ON TOLERANCE LATTICES OF ALGEBRAS IN CONGRUENCE MODULAR VARIETIES

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Abstract. We prove that the tolerance lattice Tol$A$ of an algebra A from a congruence modular variety V is 0-1 modular and satisfies the general disjointness property. If V is congruence distributive, then the lattice Tol$A$ is pseudocomplemented. If V admits a majority term, then Tol$A$ is 0-modular.

Keywords: tolerance relation, congruence modularity, congruence distributivity, pseudocomplement, 0-modular lattice, joint disjointness property.

1. Introduction

The tolerances and the congruences of an algebra A form algebraic lattices denoted by Tol$A$ = (Tol$A, \wedge, \sqcup$) and Con$A$ = (Con$A, \wedge, \vee$), respectively. The congrence lattice Con$A$ of an algebra A is an algebraic lattice but (according to the Grätzer–Schmidt theorem, cf. [9]) it has no further special properties. The same is true for the tolerance lattice Tol$A$ by [3] (for an alternative proof cf. also Theorem 2 with ρ being the identical map plus checking the construction for reflexivity in Grätzer and Lampe [8]). As a contrast to the general case, the tolarence lattice Tol$L$ of an arbitrary lattice L has many nice properties by [10] and Bandelt [1]. Bandelt [1] is also a good source to convince the reader about the importance of tolerances of lattices.

The purpose of the present paper is to extend known results on tolerance lattices of lattices to tolerance lattices of more general algebras. Some results will be extended "only" for algebras with a majority term while some others for algebras in a congruence modular variety. Surprisingly enough, the proof of our generalized statement on 0-modularity, to be stated in the last theorem here, is considerably simpler than Bandelt's original approach and seems to be the right way

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to reveal what is behind the scene in [1]. In spite of the present achievments, we are not able to generalize all properties of lattice tolerances, for example, there is still no generalization of [7].

Now, for $\varphi \in \mathrm{Tol}A$, the transitive closure of φ will be denoted by $\overline{\varphi}$. Clearly, $\overline{\varphi}$ is a congruence of A. For any $\varphi, \psi \in \mathrm{Tol}A$ the least congruence containing both φ and ψ will be denoted by $\varphi \veebar \psi$. Obviously, we have $\varphi \veebar \psi = \overline{\varphi \sqcup \psi} = \overline{\varphi} \vee \overline{\psi}$ and $\zeta \veebar \theta = \zeta \vee \theta$ for $\zeta, \theta \in \mathrm{Con}A$. We say that $\mathrm{Tol}A$ satisfies the inequality dist(tol,tol,tol) respectively mod(tol,tol,tol), if $\alpha \wedge (\beta \veebar \gamma) \leqq (\alpha \wedge \beta) \veebar (\alpha \wedge \gamma)$ respectively $\alpha \wedge (\beta \veebar (\alpha \wedge \gamma)) \leqq (\alpha \wedge \beta) \veebar (\alpha \wedge \gamma)$ is valid for all $\alpha, \beta, \gamma \in \mathrm{Tol}A$. This is a right place for a warning: the satisfaction of dist(tol,tol,tol) resp. mod(tol,tol,tol) by $\mathrm{Tol}A$ does not mean that the lattice $\mathrm{Tol}A = (\mathrm{Tol}A, \wedge, \sqcup)$ is distributive resp. modular; in virtue of Bandelt [1] and [6] this is exemplified by lattices in place of A.

In [6] the first two authors proved that if $\mathcal{V}$ is a congruence modular resp. congruence distributive variety, then, for each algebra $A \in \mathcal{V}$, $\mathrm{Tol}A$ satisfies mod(tol,tol,tol) resp. dist(tol,tol,tol). They also proved $\alpha \wedge \overline{\beta} \leqq \overline{(\alpha \wedge \beta)}$ for all $\alpha, \beta \in \mathrm{Tol}A$ and $A \in \mathcal{V}$, and pointed out that it is essential to consider a whole variety, not just a single algebra.

Now it is known that the variety of all lattices is congruence distributive. The afore-mentioned results of Bandelt [1] state that for any lattice L, $\mathrm{Tol}L$ is a pseudocomplemented and 0-modular lattice. The pseudocomplement φ^* of any $\varphi \in \mathrm{Tol}L$ is a congruence by [10]. Now the above-mentioned results of [6] provide us with the main tool to prove, for instance, that if A belongs to a congruence modular variety then $\mathrm{Con}A$ is a homomorphic image of $\mathrm{Tol}A$; if A belongs to a congruence distributive variety then $\mathrm{Tol}A$ is 0-1 modular and pseudocomplemented lattice and for any $\varphi \in \mathrm{Tol}A$ φ^* is a congruence.

