# ON COMPATIBLE ORDERING OF LATTICES

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This paper is motivated by the following theorem of Ivo G. Rosenberg [3]: For a finite lattice $L = (L;\wedge,\vee)$ and a set $\{f_\gamma : \gamma \in \Gamma\}$ of finitary operations over $L$ the algebra $(L;\{\wedge,\vee\}\cup\{f_\gamma : \gamma \in \Gamma\})$ is functionally complete iff it is simple, for any compatible bounded partial ordering $\rho$ of $(L;\wedge,\vee)$ and for any non-trivial compatible binary central relation $\theta$ of $(L;\wedge,\vee)$ there exist $\gamma$, $\delta \in \Gamma$ such that $f_\gamma$ does not preserve $\rho$ and $f_\delta$ does not preserve $\theta$. In view of this theorem it is interesting to ask how compatible bounded orderings and compatible central relations of a finite lattice can be characterized. Here we deal with orderings; binary central relations will be considered in a separate paper. Two different methods will be developed to handle the question. The first method in Sections 1 and 2 generalizes to a description of all compatible bounded orderings, or equivalently, all compatible lattice orderings of a (not necessary finite) lattice, while the second method in Section 3 is suitable to describe arbitrary (not only bounded) compatible orderings of a finite lattice. Interesting related questions occur in the theory of graph isomorphisms of lattices developed mainly by J. Jakubík and M. Kolibiar. (For a survey see Kolibiar [2] and its bibliography.) Our results generalize some of their theorems.

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## 1. BOUNDED ORDERINGS OF LATTICES

An element $d$ of a lattice $L$ is called neutral if for any $x$, $y \in L$ $\{d,x,y\}$ generates a distributive sublattice. Let $N(L)$ denote the set of complemented neutral elements of $L$. The set of compatible bounded partial orderings of $L$ will be denoted by $R(L)$.

THEOREM 1. *Let $L = (L;\wedge,\vee)$ be an arbitrary lattice. Then $R(L)$ consists of lattice orderings, the map*

$$\kappa\colon R(L) \to N(L),\ \kappa(\prec) = \text{the least element by } \prec,$$

*is bijective, and its inverse map is*

$$\rho\colon N(L) \to R(L),$$
$$\rho(d) = \{(x,y)\in L^2\colon x\wedge d \geq y\wedge d \text{ and } x\vee d \leq y\vee d\}.$$

*In particular, $R(L) = \emptyset$ iff $L$ is not bounded. For $d \in N(L)$ the supremum and infimum of $\{x,y\}$ by $\rho(d)$ are $(d'\vee x)\wedge(d'\vee y)\wedge(x\vee y)$ and $(d\vee x)\wedge(d\vee y)\wedge(x\vee y)$, respectively, where $d'$ is the (unique) complement of $d$ in $(L;\wedge,\vee)$. For $\prec\ \in R(L)$ with least element $\tilde{0}$ and greatest element $\tilde{1}$ we have $\tilde{0}\wedge\tilde{1} = 0$ and $\tilde{0}\vee\tilde{1} = 1$.*

REMARK 1. It is known (cf. Grätzer [1, Theorem III. 4.1]) that for any $d \in N(L)$, $L \cong (d]\times[d)$, and the map $L \to (d]\times[d)$, $x \to (x\wedge d, x\vee d)$ is an isomorphism. Therefore we obtain the following

COROLLARY 1. *For any* $\prec\ \in R(L)$ *there exists a direct decomposition* $L \cong L_1 \times L_2$ *of* $L$ *such that the lattice* $(L,\prec)$ *is isomorphic to* $L_1^d \times L_2$, *where* $L_1^d$ *denotes the dual of* $L_1$.

PROOF OF THEOREM 1. Let $L = (L;\wedge,\vee,\leq)$ be a lattice. First we show the following

CLAIM 1. Let $\mu(x,y,z)$ denote one of the ternary functions $(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$ and $(x\vee y)\wedge(x\vee z)\wedge(y\vee z)$. Let $\prec$ belong to $R(L)$ with least element $\tilde{0}$ and greatest element $\tilde{1}$. Then $(L,\prec)$ is a lattice, in which the infimum and supremum of $\{x,y\}$ $(\subseteq L)$ are $\mu(\tilde{0},x,y)$ and $\mu(\tilde{1},x,y)$, respectively.

