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# An irregular Horn sentence in submodule lattices

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*Dedicated to the memory of András P. Huhn*

For a ring $R$, always with 1, a lattice is said to be representable by $R$-modules if it is embeddable in the lattice of submodules of some unital left $R$-module. Let $\mathbf{L}(R)$ denote the class of lattices representable by $R$-modules. Then $\mathbf{L}(R)$ is known to be a quasivariety, i.e., to be axiomatizable by (universal) Horn sentences (cf. e.g., [5]). Let $\mathbf{HL}(R)$ denote the lattice variety generated by $\mathbf{L}(R)$. A Horn sentence $\chi$ is called *irregular* (cf. [1]) if there are rings $R_1$ and $R_2$ such that $\mathbf{HL}(R_1)=\mathbf{HL}(R_2)$ and $\chi$ holds in $\mathbf{L}(R_1)$ but $\chi$ does not hold in $\mathbf{L}(R_2)$. Although the existence of irregular Horn sentences follows from [4, p. 92], no concrete irregular Horn sentence was known previously. The aim of the present note is to give an irregular Horn sentence $\hat{\chi}$. This $\hat{\chi}$ was found by applying the techniques of [1] and generalizing the methods of HERRMANN and HUHN [3] and [8]. Note that regular Horn sentences are much easier to handle, cf. [1].

Consider the following lattice terms on the set $U=\{x, y, z, t\}$ of variables:

$$p = (x+y)(z+t), \quad h_0 = (x+z)(y+t),$$
$$h_1 = (x+t)(y+z), \quad h_2 = (x+t)(p+h_0),$$
$$h_3 = (y+t)(h_1+p), \quad p_0 = (h_2+z)y,$$
$$q_0 = x+z+h_3, \qquad q = p_0+x,$$

and let $\hat{\chi}$ be the Horn sentence

$$p_0 \leqq q_0 \Rightarrow p \leqq q.$$

Theorem. *$\hat{\chi}$ is irregular.*

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Proof. Let $\mathbf{Z}_4$ stand for the factor ring of the ring of integers modulo 4. Let $I_1$ and $I_2$ denote the ideals of $\mathbf{Z}_4[x]$ generated by $\{x^2-2, 2x\}$ and $\{x^2, 2x\}$, respectively. The rings $R_1=\mathbf{Z}_4[x]/I_1$ and $R_2=\mathbf{Z}_4[x]/I_2$ consist of eight elements. With the notations $a=x+I_1$ and $b=x+I_2$, we have

$$R_1 = \{i+ja\colon\ 0 \leqq i \leqq 3,\ 0 \leqq j \leqq 1\} \quad \text{and} \quad R_2 = \{i+jb\colon\ 0 \leqq i \leqq 3,\ 0 \leqq j \leqq 1\}.$$

Moreover, the bijection $\varphi\colon R_1 \to R_2$, $i+ja \mapsto i+jb$ preserves the unit element and the additive structure. Therefore, $\mathbf{HL}(R_1)=\mathbf{HL}(R_2)$ (cf. [8, Prop. 3]). So, it suffices to show that $\chi$ holds in $\mathbf{L}(R_1)$ but does not hold in $\mathbf{L}(R_2)$.

As Theorem 3.5 of [1] will be our main tool, we adopt the notations preceeding the theorem in [1, § 3]. First, by [1, Thm. 3.5 (A)], we prove that $\chi$ holds in $\mathbf{L}(R_1)$. Now $p_j=p_0$ and $q_j=q_0$ for $j\geqq 1$, and $F^0=\{f_1, f_2, f_3\}$ according to Figure 1. We have $X^0=[f_2]$, $Y^0=[f_1-f_2]$, $Z^0=[f_3]$ and $T^0=[f_1-f_3]$. Denoting $k(C^m\colon c\in U)$ by $K^m$ for $m\geqq 0$ and $k\in\{p, q, p_0, q_0, h_0, h_1, h_2, h_3\}$, an elementary calculation in $\mathrm{Su}\,(M^0)$ shows that $P^0=[f_1]$, $H_0^0=[f_2-f_3]$, $H_2^0=[f_1+f_2-f_3]$ and $P_1^0=P_0^0=\{r(f_1-f_2)\colon r\in R_1$ and $2r=0\}$. Since $2a=0$, we may choose $S_1=\{a(f_1-f_2)\}$. Let $F^1=\{f_1, f_2, f_3, e_1, e_2, \dots, e_8\}$ according to Figure 2.

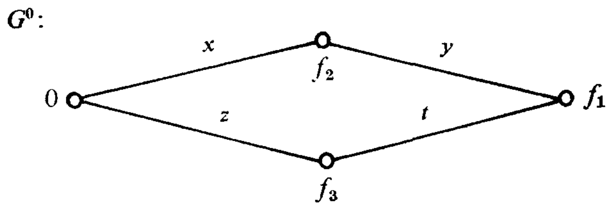


Figure 1

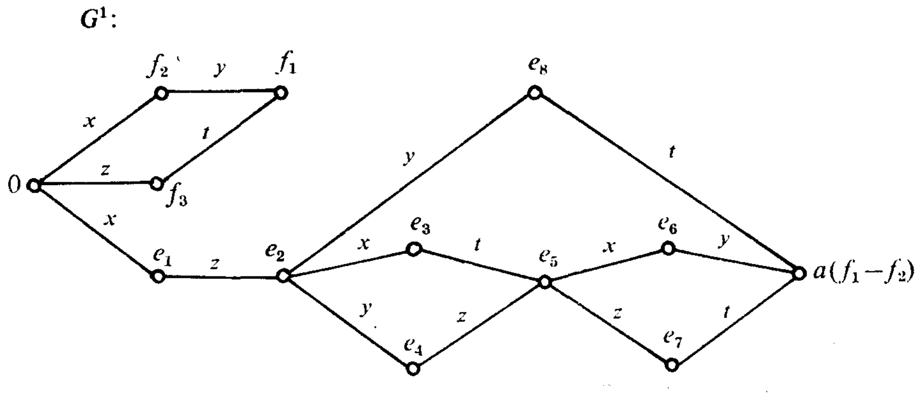


Figure 2

We obtain the following formulas, each of them an easy consequence of the previous ones or Figure 2.

