x) = f_X(x)$, $g(y) = f_Y(y)$ for each $x \in X$, $x \neq \mathrm{top}(X)$, $y \in Y$, $y \neq \mathrm{top}(Y)$;
(3) $g(\mathrm{top}(X)) = w_1$, $g(\mathrm{top}(Y)) = w_2$;
(4) $\mathrm{top}(U(X, Y)) = t = g(t) = g(u_1) = g(u_2)$, $g(w_1) = g(w_3) = u_1$, $g(w_2) = u_2$.

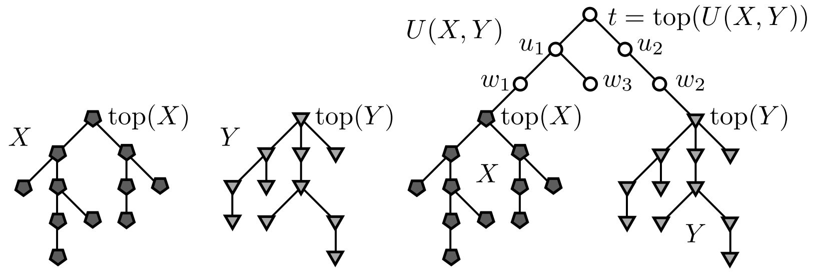

FIGURE 1. An example for $U(X, Y)$

We are only interested in algebras up to isomorphism. This allows us to apply Constructions 2.1 and 2.4 even if the monounary algebras in question are not (pairwise) disjoint: first we take disjoint isomorphic copies, and the we apply the constructions to these disjoint copies.

Lemma 2.2. *Let* $X = \langle X, f_X\rangle$, $Y = \langle Y, f_Y\rangle$, $\overline{X} = \langle \overline{X}, f_{\overline{X}}\rangle$, *and* $\overline{Y} = \langle \overline{Y}, f_{\overline{Y}}\rangle$ *be root algebras. Then* φ *is an embedding of* $U = \langle U, g\rangle = U(X,Y)$ *into* $\overline{U} = \langle \overline{U}, \overline{g}\rangle = U(\overline{X}, \overline{Y})$ *if and only if the restriction of* φ *to* X*, denoted by* $\varphi\rceil_X$*, is an embedding of* X *into* $\overline{X}$*,* $\varphi\rceil_Y$ *is an embedding of* Y *into* $\overline{Y}$*,* $\varphi(\mathrm{top}(X)) = \mathrm{top}(\overline{X})$*,* $\varphi(\mathrm{top}(Y)) = \mathrm{top}(\overline{Y})$*, and* $\varphi\rceil_{\{t,u_1,u_2,w_1,w_2,w_3\}}$ *is the identity map.*

Proof. Suppose that φ is an embedding of U into $\overline{U}$. The mapping φ is a homomorphism, thus $\varphi(t) = \varphi(g(t)) = \overline{g}(\varphi(t))$, and we obtain $\varphi(t) = t$. Next, $\{u_1, u_2\} \subseteq g^{-1}(t)$ yields $\{\varphi(u_1), \varphi(u_2)\} \subseteq \overline{g}^{\,-1}(\varphi(t)) = \overline{g}^{\,-1}(t) = \{u_1, u_2, t\}$, and by injectivity, $\{\varphi(u_1), \varphi(u_2)\} = \{u_1, u_2\}$. From $|g^{-1}(u_1)| = 2 = |\overline{g}^{\,-1}(u_1)|$, $|g^{-1}(u_2)| = 1 = |\overline{g}^{\,-1}(u_2)|$ we get $\varphi(u_1) = u_1$, $\varphi(u_2) = u_2$. Similarly, $\{w_1, w_3\} \subseteq g^{-1}(u_1) = \overline{g}^{\,-1}(\varphi(u_1)) = \{\varphi(w_1), \varphi(w_3)\}$, $|g^{-1}(w_1)| = 1 = |\overline{g}^{\,-1}(w_1)|$, $g^{-1}(w_3) = \emptyset = \overline{g}^{\,-1}(w_3)$, therefore $\varphi(w_1) = w_1$, $\varphi(w_3) = w_3$. Then it is clear that $\varphi(w_2) = w_2$, $\varphi(\mathrm{top}(X)) = \mathrm{top}(\overline{X})$ and $\varphi(\mathrm{top}(Y)) = \mathrm{top}(\overline{Y})$. Since X in U is characterized by $X = \{x \in U\colon (\exists n \in \mathbb{N}_0)(f_X^n(x) = \mathrm{top}(X))\}$ and similarly for Y, $\overline{X}$ and $\overline{Y}$, it follows that $\varphi(X) \subseteq \overline{X}$ and $\varphi(Y) \subseteq \overline{Y}$. Therefore $\varphi\rceil_X$ is an embedding of X into $\overline{X}$ and $\varphi\rceil_Y$ is an embedding of Y into $\overline{Y}$.

