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Weakly independent subsets in lattices

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It is well-known that in a finite distributive lattice the set of join-irreducible elements and any maximal chain have the same cardinality. In this note we give a generalization of this theorem, using the following notions.

DEFINITION. A subset H of a lattice L is called weakly independent iff for all $h, h_1, h_2, \dots, h_n \in H$ which satisfy $h \leq h_1 \vee h_2 \vee \cdots \vee h_n$ there exists an i $(1 \leq i \leq n)$ such that $h \leq h_i$. A maximal weakly independent set is called a basis of L.

Every subchain of L is a weakly independent subset and any maximal chain is a basis. In this paper *L denotes a finite distributive* lattice and let $J_0(L)$ denote the set of all join-irreducible elements of L. By using Lemma 1 (cf. later) it is easy to show that $J_0(L)$ is a basis of L.

THEOREM 1. *Any two bases of a finite distributive lattice have the same number of elements.*

To characterize the bases we need the following concept: a sublattice K of L is called a sublattice of maximal length iff K contains a maximal chain of L (i.e., every maximal chain of K is a maximal chain of L).

THEOREM 2. *A subset H of a finite distributive lattice L is a basis if and only if the sublattice K generated by H is a sublattice of maximal length and $H = J_0(K)$.*

The proof of Theorem 1 starts with the following well-known

LEMMA 1. *Whenever $x \in J_0(L)$ and $x \leq y_1 \vee \cdots \vee y_n$ then $x \leq y_i$ for some i* $(1 \leq i \leq n)$.

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From now on let U be a basis of L.

LEMMA 2. *For every $p \in J_0(L)$ there exists a $u \in U$ such that $p \leq u$.*

Proof. By Lemma 1 it is sufficient to show that $\bigvee U = 1$. If we had $\bigvee U \neq 1$ then $U \cup \{1\}$ would be a weakly independent subset again, contradicting the maximality of U. □

Now let p be an element of $J_0(L)$. We define the element $\bar{p} = \bigwedge (u : u \in U, p \leq u)$. By Lemma 2 $\bar{p}$ exists and $p \leq \bar{p}$.

LEMMA 3. *For every $p \in J_0(L)$ $\bar{p}$ belongs to U.*

Proof. All we have to prove is that $U \cup \{\bar{p}\}$ is weakly independent. Firstly, we consider the case $\bar{p} \leq u_1 \vee \cdots \vee u_n$ where $u_1, \ldots, u_n \in U$. From Lemma 1 and $p \leq \bar{p} \leq u_1 \vee \cdots \vee u_n$ we obtain $p \leq u_i$ for some i. Hence $\bar{p} \leq u_i$. The second case is $u \leq \bar{p} \vee u_1 \vee \cdots \vee u_n$ where $u, u_1, \ldots, u_n \in U$. We can assume that $u \not\leq u_1 \vee \cdots \vee u_n$. For any $v \in U$ satisfying $p \leq v$ we have $\bar{p} \leq v$, whence $u \leq v \vee u_1 \vee \cdots \vee u_n$. Since U is weakly independent we conclude $u \leq v$. Therefore $u \leq \bigwedge (v : v \in U, p \leq v) = \bar{p}$. □

LEMMA 4. *The map $\varphi : J_0(L) \to U$, $p \mapsto \bar{p}$ is onto.*

Proof. Let u be an arbitrary element of U. Then $u = p_1 \vee \cdots \vee p_n$ for suitable $p_1, \ldots, p_n \in J_0(L)$. From $p_i \leq \bar{p}_i$ $(i = 1, \ldots, n)$ we have $u \leq \bar{p}_1 \vee \cdots \vee \bar{p}_n$. Since U is weakly independent, $u \leq \bar{p}_i$ holds for some i. Finally, $p_i \leq u$ yields $\bar{p}_i \leq u$, i.e. $u = \bar{p}_i$. □

The following lemma is not only to complete the proof of Theorem 1, it is interesting in itself.

