Weakly independent subsets in lattices

by

G. Czédli, A. P. Huhn, and E. T. Schmidt

It is well-known that in a finite distributive lattice the set of join-irreducible elements and any maximal chain have the same cardinality. In this note we give a generalization of this theorem, using the following notions.

Definition. A subset H of a lattice L is called weakly independent iff for all $h, h_1, h_2, \ldots, h_n \in H$ which satisfy $h \leq h_1 \vee h_2 \vee \ldots \vee h_n$ there exists an i $(1 \leq i \leq n)$ such that $h \leq h_i$. A maximal weakly independent set is called a basis of L.

Every subchain of L is a weakly independent subset and any maximal chain is a basis. In this paper L denotes a finite distributive lattice and let $J_0(L)$ denote the set of all join-irreducible elements of L. By using Lemma 1 (cf. later) it is easy to show that $J_0(L)$ is a basis of L.

Theorem 1. *Any two bases of a finite distributive lattice have the same number of elements.*

To characterize the bases we need the following concept: a sublattice K of L is called a sublattice of maximal length iff K contains a maximal chain of L (i.e., every maximal chain of K is a maximal chain of L).

Theorem 2. A subset H *of a finite distributive lattice* L *is a basis if and only if the sublattice* K *generated by* H *is a sublattice of maximal length and* $H = J_0(K)$.

The proof of Theorem 1 starts with the following well-known

Lemma 1. Whenever $x \in J_0(L)$ and $x \leq y_1 \vee \dots \vee y_n$ then $x \leq y_i$ for some i $(1 \leq i \leq n)$.

From now on let U be a basis of L.

Lemma 2. For every $p \in J_0(L)$ there exists a $u \in U$ such that $p \leq u$.

Proof. By Lemma 1 it is sufficient to show that $\bigvee U = 1$. If we had $\bigvee U \neq 1$ then $U \cup \{1\}$ would be a weakly independent subset again, contradicting the maximality of U. ●

Now let p be an element of $J_0(L)$. We define the element $\bar{p} = \bigwedge(u: u \in U, p \leq u)$. By Lemma 2 $\bar{p}$ exists and $\bar{p} \leq p$.

Lemma 3. For every $p \in J_0(L)$ $\bar{p}$ belongs to U.

Proof. All we have to prove is that $U \cup \{\bar{p}\}$ is weakly independent. Firstly, we consider the case $\bar{p} \leq u_1 \vee \dots \vee u_n$ where $u_1, \dots, u_n \in U$. From Lemma 1 and $p \leq \bar{p} \leq u_1 \vee \dots \vee u_n$ we obtain $p \leq u_i$ for some i. Hence $\bar{p} \leq u_i$. The second case is $u \leq \bar{p} \vee u_1 \vee \dots \vee u_n$ where $u, u_1, \dots, u_n \in U$. We can assume that $u \not\leq u_1 \vee \dots \vee u_n$. For any $v \in U$ satisfying $p \leq v$ we have $\bar{p} \leq v$, whence $u \leq v \vee u_1 \vee \dots \vee u_n$. Since U is weakly independent we conclude $u \leq v$. Therefore $u \leq \bigwedge(v\colon v \in U,\ p \leq v) = \bar{p}$. ●

Lemma 4. The map $\varphi\colon J_0(L) \to U$, $p \mapsto \bar{p}$ is onto.

Proof. Let u be an arbitrary element of U. Then $u = p_1 \vee \dots \vee p_n$ for suitable $p_1, \dots, p_n \in J_0(L)$. From $p_i \leq \bar{p}_i$ $(i = 1, \dots, n)$ we have $u \leq \bar{p}_1 \vee \dots \vee \bar{p}_n$. Since U is weakly independent, $u \leq \bar{p}_i$ holds for some i. Finally, $p_i \leq u$ yields $\bar{p}_i \leq u$, i.e. $u = \bar{p}_i$. ●

The following lemma is not only to complete the proof of Theorem 1, it is interesting in itself.