PROOF OF CLAIM 1. By the Duality Principle it is enough to show that $\mu(\tilde{0},x,y)$ is the infimum of $\{x,y\}$ in $(L,\prec)$. Since the operations $\wedge$ and $\vee$ are monoton by $\prec$, so is $\mu$. Hence $\mu(\tilde{0},x,y) \prec \mu(x,x,y) = x$ and $\mu(\tilde{0},x,y) \prec$ $\prec \mu(y,x,y) = y$, showing that $\mu(\tilde{0},x,y)$ is a lower bound in $(L,\prec)$. Suppose $z$ is another lower bound. Then $z =$ $= \mu(\tilde{0},z,z) \prec \mu(\tilde{0},x,y)$, which completes the proof.

Now if $d \in L$ and $(d\wedge x)\vee(d\wedge y)\vee(x\wedge y) = (d\vee x)\wedge(d\vee y)\wedge$ $\wedge(x\vee y)$ holds for any $x$, $y \in L$ then $d$ is a neutral element (cf. Grätzer [1, Theorem II. 2.4]). Since the infimum is unique in $(L,\prec)$, Claim 1 yields that $\tilde{0}$, the least element by $\prec$ for $\prec$ in $R(L)$, is neutral in $L = (L;\wedge,\vee,\leq)$. Let $\prec$ belong to $R(L)$ with least element $\tilde{0}$ and gratest element $\tilde{1}$. Then, computing by Claim 1 and using the distributivity law, we have $x = \mu(\tilde{0},x,\tilde{1}) = (\tilde{0}\wedge x)\vee(\tilde{0}\wedge\tilde{1})\vee$ $\vee(x\wedge\tilde{1}) = (x\wedge(\tilde{0}\vee\tilde{1}))\vee(\tilde{0}\wedge\tilde{1})$, for any $x \in L$. This easily implies that $(L;\wedge,\vee,\leq)$ is bounded with $1 = \tilde{0}\vee\tilde{1}$ and $0 =$ $= \tilde{0}\wedge\tilde{1}$, whence $\tilde{0}$ is a complemented element. Therefore $\kappa$ really maps $R(L)$ into $N(L)$. Since any lattice ordering is determined by the infimums, the injectivity of $\kappa$ can be concluded from Claim 1. It remains to show that $\rho$ maps $N(L)$ into $R(L)$ and for any $d \in N(L)$ we have $\kappa(\rho(d)) = d$. The reflexivity and transitivity of $\rho(d)$ is evident. The neutrality of $d$ yields that $\rho(d)$ is antisymmetric. If $x_i\ \rho(d) y_i$, for $i = 1, 2$, then $(x_1 \wedge x_2) \wedge d = (x_1 \wedge d) \wedge (x_2 \wedge d) \geq (y_1 \wedge d) \wedge (y_2 \wedge d) = (y_1 \wedge y_2) \wedge d$, while, by making use of the neutrality of $d$, $(x_1 \wedge x_2) \vee d = (x_1 \vee d) \wedge (x_2 \vee d) \leq (y_1 \vee d) \wedge \wedge (y_2 \vee d) = (y_1 \wedge y_2) \vee d$. Therefore $\rho(d)$ is compatible. It is evident that $d$ and its complement are the least and greatest elements by $\rho(d)$, respectively. Therefore $\rho(d) \in R(L)$, and $\kappa(\rho(d)) = d$. Q.E.D.

## 2. LATTICE ORDERINGS OF LATTICES

THEOREM 2. *Let $L = (L;\wedge,\vee)$ be a lattice and let $\prec$ be a compatible partial ordering of $(L;\wedge,\vee)$ such that $(L,\prec)$ is also a lattice. Then there exist lattices $L_1$, $L_2$ and a lattice isomorphism $\psi\colon L \to L_1 \times L_2$ such that $\prec\ = \{(x,y) \in L^2\colon x\psi_1 \geq y\psi_1 \text{ and } x\psi_2 \leq y\psi_2\}$. (Thus $(L,\prec) \cong L_1^d \times L_2$.) On the other hand, if $\psi\colon L \to L_1 \times L_2$ is a lattice isomorphism between $L$ and $L_1 \times L_2$, then $\{(x,y) \in L^2\colon x\psi_1 \geq y\psi_1 \text{ and } x\psi_2 \leq y\psi_2\}$ is a compatible lattice ordering of $L$.*

PROOF. It is enough to prove the first statement, because the second one is trivial. Let $L = (L;\wedge,\vee,\leq)$ be a lattice, and let $\prec$ be a compatible lattice ordering of it. Let $\cap$ and $\cup$ denote the infimum and supremum by $\prec$, respectively.