The converse implication is obvious. □

From Lemma 2.2 we easily obtain

Corollary 2.3. *If* $\{X_i : i \in I\}$ *and* $\{Y_i : i \in I\}$ *are e-rigid sets of root monounary algebras, then* $\{U(X_i, Y_i) : i \in I\}$ *is also an e-rigid set. In particular, if* X *and* Y *are e-rigid root monounary algebras, then so is* $U(X,Y)$*.*

Construction 2.4. Let $X_i = \langle X_i, f_i\rangle$, $i \in I$, be a set of pairwise disjoint root algebras and let t be a fixed element belonging to none of them. We denote by $\sum_{i\in I}^t X_i$ the root algebra $\langle X, f\rangle$ defined by $X = \bigcup_{i\in I} X_i \cup \{t\}$ and

$$f(x) = \begin{cases} f_i(x) & \text{if } x \in X_i \setminus \{\mathrm{top}(X_i)\},\ i \in I,\\ t & \text{otherwise.}\end{cases}$$

Lemma 2.5. *Let* $X_i = \langle X_i, f_i\rangle$*,* $i \in I$*, and* $Y_j = \langle Y_j, f_j\rangle$*,* $j \in J$*, be root algebras. Then* $\sum_{i\in I}^t X_i$ *is embeddable into* $\sum_{j\in J}^t Y_j$ *if and only if there exists an injective map* $\psi : I \to J$ *such that, for all* $i \in I$*,* X_i *is embeddable into* $Y_{\psi(i)}$*.*

Proof. The straightforward proof is analogous to the proof of Lemma 2.2; in fact, it is much easier. □

Corollary 2.6. *Let* $\{X_i = \langle X_i, f_i\rangle : i \in I\}$ *be a nonempty e-rigid set of root algebras. Assume that* $\mathcal{K}$ *is an antichain in the powerset lattice* $\langle P(I), \subseteq\rangle$*. Then* $\{\sum_{i\in Z}^t X_i : Z \in \mathcal{K}\}$ *is an e-rigid set of root algebras.*

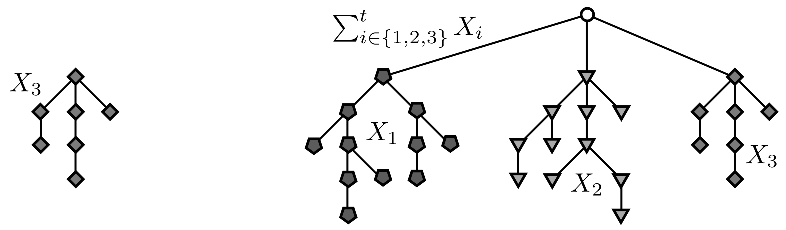

FIGURE 2. An example for the sum construct; here $X_1 = X$ and $X_2 = Y$ are given in Figure 1

The following lemma is well-known; we only give its short proof for the reader's convenience.

Lemma 2.7. *Let I be an infinite set. Then $\langle P(I), \subseteq\rangle$ contains an antichain of size $2^{|I|}$.*

Proof. Clearly, there are $I_1, I_2 \subseteq I$ such that $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = I$ and $|I_1| = |I_2|$. Pick a bijection $\psi : I_1 \to I_2$, and observe that $\{X \cup (I_2 \setminus \psi(X)) : X \in P(I_1)\}$ is an antichain. $\square$

Now we are ready to prove our theorem.

Proof of Theorem 1.3. Suppose, for a contradiction, that the theorem fails. Let $\mathfrak{m}$ be the smallest cardinal witnessing this failure.

There are many easy ways to construct a countable e-rigid set. E.g., for $2 < n \in \mathbb{N}$, take $A_n = \{0, 1, \ldots, n\}$ and put $f_n(x) = x - 1$ for $x = 1, 2, \ldots, n-1$, $f_n(0) = 0$, $f_n(n) = n - 3$. The A_n for the first few n are given in Figure 3. Note that all monounary algebras in our figures are e-rigid.