Lemma 5. The map $\varphi\colon J_0(L) \to U$, $p \mapsto \bar{p}$ is bijective.

Proof. Let us assume that although $q, r \in J_0(L)$ and $q \nleq r$, we have $\bar{q} = \bar{r}$. Put $x = \bigvee H$ where $H = \{y\colon y \in J_0(L),\ \bar{y} \leq \bar{q}, \text{ and } q \nleq y\}$. Then $x \leq \bigvee(\bar{y}\colon y \in H) \leq \bar{q}$. By Lemma 1 $q \nleq x$. We claim that $x \notin U$. Indeed, $r \in H$ implies $r \leq x$, whence if $x \in U$ then $q \leq \bar{q} = \bar{r} \leq x$ contradicts $q \nleq x$. Now we prove that $U \cup \{x\}$ is weakly independent. We have to discuss two cases. Firstly, $x \leq u_1 \vee \dots \vee u_n$ where $u_1, \dots, u_n \in U$. From Lemma 1 and $r \leq x$ we have $r \leq u_i$ for some i $(1 \leq i \leq n)$, whence $\bar{r} \leq u_i$. We have already seen that $x \leq \bar{q} = \bar{r}$, so $x \leq u_i$. Secondly, $u \leq x \vee u_1 \vee \dots \vee u_n$ $(u, u_i \in U)$. We can assume that $u \nleq u_1 \vee \dots \vee u_n$. Then $x \leq \bar{q}$ implies $u \leq \bar{q} \vee u_1 \vee \dots \vee u_n$. Since U is weakly independent, $u \leq \bar{q}$. We have $u = p_1 \vee \dots \vee p_k$ for suitable $p_1, \dots, p_k \in J_0(L)$. If $q \nleq u$ then, for all i, $q \nleq p_i$ and $\bar{p}_i \leq u \leq \bar{q}$, whence $p_i \in H$ and $u = p_1 \vee \dots \vee p_k \leq \bigvee H = x$. Finally, if $q \leq u$, i.e. $\bar{q} \leq u$, then Lemma 1 yields that either $q \leq x$ or $\bar{q} \leq u_i$ for some i. But we have shown that $q \nleq x$. Therefore $u \leq u_i$ follows from $\bar{q} \leq u_i$ and $u \leq \bar{q}$. This proves Theorem 1. ●

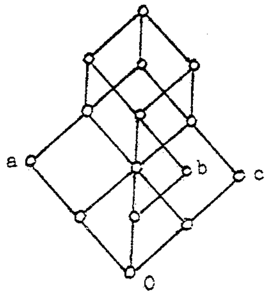
Proof of Theorem 2. Let H be a basis of L and let $H^{\wedge} = \{h_1 \wedge \dots \wedge h_n : 1 \leq n < \omega \text{ and } h_1, \dots, h_n \in H\}$. Further, let K be the sublattice generated by H. Then every element of K has a representation as a join of elements of $H^{\wedge}$. Now we prove that $H \subseteq J_0(K)$. Let $z = x \vee y$ where $z \in H$, $x, y \in K$. Then $x = a_1 \vee \dots \vee a_k$, $y = a_{k+1} \vee \dots \vee a_m$ with $a_1, \dots, a_m \in H^{\wedge}$. Therefore $a_i = \bigwedge_j h_{ij}$ for suitable $h_{ij} \in H$. By distributivity, z is the meet of all possible $h_{1j_1} \vee \dots \vee h_{mj_m}$. If there existed an ℓ_n for each n $(1 \leq n \leq m)$ such that $z \nleq h_{n\ell_n}$ then the weak independence of H would yield $z \nleq h_{1\ell_1} \vee \dots \vee h_{m\ell_m}$, a contradiction. Therefore there is a fixed n such that $z \leq h_{nj}$ holds for all j. Consequently $z \leq a_n$, which yields $x = z$ (if $1 \leq n \leq k$) or $y = z$ (if $k < n \leq m$). This proves $z \in J_0(K)$, i.e. $H \subseteq J_0(K)$. Since $J_0(K)$ is weakly independent in K and therefore in L, we conclude $H = J_0(K)$.