STEP 1. Any of the operations $\wedge$, $\vee$, $\cap$, $\cup$ preserves both $\leq$ and $\prec$.

By the Duality Principle it is enough to show that $\cup$ preserves $\leq$. Let the ternary function $(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$ be denoted by $\mu(x,y,z)$. Then $\mu$ is monotone concerning both $\leq$ and $\prec$. We claim that $x\cup y = \mu(x,y,u)$ whenever $x \prec u$ and $y \prec u$. Indeed, $\mu(x,y,u) \prec \mu(x\cup y,x\cup y,u) =$ $= x\cup y = \mu(x,y,x) \cup \mu(x,y,y) \prec \mu(x,y,u) \cup \mu(x,y,u) =$ $= \mu(x,y,u)$. Suppose $a_i \leq b_i$ for $i = 1, 2$, and let $u$ be defined as $a_1\cup a_2\cup b_1\cup b_2$. Then $a_1\cup a_2 = \mu(a_1,a_2,u) \leq$ $\leq \mu(b_1,b_2,u) = b_1\cup b_2$, which completes the proof of Step 1.

STEP 2. For any $H \subseteq L$ $H$ is an interval in $(L;\wedge,\vee,\leq)$ if and only if it is an interval in $(L;\cap,\cup,\prec)$.

Since, by Step 1, the role of $(L;\wedge,\vee,\prec)$ and that of $(L;\cap,\cup,\prec)$ can be interchanged, it suffices to show that any $H = \{x\in L\colon a \leq x \leq b\}$ is an interval in $(L;\cap,\cup,\prec)$. If $x, y \in H$ then $a = a\cap a \leq x\cap y \leq b\cap b = b$, therefore $H$ is a sublattice of $(L;\cap,\cup,\prec)$. If $x \prec z \prec y$ and $x, y \in H$, then $a = x\wedge a \prec z\wedge a \prec y\wedge a = a$ and $b = x\vee b \prec z\vee b \prec y\vee$ $\vee b = b$, i.e. $a = z\wedge a$ and $b = z\vee b$. Thus $z \in H$ and $H$ is a convex sublattice of $(L;\cap,\cup,\prec)$. Since the restriction of $\leq$ is a bounded compatible ordering of $(H;\cap,\cup,\prec)$, $(L;\cap,$ $\cup,\prec)$ is also bounded by Theorem 1. Consequently $H$ is an interval in $(L;\cap,\cup,\prec)$, which completes Step 2.

Now let $\Gamma$ be the set of intervals of $L$, and let $a_\alpha$, $b_\alpha$, $d_\alpha$, and $e_\alpha$ denote the endpoints of $\alpha \in \Gamma$ such that $\alpha = \{x\in L\colon a_\alpha \leq x \leq b_\alpha\} = \{x\in L\colon d_\alpha \prec x \prec e_\alpha\}$. Let us define two binary relations $\Theta$ and $\Phi$ of $L$ as follows: $\Theta =$ $= \{(x,y)\in L^2$: there exists $\alpha \in \Gamma$ such that $x, y \in \alpha$ and $d_\alpha\wedge x = d_\alpha\wedge y\}$ and $\Phi = \{(x,y)\in L^2$: there exists $\alpha \in \Gamma$ such that $x, y \in \alpha$ and $d_\alpha\vee x = d_\alpha\vee y\}$.

STEP 3. For $(x,y) \in L^2$ $(x,y) \in \Theta$ iff for any $\beta$, $\{x,y\} \subseteq \beta \in \Gamma$ implies $d_\beta\wedge x = d_\beta\wedge y$; and $(x,y) \in \Phi$ iff for any $\beta$, $\{x,y\} \subseteq \beta \in \Gamma$ implies $d_\beta \vee x = d_\beta \vee y$.