$$X^1 = [f_2, e_1, e_2-e_3, e_5-e_6],$$

$$Y^1 = [f_1-f_2, e_2-e_4, e_2-e_8, e_6],$$

$$Z^1 = [f_3, e_1-e_2, e_4-e_5, e_5-e_7],$$

$$T^1 = [f_1-f_3, e_3-e_5, e_7-e_8, a(f_1-f_2)-e_7],$$

$$P^1 \supseteqq [f_1, e_3-e_4, af_2+e_5],$$

$$H_0^1 \supseteqq [f_2-f_3, e_3-e_5+e_6, e_4-e_6],$$

$$H_2^1 \supseteqq [f_1+f_2-f_3, e_3-e_6, a(f_1-f_3)+e_3-2e_5+e_6].$$

Since $a^2=2$ and $2a=0$,

$$f_1-f_2 = -(f_1+f_2-f_3)+a(e_3-e_6)+a\big(a(f_1-f_3)+e_3-2e_5+e_6\big)+f_3 \in H_2^1+Z^1.$$

Therefore, we have $f_1=(f_1-f_2)+f_2 \in P_0^1+X^1=Q^1=q(C^1\colon c\in U)$. Hence $\chi$ holds in $\mathbf{L}(R_1)$ by [1, Thm. 3.5 (A)].

Now observe that $I_2$ is included in the ideal $I$ of $\mathbf{Z}_4[x]$ generated by $x$, whence $\mathbf{Z}_4 \approx \mathbf{Z}_4[x]/I$ is a homomorphic image of $R_2$. Therefore, if $\chi$ held in $\mathbf{L}(R_2)$, it would also hold in $\mathbf{L}(\mathbf{Z}_4)$ by [4, Prop. 2] (or by [1, Cor. 6.1]). Hence it suffices to show that $\chi$ does not hold in $\mathbf{L}(\mathbf{Z}_4)$. As suggested by [1, Thm. 3.5 (B)], we let $x=\mathbf{Z}_4 f_2$, $y=\mathbf{Z}_4(f_1-f_2)$, $z=\mathbf{Z}_4 f_3$ and $t=\mathbf{Z}_4(f_1-f_3)$ in a free $\mathbf{Z}_4$-module with three generators $f_1$, $f_2$ and $f_3$. Calculation shows that $p_0=\mathbf{Z}_4(2f_1+2f_2)$, $q_0=\mathbf{Z}_4 2f_1+\mathbf{Z}_4 f_2+\mathbf{Z}_4 f_3$, $p=\mathbf{Z}_4 f_1$ and $q=\mathbf{Z}_4 2f_1+\mathbf{Z}_4 f_2$. Therefore, $\chi$ fails in $\mathbf{L}(\mathbf{Z}_4)$, proving the theorem.

In [4, p. 92], it was shown that no $R_1$-module is a free $\mathbf{Z}_4$-module (a direct sum of cyclic groups of order 4). This is the key property allowing construction of an irregular Horn sentence, as observed below.

Let $S$ denote $\mathbf{Z}/p^k\mathbf{Z}$, the ring of integers modulo $p^k$ for $p$ prime and $k \geqq 2$. We show that $\mathbf{L}(R)=\mathbf{L}(S)$ if and only if $R$ has characteristic $p^k$ and some (nontrivial) $R$-module $M$ is free as an $S$-module (that is, $M$ is a direct sum of cyclic groups of order $p^k$).

Supposing $\mathbf{L}(R)=\mathbf{L}(S)$, $R$ has characteristic $p^k$ (cf. [1, Thm. 2.1]). By [6, Thm. 1, p. 108], there is an exact embedding functor $F$ from $S$-Mod into $R$-Mod. For $n\cdot f=f+\ldots+f$ ($n$ times), we see that $\langle p\cdot 1_A, p^{k-1}\cdot 1_A\rangle$ is exact in $R$-Mod for $A=F({}_S S)\neq 0$. Since $A$ is a direct sum of cyclic groups, each with order dividing $p^k$ (PRÜFER, see [2, Thm. 17.2, p. 88], it follows that $A$ is free as an $S$-module.

For the converse, note that an $R$-module $M$ which is free as an $S$-module can be regarded as a bimodule ${}_R M_S$, which induces an exact embedding $S$-Mod $\to$ $R$-Mod by the tensor product functor ${}_R M_S \otimes_S -$, yielding $\mathbf{L}(S) \subseteqq \mathbf{L}(R)$ by [6, Thm. 1,

p. 108]. Since $R$ has characteristic $p^k$, there is a ring homomorphism $S \to R$. Then $\mathbf{L}(R)=\mathbf{L}(S)$ (cf. [1, Cor. 6.1]).

This result can be regarded as a corollary of the ring theory result proved in [7]: If $R$ and $S$ are nontrivial rings with $S$ left artinian, then there exists an exact embedding functor $S$-Mod$\to R$-Mod if and only if there exists a nontrivial bimodule ${}_R A_S$ such that $A_S$ is a free right $S$-module.

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