2. Preliminary notions

A lattice L with 0 is called *0-modular*, cf. Stern [14], if there is no N_5 sublattice of L including 0. A bounded lattice L is called 0-1 *modular* if no N_5 of L includes both 0 and 1. Clearly, this is equivalent to the condition that none of the elements of L has comparable complements. A complete lattice L is called *upper continuous*, cf. Schmidt [13], if any directed family of elements $\{a_\delta \mid \delta \in D\} \subseteqq L$ and any $a \in L$ satisfies $a \wedge (\bigvee \{a_\delta \mid \delta \in D\}) = \bigvee \{a \wedge a_\delta \mid \delta \in D\}$. It is well-known that any algebraic lattice is upper continuous.

For $a, b \in L$ set $\mathrm{SC}(a/b) = \{x \in L \mid a \wedge x \leqslant b\}$. If L is an upper continuous lattice, then the set $\mathrm{SC}(a/b)$ contains at least one maximal element [5], which is called a *weak pseudocomplement* of *a relative to b* and it is denoted by $a_w b$. It is easy to see that $a_w b$ is not necessarily unique and for any $x \in$SC(a/b) there exists at least one $a_w b$ such that $x \leq a_w b$. If $0 \in L$, then $a_w 0$ is called a *weak pseudocomplement* of a and it is denoted by a^w. If a^w is unique, i.e. if a^w is the greatest element of SC$(a/0)$, then it is called the *pseudocomplement* of a and usually it is denoted by a^*. L is called a *pseudocomplemented lattice* if for each $a \in L$ there exists $a^* \in L$. In other words, L is pseudocomplemented if for any $a \in L$ there exists an $a^* \in L$ such that for any $x \in L$, $x \wedge a = 0 \Leftrightarrow x \leqq a^*$. It is well-known that any algebraic distributive lattice is pseudocomplemented. If L is a pseudocomplemented lattice then $(L, \wedge, \vee, ^*, 0, 1)$ is called a *p-algebra.* $\triangle$ and ∇ stands for the identity relation and the all relation on A, respectively. The algebra A is called *tolerance-simple,* cf. e.g. Chajda [2], if Tol$A = \{\triangle, \nabla\}$.

The following lemma will be useful in our proofs:

Lemma 2.1. *Let A be an arbitrary algebra and φ_1,$\varphi_2 \in$ Tol$A$. Then* $\varphi_1 \sqcup \varphi_2 = \nabla$ implies $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = \nabla$.

Proof. Since $(\varphi_1 \circ \varphi_2) \cap (\varphi_2 \circ \varphi_1)$ is clearly a tolerance of A, cf. e.g. [11], and it includes φ_1 and φ_2, we obtain $\nabla = \varphi_1 \sqcup \varphi_2 \subseteqq (\varphi_1 \circ \varphi_2) \cap (\varphi_2 \circ \varphi_1)$. Hence $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = \nabla$. *Q.E.D.*

3. The proofs

Lemma 3.1. *Let A be a congruence modular (congruence distributive) algebra. Then the following statements are equivalent:*

(i) *For any $\theta \in$ Con$A$ and any $\varphi \in$ Tol$A$ we have $\varphi_w \theta \in$ Con$A$.*

(ii) $\overline{\varphi} \wedge \overline{\psi} = \overline{(\varphi \wedge \psi)}$, *for all $\varphi, \psi \in$ Tol$A$.*

(iii) *The map h: Tol$A$ $\rightarrow$ Con$A$, $\varphi \mapsto \overline{\varphi}$, is a surjective lattice homomorphism.*