Suppose $(x,y) \in \Theta$, i.e., $x$, $y \in \alpha$ and $d_\alpha \wedge x = d_\alpha \wedge y$ for some $\alpha \in \Gamma$, and $x$, $y \in \beta \in \Gamma$. Choose an interval $\gamma \in \Gamma$ such that $\alpha \subseteq \gamma$ and $\beta \subseteq \gamma$. Then, by Theorem 1 and Remark 1, the map $\phi = \phi_1 \times \phi_2 \colon \gamma \to \{x \in L\colon a_\gamma \leq x \leq d_\gamma\} \times \{x \in L\colon d_\gamma \leq x \leq b_\gamma\}$, $\phi(x) = (\phi_1(x), \phi_2(x)) = (d_\gamma \wedge x, d_\gamma \vee x)$ is an isomorphism; and for $u$, $v \in \gamma$ $u \leq v$ is equivalent to $d_\gamma \wedge u \leq d_\gamma \wedge v$ and $d_\gamma \vee u \leq d_\gamma \vee v$, while $u \prec v$ is equivalent to $d_\gamma \wedge u \geq d_\gamma \wedge v$ and $d_\gamma \vee u \leq d_\gamma \vee v$. In what follows in this proof let $\varepsilon$ be an arbitrary element of $\{\alpha,\beta\}$. Then $d_\varepsilon \leq b_\varepsilon$ and $d_\varepsilon \prec b_\varepsilon$ imply $d_\gamma \wedge d_\varepsilon = d_\gamma \wedge b_\varepsilon$, while $d_\varepsilon \prec a_\varepsilon$ and $a_\varepsilon \leq d_\varepsilon$ imply $d_\gamma \vee d_\varepsilon = d_\gamma \vee a_\varepsilon$. For any $u \in \varepsilon$ we have $d_\gamma \wedge u = d_\gamma \wedge b_\varepsilon \wedge u = d_\gamma \wedge d_\varepsilon \wedge u$. Therefore the "if" part of the following observation evidently holds: (*) For $u$, $v \in \varepsilon$ $d_\gamma \wedge u = d_\gamma \wedge v$ if and only if $d_\varepsilon \wedge u = d_\varepsilon \wedge v$. On the other hand, if $d_\gamma \wedge u = d_\gamma \wedge v$ then $d_\gamma \wedge (d_\varepsilon \wedge u) = d_\gamma \wedge (d_\varepsilon \wedge v)$ and, from $d_\gamma \vee a_\varepsilon \leq d_\gamma \vee (d_\varepsilon \wedge u) \leq d_\gamma \vee d_\varepsilon = d_\gamma \vee a_\varepsilon \leq d_\gamma \vee (d_\varepsilon \wedge v) \leq \leq d_\gamma \vee d_\varepsilon = d_\gamma \vee a_\varepsilon$, we have $d_\gamma \vee (d_\varepsilon \wedge u) = d_\gamma \vee (d_\varepsilon \wedge v)$. Hence the injectivity of $\phi$ yields (*). Now (*) and $d_\alpha \wedge x = d_\alpha \wedge y$ imply $d_\beta \wedge x = d_\beta \wedge y$, and the Duality Principle completes the proof of Step 3.

By Theorem 1, Remark 1, and Step 3 for any $\alpha \in \Gamma$ $\Theta_\alpha$ and $\Phi_\alpha$ (the restrictions of $\Theta$ and $\Phi$ to $\alpha$) are congruences, $\Theta_\alpha \circ \Phi_\alpha = \alpha \times \alpha$, and $\Theta_\alpha \cap \Phi_\alpha = \omega_\alpha$ (the equality relation on $\alpha$). Since $\Theta = \bigcup_{\alpha \in \Gamma} \Theta_\alpha$, $\Phi = \bigcup_{\alpha \in \Gamma} \Phi_\alpha$, and $\Gamma$ is a directed partially ordered set under the set-theoretic inclusion, from Step 3 it follows that $\Theta$ and $\Phi$ are congruences of $L = (L;\wedge,\vee,\leq)$, $\Theta \cap \Phi = \omega$ and $\Theta \circ \Phi = L \times L$. Therefore $L \cong L/\Theta \times L/\Phi$. For $x$, $y \in L$, by Theorem 1 and Step 3, $[x]\Theta \geq [y]\Theta$ and $[x]\Phi \leq [y]\Phi$ iff $[x]\Theta_\alpha \geq [y]\Theta_\alpha$ and