LEMMA 5. *The map $\varphi : J_0(L) \to U$, $p \mapsto \bar{p}$ is bijective.*

Proof. Let us assume that although $q, r \in J_0(L)$ and $q \not\leq r$, we have $\bar{q} = \bar{r}$. Put $x = \bigvee H$ where $H = \{y : y \in J_0(L),\ \bar{y} \leq \bar{q}, \text{ and } q \not\leq y\}$. Then $x \leq \bigvee (\bar{y} : y \in H) \leq \bar{q}$. By Lemma 1 $q \not\leq x$. We claim that $x \notin U$. Indeed, $r \in H$ implies $r \leq x$, whence if $x \in U$ then $q \leq \bar{q} = \bar{r} \leq x$ contradicts $q \not\leq x$. Now we prove that $U \cup \{x\}$ is weakly independent. We have to discuss two cases. Firstly, $x \leq u_1 \vee \cdots \vee u_n$ where $u_1, \ldots, u_n \in U$. From Lemma 1 and $r \leq x$ we have $r \leq u_i$ for some i $(1 \leq i \leq n)$, whence $\bar{r} \leq u_i$. We have already seen that $x \leq \bar{q} = \bar{r}$, so $x \leq u_i$. Secondly, $u \leq x \vee u_1 \vee \cdots \vee u_n$

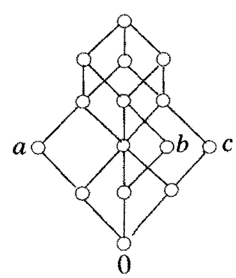
$(u, u_i \in U)$. We can assume that $u \not\leq u_1 \vee \cdots \vee u_n$. Then $x \leq \bar{q}$ implies $u \leq \bar{q} \vee u_1 \vee \cdots \vee u_n$. Since U is weakly independent, $u \leq \bar{q}$. We have $u = p_1 \vee \cdots \vee p_k$ for suitable $p_1, \ldots, p_k \in J_0(L)$. If $q \not\leq u$ then, for all i, $q \not\leq p_i$ and $\bar{p}_i \leq u \leq \bar{q}$, whence $p_i \in H$ and $u = p_1 \vee \cdots \vee p_k \leq \bigvee H = x$. Finally, if $q \leq u$, i.e. $\bar{q} \leq u$, then Lemma 1 yields that either $q \leq x$ or $\bar{q} \leq u_i$ for some i. But we have shown that $q \not\leq x$. Therefore $u \leq u_i$ follows from $\bar{q} \leq u_i$ and $u \leq \bar{q}$. This proves Theorem 1.

Proof of Theorem 2. Let H be a basis of L and let $H^\wedge = \{h_1 \wedge \cdots \wedge h_n : 1 \leq n < \omega$ and $h_1, \ldots, h_n \in H\}$. Further, let K be the sublattice generated by H. Then every element of K has a representation as a join of elements of $H^\wedge$. Now we prove that $H \subseteq J_0(K)$. Let $z = x \vee y$ where $z \in H$, $x, y \in K$. Then $x = a_1 \vee \cdots \vee a_k$, $y = a_{k+1} \vee \cdots \vee a_m$ with $a_1, \ldots, a_m \in H^\wedge$. Therefore $a_i = \bigwedge_j h_{ij}$ for suitable $h_{ij} \in H$. By distributivity, z is the meet of all possible $h_{1j_1} \vee \cdots \vee h_{mj_m}$. If there existed an l_n for each n $(1 \leq n \leq m)$ such that $z \not\leq h_{nl_n}$ then the weak independence of H would yield $z \not\leq h_{1l_1} \vee \cdots \vee h_{ml_m}$, a contradiction. Therefore there is a fixed n such that $z \leq h_{nj}$ holds for all j. Consequently $z \leq a_n$, which yields $x = z$ (if $1 \leq n \leq k$) or $y = z$ (if $k < n \leq m$). This proves $z \in J_0(K)$, i.e. $H \subseteq J_0(K)$. Since $J_0(K)$ is weakly independent in K and therefore in L, we conclude $H = J_0(K)$.

Since all maximal chains in K have the same cardinality as $J_0(K)$, it follows from Theorem 1 and the above that K is a sublattice of maximal length in L.

Conversely, suppose K is a sublattice of maximal length and $H = J_0(K)$. Since H is a basis in K, it is weakly independent in L. For any maximal chain C in K, $|C| = |J_0(K)| = |H|$ and C is a basis in L. Now Theorem 1 yields that H is a maximal weakly independent set in L. The proof of Theorem 2 is complete.

Finally we give a modular lattice M in which both Theorems fail:



Indeed, $H = \{0, a, b, c\}$ is a basis of four elements and every maximal chain is a basis with six elements. Further, $H \neq J_0([H]) = J_0(M)$.

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