(iv) Tol$A$ *satisfies mod(tol,tol,tol) (dist(tol,tol,tol)).*

Proof. (i)$\Rightarrow$(ii). Let $\varphi, \psi \in$ Tol$A$ and consider $\theta = \overline{(\varphi \wedge \psi)} \in$ Con$A$. Then $\varphi \wedge \psi \leq \theta$. As Tol$A$ is an algebraic lattice, there exists a $\varphi_w \theta$ such that $\psi \leq \varphi_w \theta$. Since by the assumption of (i) $\varphi_w \theta \in$ Con$A$, we obtain $\overline{\psi} \leq \varphi_w \theta$, and this implies $\varphi \wedge \overline{\psi} \leq \overline{(\varphi \wedge \psi)}$. As this relation is valid for any pair of tolerances, we obtain $\overline{\varphi} \wedge \overline{\psi} \leq \overline{(\overline{\varphi} \wedge \psi)} \leq \overline{\overline{(\varphi \wedge \psi)}} = \overline{(\varphi \wedge \psi)}$. Since $\overline{(\varphi \wedge \psi)} \leq \overline{\varphi} \wedge \overline{\psi}$, we obtain $\overline{\varphi} \wedge \overline{\psi} = \overline{(\varphi \wedge \psi)}$.

(ii)$\Rightarrow$(iii). Since for any $\theta \in$ Con$A$ we have $h(\theta) = \theta$, the map h :Tol$A$ $\rightarrow$Con$A$ is onto. Take $\varphi, \psi \in$Tol$A$. Then $h(\varphi \sqcup \psi) = \overline{(\varphi \sqcup \psi)} =$

$\overline{\varphi}\vee\overline{\psi} = h(\varphi)\vee h(\psi)$, moreover (ii) implies $h(\varphi\wedge\psi) = \overline{(\varphi \wedge \psi)} = \overline{\varphi}\wedge\overline{\psi} = h(\varphi) \wedge h(\psi)$. Thus h is a homomorphism.

(iii)⇒(iv). Take $\alpha, \beta, \gamma \in \mathrm{Tol}A$. Then we have $\alpha \wedge (\beta \underline{\vee} (\alpha \wedge \gamma)) \leq \overline{\alpha} \wedge (\overline{\beta} \vee \overline{(\alpha \wedge \gamma)}) \leq \overline{\alpha} \wedge (\overline{\beta} \vee (\overline{\alpha} \wedge \overline{\gamma}))$. If Con$A$ is a modular lattice, then we obtain $\overline{\alpha} \wedge (\overline{\beta} \vee (\overline{\alpha} \vee \overline{\gamma})) \leq (\overline{\alpha} \wedge \overline{\beta}) \vee (\overline{\alpha} \wedge \overline{\gamma})$. Since $h(\varphi) = \overline{\varphi}$ is a homomorphism, we have $\overline{(\alpha \wedge \beta)} = \overline{\alpha} \wedge \overline{\beta}$ and $\overline{(\alpha \wedge \gamma)} = \overline{\alpha} \wedge \overline{\gamma}$. Thus we obtain $\alpha \wedge (\beta \underline{\vee} (\alpha \wedge \gamma)) \leq (\overline{\alpha} \wedge \overline{\beta}) \vee (\overline{\alpha} \wedge \overline{\gamma}) = \overline{(\alpha \wedge \beta)} \vee \overline{(\alpha \wedge \gamma)} = (\alpha \wedge \beta) \underline{\vee} (\alpha \wedge \gamma)$, and so Tol$A$ satisfies mod(tol,tol,tol).

The case when Con$A$ is distributive is similar: $\alpha \wedge (\beta \underline{\vee} \gamma) \leq \overline{\alpha} \wedge (\overline{\beta} \vee \overline{\gamma}) = (\overline{\alpha} \wedge \overline{\beta}) \vee (\overline{\alpha} \wedge \overline{\gamma}) = \overline{(\alpha \wedge \beta)} \vee \overline{(\alpha \wedge \gamma)} = (\alpha \wedge \beta) \underline{\vee} (\alpha \wedge \gamma)$, and this proves that Tol$A$ satisfies dist(tol,tol,tol).

(iv)⇒(i). Clearly, dist(tol,tol,tol) implies mod(tol,tol,tol) and mod(tol, tol,tol), according to [6] or substituting 0 for the "third tol", implies $\alpha \wedge \overline{\beta} \leq \overline{(\alpha \wedge \beta)}$ for all $\alpha, \beta \in \mathrm{Tol}A$. Take any $\theta \in \mathrm{Con}A$ and $\varphi \in \mathrm{Tol}A$. Then $\varphi \wedge \varphi_w\theta \leq \theta$ implies $\varphi \wedge \overline{\varphi_w\theta} \leq \overline{(\varphi \wedge \varphi_w\theta)} \leq \overline{\theta} = \theta$, i.e. $\overline{\varphi_w\theta} \in$SC$(\varphi/\theta)$. As $\varphi_w\theta$ is a maximal element of SC(φ/θ) and since $\varphi_w\theta \leq \overline{\varphi_w\theta}$, we obtain $\varphi_w\theta = \overline{\varphi_w\theta} \in \mathrm{Con}A$. *Q.E.D.*