$[x]\Phi_\alpha \leq [y]\Phi_\alpha$ for some $\alpha \in \Gamma$ iff $x \prec y$. Thus by letting $L_1$ and $L_2$ be equal to $L/\Theta$ and $L/\Phi$, respectively, the proof of Theorem 2 is complete.

The section is concluded with the following corollary of Theorems 1 and 2.

COROLLARY 2. *If $L$ is a bounded lattice then the set of its compatible lattice orderings and that of its compatible bounded orderings coincide.*

## 3. ON ORDERINGS OF FINITE PRODUCTS OF SUBDIRECTLY IRREDUCIBLE LATTICES

THEOREM 3. *Let $L$ be a lattice and let $\prec$ be a compatible ordering on $L$. If $L$ is subdirectly irreducible, then there exists a congruence $\Theta$ on $L$ such that either for every $a$, $b$ in $L$ we have $a \prec b$ iff $a \leq b$ and $a \;\Theta\; b$ or for every $a$, $b$ in $L$ we have $a \prec b$ iff $b \leq a$ and $a \;\Theta\; b$. If $L$ is a subdirect product of a finite number of subdirectly irreducible factors $L_i$, $i = 1, 2, \ldots, n$, then there exist compatible orderings $\prec_i$ of the lattices $L_i$ such that for each $a$, $b$ in $L$ we have $a \prec b$ iff $a_i \prec_i b_i$ for all $i$, where the image of an element $x$ in $L$ under the $i$-th projection is denoted by $x_i$. Conversely, if $\prec$ is as described above, then $\prec$ is compatible.*

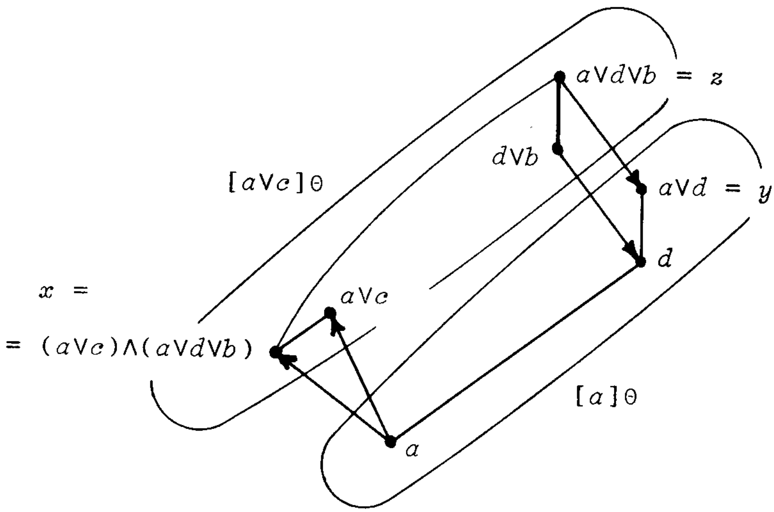
We prove the Theorem via the following four lemmata. By an orientation of a lattice we mean a reflexive, antisymmetric, and compatible relation.

LEMMA 1. *Let $L$ be subdirectly irreducible and let $\rightarrow$ be an orientation of $L$. Then $\rightarrow$ is a part of the lattice order of $L$ or its dual.*