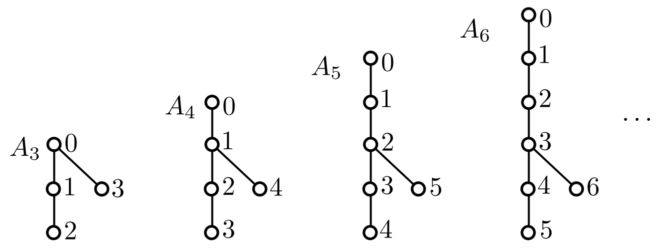

FIGURE 3. A countably infinite e-rigid set

Therefore, $\aleph_0 < \mathfrak{m}$. Since $\mathfrak{m}$ witnesses a failure, $\mathfrak{m}$ is not inaccessible. Hence, there are two cases: $\mathfrak{m} \leq 2^{\mathfrak{k}}$ for some cardinal $\mathfrak{k}$ such that $\mathfrak{k} < \mathfrak{m}$, or there is a set I of cardinals such that $|I| < \mathfrak{m}$, $\mathfrak{n} < \mathfrak{m}$ for all $\mathfrak{n} \in I$, but $\mathfrak{m} \leq \sum_{\mathfrak{n}\in I} \mathfrak{n}$.

First, assume that $\mathfrak{m} \le 2^{\mathfrak{k}}$. By the assumption, there exists an e-rigid set consisting of $\mathfrak{k}$ root algebras. Combining Corollary 2.6 and Lemma 2.7, we obtain an e-rigid set of size $2^{\mathfrak{k}}$. Since every subset of and e-rigid set is also e-rigid and $\mathfrak{m} \le 2^{\mathfrak{k}}$, we also have an e-rigid set of size $\mathfrak{m}$, which is a contradiction.

Second, assume that we have a set I of cardinals such that $|I| < \mathfrak{m}$, $\mathfrak{n} < \mathfrak{m}$ for each $\mathfrak{n} \in I$ and $\mathfrak{m} \le \sum_{\mathfrak{n}\in I} \mathfrak{n}$. Basic cardinal arithmetics yields that $\mathfrak{m} = \sum_{\mathfrak{n}\in I} \mathfrak{n}$. Let $J_{\mathfrak{n}}$ denote an (index) set of size $\mathfrak{n}$, for $\mathfrak{n} \in I$. In view of the minimality of $\mathfrak{m}$, there exist e-rigid sets of root algebras $F = \{A_{\mathfrak{n}} : \mathfrak{n} \in I\}$ and $G_{\mathfrak{n}} = \{B_j^{(\mathfrak{n})} : j \in J_{\mathfrak{n}}\}$ for each $\mathfrak{n} \in I$. We assert that

$$H = \{U(A_{\mathfrak{n}}, B_j^{(\mathfrak{n})}) : \mathfrak{n} \in I,\, j \in J_{\mathfrak{n}}\}$$

is an e-rigid set. To show this, assume that $U(A_{\mathfrak{n}}, B_j^{(\mathfrak{n})})$ is embeddable by φ into $U(A_{\overline{\mathfrak{n}}}, B_{\overline{j}}^{(\overline{\mathfrak{n}})})$. It follows from Lemma 2.1 that $\varphi\rceil_{A_{\mathfrak{n}}}$ is an $A_{\mathfrak{n}} \to A_{\overline{\mathfrak{n}}}$ isomorphism. Since F is e-rigid, $\mathfrak{n} = \overline{\mathfrak{n}}$. Applying Lemma 2.1 again and using the e-rigidity of $G_{\mathfrak{n}}$, we obtain that φ is the identity map. Hence, H is an e-rigid set. Finally,

$$|H| = \sum_{\mathfrak{n}\in I} |J_{\mathfrak{n}}| = \sum_{\mathfrak{n}\in I} \mathfrak{n} = \mathfrak{m},$$

which contradicts the assumption that $\mathfrak{m}$ is a failure. $\square$

Concluding remarks. It remains an open problem whether all cardinals, including the inaccessible ones, are the cardinalities of e-rigid sets of monounary algebras. Although the e-rigid sets we have constructed consist of very special (namely, connected root) monounary algebras, the methods presented in the paper do not lead to inaccessible cardinals even if arbitrary monounary algebras are considered.

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