Proposition 3.2. *Let A be an algebra in a congruence modular variety $\mathcal{V}$. Then the following two statements hold:*

(i) *For any $\varphi \in \mathrm{Tol}A$ each $\varphi^w \in \mathrm{Con}A$.*

(ii) *If φ and ψ are complements of each other in* Tol$A$, *then they are weak pseudocomplements of each other and form a factor congruence pair of A.*

Proof. (i*)* Since $\mathcal{V}$ is congruence modular, Tol$A$ satisfies mod(tol,tol,tol) according to [6]. As $\varphi^w = \varphi_w 0$, applying Lemma 3.1 we infer (i).

(ii) Let φ and ψ be complements of each other in Tol$A$. Then, by Lemma 2.1, $\varphi \sqcup \psi = \nabla$ implies $\varphi \circ \psi = \psi \circ \varphi = \nabla$. As $\varphi \wedge \psi = \triangle$, there is a φ^w such that $\psi \leq \varphi^w$. We have to prove $\psi = \varphi^w$, i.e. $\varphi^w \leq \psi$.

Take any $(x, y) \in \varphi^w$. Since $(x, y) \in \varphi \circ \psi$, there exists a $z \in A$ such that $(x, z) \in \varphi$ and $(z, y) \in \psi$. However $\psi \leq \varphi^w$ implies $(z, y) \in \varphi^w$. As $\varphi^w \in \mathrm{Con}A$, we obtain $(x, z) \in \varphi^w \wedge \varphi = \triangle$, i.e. $x = z$. Therefore we obtain $(x, y) \in \psi$ proving $\varphi^w \leq \psi$. Thus, we conclude that $\psi = \varphi^w \in \mathrm{Con}A$. Interchanging the role of φ and ψ we obtain $\varphi = \psi^w \in \mathrm{Con}A$. As $\varphi \wedge \psi = \triangle$ and $\varphi \circ \psi = \psi \circ \varphi = \nabla$, φ and ψ are factor congruences of A. *Q.E.D.*

Definition 3.3. The lattice L with 0 satisfies the *general disjointness property* (GD) if $a \wedge b = 0$ and $(a \vee b) \wedge c = 0$ imply $a \wedge (b \vee c) = 0$. (See [12] or [14].)

It is easy to check that any pseudocomplemented lattice has the (GD) property. It was proved in [12] that any 0-modular lattice satisfies the (GD) property, too.

Theorem 3.4. *Let A be an algebra in a congruence modular variety $\mathcal{V}$. Then the following statements hold:*

(i) *The map h: Tol$A \to$ Con$A$, $\varphi \mapsto \overline{\varphi}$, is a surjective lattice homomorphism and Tol$A$ is a* 0-1 *modular lattice having the (GD) property.*

(ii) Tol$A$ *is pseudocomplemented if and only if* Con$A$ *is pseudocomplemented.*

Proof. (i) Since $\mathcal{V}$ is a congruence modular variety and $A \in \mathcal{V}$, by [6] Tol$A$ satisfies mod(tol,tol,tol). Therefore by applying Lemma 3.1 we obtain the required homomorphism.

Now, by way of contradiction, suppose that Tol$A$ is not 0-1 modular. Then an N_5 sublattice of Tol$A$ includes Δ and ∇. Hence each element of this N_5 has a complement in Tol$A$. Since complements are weak pseudocomplements as well, we conclude from Proposition 3.2(ii) that $N_5 \subseteq$ Con$A$. Hence the homorphism h acts identically on N_5 and we infer that N_5, as a homomorphic image, is a sublattice of Con$A$, contradicting congruence modularity.