PROOF. We may assume that there exist elements $a$, $b \in L$, $a \neq b$ with $a \to b$. Then, by the compatibility of $\to$, $a = a \vee a \to a \vee b \to b \vee b = b$, that is, either there exist elements $a$, $b$ in $L$ with $a \to b$ and $a < b$ or dually. Consider the case $a \to b$, $a < b$. Let $\phi$ be the minimal non-zero congruence on $L$. We prove that there exist elements $c$, $d$ in $L$, such that $(c,d)$ generates $\phi$, $c < d$, and $c \to d$. In fact, choose an arbitrary pair $(c_1,d_1)$, $c_1 < d_1$ generating $\phi$. Clearly $c_1 \; \Theta_{ab} \; d_1$, thus there is a chain $x_1 < x_2 \dots < x_n$ with $x_1 = c_1$, $x_n = c_n$, such that each subinterval $[x_i,x_{i+1}]$ is weakly projective to $[a,b]$. Let $[c,d]$ be any of the subintervals $[x_i,x_{i+1}]$. We only have to show that $c \to d$. By the weak projectivity of $[c,d]$ to $[a,b]$ there exist intervals $[y_1,z_1]$, $[y_2,z_2]$, ..., ..., $[y_m,z_m]$ such that $[y_1,z_1] = [c,d]$, $[y_m,z_m] = [a,b]$, and, for all $i$, $[y_i,z_i]$ is transposed to a subinterval $[y'_{i+1},z'_{i+1}]$ of $[y_{i+1},z_{i+1}]$. Now $a \to b$, that is, $y_m \to z_m$. Hence $y'_m = (y'_m \vee y_m) \wedge z'_m \to (y'_m \vee z_m) \wedge z'_m = z'_m$. $[y_{m-1},z_{m-1}]$ is transposed to $[y_m,z_m]$, thus $y_{m-1} \to z_{m-1}$, whence, by induction, we have $c \to d$. Now let $e$, $f$ be arbitrary elements of $L$ with $e \to f$. We prove that $e \leq f$. Assume not. Then $f < e \vee f$, and, according to an earlier observation $e \vee f \to f$. Thus a modification of the above considerations yields that there exists a subinterval $[c_2,d_2]$ of $[c,d]$ such that $c_2 < d_2$ and $d_2 \to c_2$. On the other hand $c \to d$, whence $c_2 = (c_2 \vee c) \wedge d_2 \to (c_2 \vee d) \wedge d_2 = d_2$, contradicting the antisymmetry of $\to$.

LEMMA 2. *Let $L$ be a lattice, let $\Theta$ be a congruence on $L$, and let $\to$ be an orientation of $L$. For elements $a$, $b \in L/\Theta$ define $a \to_\Theta b$ if there exist elements $c$, $d$ in $L$ such that $[c]\Theta = a$, $[d]\Theta = b$ and $c \to d$. Then $\to_\Theta$ is an orientation of $L/\Theta$.*

PROOF. We only have to prove antisymmetry. Let $a$, $b$, $c$, $d \in L$, let $a \; \theta \; b$, $c \; \theta \; d$, and let $a \to c$, $d \to b$. We have to prove $a \; \theta \; c$, but by symmetry, it suffices to prove $a \; \theta \; a \vee c$. From what we have said so far, it follows that $a \to a \vee c$, $d \vee b \to b$, and $a \vee c \; \theta \; d \vee b$ (see Figure).




Then, clearly, $a \vee d \vee b \to a \vee d$, furthermore, forming the meet of $a \vee c$ and of a by $a \vee d \vee b$ we obtain that $a \to (a \vee c) \wedge (a \vee d \vee b)$. Now consider the elements denoted by $x$, $y$, $z$ and $a$ on the Figure. We have $a \leq x$, $a \leq y$, $x \leq z$, $y \leq z$, $a \; \theta \; y$, $x \; \theta \; z$, $a \to x$, and $z \to y$. Notice that $x \leq y$. Indeed, if $x \nleq y$, then the endpoints of the intervals $[x \wedge y, x]$ and $[y, x \vee y]$ must be oppositely oriented by $\to$, as they are subintervals of the oppositely oriented intervals $[a,x]$ and $[y,z]$, respectively, but this is impossible, for $[x \wedge y, x]$ and $[y, x \vee y]$ are transposed. Thus we have $a \leq x \leq y \leq z$. Now $a \ \Theta \ y$ yields $a \ \Theta \ x$, and $a \ \Theta \ x \ \Theta \ z$ yields $a \ \Theta \ z$. From Figure, $z \ \Theta \ a \vee c$, whence $a \ \Theta \ a \vee c$ as claimed.