Now suppose the length of K is smaller than that of L. Choose a minimal b in K having the property: there exists an $a \in K$ which is covered by b in K but not in L. From the fact that transposed intervals are isomorphic we conclude that b is a join-irreducible element in K, i.e., $b \in J_0(K) = H$. Let c be a cover of a in L such that $a < c < b$. Then $c \notin H$ and to get a required contradiction it is sufficient to show that $H\cup\{c\}$ is a weakly independent subset of L. Firstly, consider the case $c \leq h_1 \vee \ldots \vee h_n$ where $h_1, \ldots, h_n \in H$. Then $a < c \leq b \wedge (h_1 \vee \ldots \vee h_n) \leq b$. Since $b \wedge (h_1 \vee \ldots \vee h_n) \in K$, we have $b \wedge (h_1 \vee \ldots \vee h_n) = b$, i.e. $b \leq h_1 \vee \ldots \vee h_n$. From $b \in J_0(K)$ and Lemma 1 we conclude $c < b \leq h_i$ for some i. Secondly, let $h \leq c \vee h_1 \vee \ldots \vee h_n$ and $h \nleq h_1 \vee \ldots \vee h_n$ where $h, h_1, \ldots, h_n \in H$. Then $h \leq b \vee h_1 \vee \ldots \vee h_n$ and the weak independence of $J_0(K) = H$ (or Lemma 1) yields $h \leq b$. In order to show $h \neq b$ suppose $h = b$. Then $b = h \nleq h_1 \vee \ldots \vee h_n$ implies $b \wedge (h_1 \vee \ldots \vee h_n) < b$, whence $b \in J_0(K)$ yields $b \wedge (h_1 \vee \ldots \vee h_n) \leq a < c$. Consequently $c = c \vee (b \wedge (h_1 \vee \ldots \vee h_n)) = (c \vee b) \wedge (c \vee \vee h_1 \vee \ldots \vee h_n) \geq b \wedge h = b \wedge b = b$, a contradiction. Thus $h < b$, yielding $h \leq a < c$.

Now, to prove the converse, suppose K is a sublattice of maximal length and $H = J_0(K)$. Since H is a basis in K, it is weakly independent in L. To prove that H is a basis in L suppose $H\cup\{a\}$ is weakly independent for some $a \in L \setminus K$. Put $b = \bigwedge(y\colon y \in K,\ a \leq y) \in K$ and choose a lower cover c of b in K. Then b covers c in L, too, whence $b = a \vee c$. Further, $c = h_1 \vee \ldots \vee h_n$ for suitable $h_1, \ldots, h_n \in H = J_0(K)$. Since $b \leq a \vee h_1 \vee \ldots \vee h_n$ and $H\cup\{a\}$ is weakly independent, $b \notin H$. Let $\{g_1, \ldots, g_k\}$ be a minimal subset of $H = J_0(K)$ for which $b = g_1 \vee \ldots \vee g_k$. Then $k \geq 2$ and $a \leq g_1 \vee \ldots \vee g_k$ contradicts the weak independence of $H\cup\{a\}$. The proof of Theorem 2 is complete.

<u>Remarks</u>. So far we have not used the well-known fact that $J_0(L)$ has the same cardinality as any maximal subchain of L. Hence this can be a corollary to Theorem 1.

On the other hand, Theorem 2 together with this corollary yield another proof of Theorem 1, but this proof avoids Lemma 5. Finally we give a modular lattice M in which both Theorems fail:



Indeed, $H = \{0,a,b,c\}$ is a basis of four elements and every maximal chain is a basis with six elements. Further, $H \neq J_0([H]) = J_0(M)$.