Finally, take $\alpha, \beta, \gamma \in$ Tol$A$ and assume that $\alpha \wedge \beta = \triangle$ and $(\alpha \sqcup \beta) \wedge \gamma = \triangle$. Applying the homomorphism h to these two equations we obtain $h(\alpha) \wedge h(\beta) = h(\triangle) = \triangle$ and $(h(\alpha) \vee h(\beta)) \wedge h(\gamma) = \triangle$. Since Con$A$ is a modular lattice, it has the (GD) property as well, and this gives $\alpha \wedge (\beta \sqcup \gamma) \leq h(\alpha \wedge (\beta \sqcup \gamma)) = h(\alpha) \wedge (h(\beta) \vee h(\gamma)) = \triangle$. Thus Tol$A$ has the (GD) property.

(ii) Assume that Tol$A$ is a pseudocomplemented lattice. Since now for any $\theta \in$ Con$A$, θ^* is its (unique) weak pseudocomplement in Tol$A$, Proposition 3.2(i) gives $\theta^* \in$ Con$A$. As any $\zeta \in$Con$A$ is also a tolerance, we have $\theta \wedge \zeta = \triangle \Leftrightarrow \zeta \leq \theta^*$. Hence θ^* is the pseudocomplement of θ in the lattice Con$A$ as well. Thus Con$A$ is pseudocomplemented.

Conversely, assume that Con$A$ is pseudocomplemented and denote by θ^* the pseudocomplement of a $\theta \in$ Con$A$. We prove that for each $\varphi \in$ Tol$A$ the congruence $(\overline{\varphi})^*$ is the pseudocomplement of φ in Tol$A$.

Let $\psi \in$ Tol$A$, $\psi \leqq (\overline{\varphi})^*$. Then $\varphi \wedge \psi \leqq \overline{\varphi} \wedge (\overline{\varphi})^* = \triangle$. Take a $\psi \in$ Tol$A$ with $\varphi \wedge \psi = \triangle$. Then, in view of Lemma 3.1(ii), we have $\overline{\varphi} \wedge \overline{\psi} = \overline{(\varphi \wedge \theta)} = \triangle$. Thus we obtain $\overline{\psi} \leq (\overline{\varphi})^*$and so $\psi \leq (\overline{\varphi})^*$. Hence $\varphi \wedge \psi = 0 \Leftrightarrow \psi \leqq (\overline{\varphi})^*$ and this proves that Tol$A$ is pseudocomplemented and the pseudocomplement φ^* of φ in Tol$A$ is the same as $(\overline{\varphi})^*$. *Q.E.D.*

Remark 3.5. Observe that it is implicit in the proof of Theorem 3.5(ii) the following: The pseudocomplement in Con$A$ of a $\theta \in$ Con$A$ is the same as its pseudocomplement in Tol$A$. As a consequence, the pseudocomplementation operation will be denoted by the same symbol "$*$" in both of the lattices Con$A$ and Tol$A$. It is also clear that in this case $(\text{Con}A, \wedge, ^*)$ is a subalgebra of $(\text{Tol}A, \wedge, ^*)$. Notice that in the proof of the Theorem 3.5(ii) it was also deduced that $\varphi^* = (\overline{\varphi})^*$.

Proposition 3.6. *Let $\mathcal{V}$ be a congruence distributive variety and let $A \in \mathcal{V}$. Then the following hold:*

(i) Tol$A$ *is a pseudocomplemented* 0-1 *modular lattice and for any* $\varphi \in$ Tol$A$ *we have* $\varphi^* \in$Con$A$.

(ii) *The map* h: *Tol*A$\rightarrow$ Con$A$, $\varphi \mapsto \overline{\varphi}$, *is a homomorphism of the p-algebra* $(\text{Tol}A, \wedge, \sqcup, ^*, \triangle, \nabla)$ *onto the p-algebra* $(\text{Con}A, \wedge, \vee, ^*, \triangle, \nabla)$.

Proof. Now Con$A$, as an algebraic distributive lattice, is pseudocomplemented as well. Therefore (i) is an obvious consequence of Theorem 3.4 and Proposition 3.2(i).

(ii) In view of Theorem 3.4(i) h is a lattice homomorphism and h is onto. We have also $h(\triangle) = \triangle$ and $h(\nabla) = \nabla$. Since $\varphi^* \in$ Con$A$, $h(\varphi^*) = \varphi^*$. On the other hand, we have $(h(\varphi))^* = (\overline{\varphi})^* = \varphi^*$, according to Remark 3.5. Thus we obtain $h(\varphi^*) = (h(\varphi))^*$, and hence h is a homomorphism of p-algebras. *Q.E.D.*

Corollary 3.7. *Let A be an algebra of a variety $\mathcal{V}$.*

(i) *If $\mathcal{V}$ is congruence modular and* Tol$A$ *is a simple or complemented lattice then* Tol$A$ =Con$A$.