LEMMA 3. *Let $\rightarrow$ be an orientation of the subdirectly irreducible lattice $L$. Then the transitive closure $\prec$ of $\rightarrow$ is a compatible ordering.*

PROOF. We have to prove that, whenever $a_1, a_2, \ldots \ldots, a_n \in L$ are such that $a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_n \rightarrow a_1$, then $a_1 = a_2 = \ldots = a_n$. Assume not, say, $a_1 \neq a_2$. By Lemma 1, we may assume that $a_1 < a_2$. Applying Lemma 1 again, we have $a_1 < a_2 \leq a_3 \ldots \leq a_n \leq a_1$, a contradiction.

LEMMA 4. *Let $L$ be a subdirect product of the lattices $L_1$ and $L_2$, and let $\rightarrow$ be an orientation of $L$. Let $\rightarrow_i$, $i = 1, 2$ denote the relation $\rightarrow_{\Theta_i}$ of Lemma 2, with $\Theta_i$ the congruence associated with the projection to $L_i$. Finally let $\prec$, $\prec_i$ be the transitive closures of $\rightarrow$, $\rightarrow_i$, respectively. Then, for any $a$, $b \in L$, $a \prec b$ in $L$ if and only if $a_1 \prec_1 b_1$ and $a_2 \prec_2 b_2$.*

Here and from now on, for an element $x$ of a subdirect product, $x_i$ denotes the images of $x$ under the $i$-th projection.

PROOF. The non-trivial part of the assertion is that $a_1 \prec_1 b_1$ and $a_2 \prec_2 b_2$ implies $a \prec b$. We may assume that $a \leq b$. (Indeed, $a_i \prec_i b_i$ ensures that there exist elements $x_{i1}, x_{i2}, \ldots, x_{in_i}$ in $L_i$ with $a_i = x_{i1} \rightarrow_i x_{i2} \rightarrow_i \ldots \rightarrow_i x_{in_i} = b_i$, whence we have $a_i = a_i \vee a_i = x_{i1} \vee a_i \rightarrow_i x_{i2} \vee a_i \rightarrow_i \ldots \rightarrow_i x_{in_i} \vee a_i = b_i \vee a_i$, that is

$a_i \prec_i a_i \vee b_i$, and analogously $a_i \vee b_i \prec b_i$. We only have to show that $a_i \prec a_i \vee b_i$, $i = 1, 2$ implies $a \prec a \vee b$, for $a \vee b \prec b$ follows analogously, which is the same as to assume $a \leq b$.) According to the hypotheses there exist elements $x^{i1}, x^{i2}, \ldots, x^{im_i}$ and $y^{i1}, y^{i2}, \ldots$ $\ldots, y^{im_i}$ in $L$, $i = 1, 2$, with $a_i = x_i^{i1}$, $x^{i1} \to y^{i1}$, $y_i^{i1} = x_i^{i2}$, $x^{i2} \to y^{i2}, \ldots, x^{im_i} \to y^{im_i}$, $y^{im_i} = b_i$. We may assume that all the elements $x^{ij}$ and $y^{ij}$ are in between $a$ and $b$ in the lattice ordering $\leq$, or else we could replace them by $(x^{ij} \wedge b) \vee a$ and $(y^{ij} \wedge b) \vee a$, respectively. By the same token, we may assume that $x^{i1} \leq$ $\leq y^{i1} \leq x^{i2} \leq y^{i2} \leq \ldots \leq x^{im_i}$, or else they could be replaced by $x^{i1}$, $x^{i1} \vee y^{i1}$, $x^{i1} \vee y^{i1} \vee x^{i2}$, $x^{i1} \vee y^{i1} \vee x^{i2} \vee$ $\vee y^{i2}, \ldots, x^{i1} \vee y^{i1} \vee \ldots \vee x^{im_i} \vee y^{im_i}$, respectively. We are going to prove that, for each $k \in \{0, 1, \ldots, m_1\}$, $y^{ik} \prec x^{1(k+1)}$, where $y^{10} = a$ and $x^{1(m_1+1)} = b$. Hence $a \prec x^{11} \to y^{11} \prec x^{12} \to y^{12} \prec \ldots \prec x^{1m_1} \to y^{1m_1} \prec b$, and, by transitivity, $a \prec b$, proving the lemma. Now, to prove $y^{1k} \prec x^{1(k+1)}$, let $y^{2j}(k) = (y^{2j} \vee y^{1k}) \wedge x^{1(k+1)}$, $j = 0, 1, \ldots, m_2$ and let $x^{2j}(k) = (x^{2j} \vee y^{1k}) \wedge x^{1(k+1)}$, $j = 1, 2, \ldots, m_2+1$, where $a = y^{20}$ and $b = x^{2(m_2+1)}$. By the compatibility of the relation $\to$, we have $x^{2j}(k) \to y^{2j}(k)$, $j = 1, 2, \ldots, m_2$. Thus the proof is completed once the following relations are proved

(1) $\qquad y^{20}(k) = y^{1k}$,

(2) $\qquad x^{2(m_2+1)}(k) = x^{1(k+1)}$,

(3) $\qquad x^{2j}(k) = x^{2(j+1)}(k)$, $j = 0, 1, \ldots, m_2$.

All these relations can be proved via componentwise calculations, let us check, for instance, (3): $y_1^{2j}(k) = (y_1^{2j} \vee y_1^{1k}) \wedge x_1^{1(k+1)} = (y_1^{2j} \vee y_1^{1k}) \wedge y_1^{1k} = y_1^{1k}$, as $x_1^{1(k+1)} = y_1^{1k}$, and similarly $x_1^{2(j+1)}(k) = y_1^{1k}$. Furthermore, $y_2^{2j}(k) = (y_2^{2j} \vee y_2^{1k}) \wedge x_2^{1(k+1)} = (x_2^{2(j+1)} \vee y_2^{1k}) \wedge x_2^{1(k+1)} = x_2^{2(j+1)}(k)$, which was to be proved.

Now we are ready to prove Theorem 3.

We first prove the assertion concerning subdirectly irreducible lattices. Let $L$ be subdirectly irreducible, let $\prec$ be a compatible ordering on $L$, and let $\Theta$ be the equivalence closure of $\prec$. Clearly $a \,\Theta\, b$ iff there is a sequence $x_1, x_2, \ldots, x_{2n}$ of elements in $L$ such that $a = x_1 \prec x_2 \succ x_3 \prec x_4 \succ \ldots \prec x_{2n} = b$. Hence the compatibility of $\Theta$ is immediate. It is also clear that elements in different $\Theta$-classes are incomparable. Now applying Lemma 1, let, say, $\prec$ be a part of the lattice order of $L$. We show that, whenever $a \leq b$ and $a \,\Theta\, b$, then $a \prec b$. In fact $a \,\Theta\, b$ means that there exist elements $x_1, x_2, \ldots, x_{2n}$ in $L$ with $a = x_1 \prec x_2 \succ x_3 \prec \ldots \ldots \prec x_{2n} = b$. We have $x_2 = x_2 \vee x_3 \prec x_2 \vee x_4$, $x_2 \vee x_4 = x_2 \vee x_4 \vee x_5 \prec x_2 \vee x_4 \vee x_6$, etc., that is, $a = x_1 \prec x_2 \prec x_2 \vee x_4 \prec x_2 \vee x_4 \vee x_6 \prec \ldots \prec x_2 \vee x_4 \vee \ldots \vee x_{2n}$. Summarizing $a \prec x_2 \vee x_4 \vee \ldots \vee x_{2n}$. Applying $b = x_{2n}$, we have $a \leq b \leq x_2 \vee x_4 \vee \ldots \vee x_{2n}$. Hence $a = a \wedge b \prec (x_2 \vee x_4 \vee \ldots \ldots \vee x_{2n}) \wedge b = b$, as claimed.

Now consider the case that $L$ is a subdirect product of finitely many subdirectly irreducible lattices. Let $\prec$ be a compatible ordering on $L$. Then $\prec$ is an orientation whose transitive closure is itself. Thus Lemma 4, which applies for finitely many components, too, provides a representation of $\prec$. To obtain the assertion of Theorem 3 we only have to show that the relations $\prec_i$ of Lemma 4 are compatible order relations. But they are orientations by Lemma 2, whence they are compatible orderings by Lemma 3.

Finally, if the ordering $\prec$ is as described in Theorem 3, then $\prec$ is obviously compatible completing the proof.

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