(ii) *If $\mathcal{V}$ is congruence distributive and the lattice* Tol$A$ *is simple, then the algebra A itself is tolerance-simple.*

Proof. We may assume that $|A| \geq 2$.

(i) If Tol$A$ is complemented, then Proposition 3.2(ii) gives Tol$A$ = Con$A$. If Tol$A$ is simple, then the congruence $\Theta \subseteqq$Tol$A\times$Tol$A$ defined by $(\varphi_1, \varphi_2) \in \Theta \Leftrightarrow \overline{\varphi_1} = \overline{\varphi_2}$ is either the identity relation or the total relation on Tol$A$. The latter case can be excluded, as $\overline{\triangle} = \triangle \neq \nabla = \overline{\nabla}$. Since we have $(\varphi, \overline{\varphi}) \in \Theta$, we obtain $\varphi = \overline{\varphi}$ for all $\varphi \in$Tol$A$, i.e. Tol$A$ =Con$A$.

(ii) We have Tol$A$ =Con$A$, according to the above (i). As now Con$A$ is a simple distributive lattice, it is a two-element chain. Hence Tol$A$ is also a two-element chain, i.e. A is tolerance-simple. $\diamondsuit$

A term function $m(x, y, z)$ of an algebra A is called a *majority term* if $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$ holds for all $x, y \in A$. For instance, any lattice $(L, \wedge, \vee)$ admits a majority term. It is well-known that the variety $\mathcal{V}(A)$ generated by an algebra A with a majority term is congruence distributive.

Now let A be an arbitrary algebra and $\alpha, \beta \in \mathrm{Tol}A$. By an (α,β)-*circle* we mean a 4-tuple $(a,b,c,d) \in A^4$ such that $(a,b),(c,d) \in \alpha$ and $(b,c),(d,a) \in \beta$.

Lemma 3.8. *Let A be an algebra with a majority term m, and let $\alpha, \beta \in \mathrm{Tol}A$ with $\alpha \wedge \beta = \triangle$.*

(i) *If $(a,b,c,d) \in A^4$ is an (α,β)-circle, then*

$$m(a,b,c) = b,\ m(b,c,d) = c,\ m(c,d,a) = d,\ m(d,a,b) = a. \qquad (1)$$

(ii) *We have $\alpha \sqcup \beta = (\alpha \circ \beta) \cap (\beta \circ \alpha)$.*

Proof. (i) Because of symmetry, it suffices to prove the first equality. Since we have $(m(a,b,c), m(b,b,c)) \in \alpha$, $((m(a,b,c), m(a,b,b)) \in \beta$ and $m(b,b,c) = m(a,b,b) = b$, the first equality comes from $\alpha \wedge \beta = \triangle$.

(ii) As it was pointed out in the argument of Lemma 2.1, we have $(\alpha \circ \beta) \cap (\beta \circ \alpha) \in \mathrm{Tol}A$ and $\alpha \sqcup \beta \subseteqq (\alpha \circ \beta) \cap (\beta \circ \alpha)$. Now let δ be a tolerance with $\alpha \leq \delta$ and $\beta \leq \delta$ and take any $a, c \in A$ with $(a,c) \in (\alpha \circ \beta) \cap (\beta \circ \alpha)$. Then there exist $b, d \in A$ such that $(a,b) \in \alpha$, $(b,c) \in \beta$ and $(a,d) \in \beta$, $(d,c) \in \alpha$. Then (a,b,c,d) is an (α,β)-circle. Therefore (1) gives $m(d,a,b) = a$. On the other hand, $(d,c),(b,c) \in \delta$ implies $(m(d,a,b), m(c,a,c)) \in \delta$. As $m(c,a,c) = c$, we obtain $(a,c) \in \delta$. Thus we conclude $(\alpha \circ \beta) \cap (\beta \circ \alpha) \leq \delta$ and this proves $(\alpha \circ \beta) \cap (\beta \circ \alpha) = \alpha \sqcup \beta$. *Q.E.D.*

Theorem 3.9. *Let A be an algebra. If A has a majority term then:*

(i) $\mathrm{Tol}A$ *is a 0-modular pseudocomplemented lattice.*

(ii) *The tolerances α, β are complements of each other in TolA if and only if they form a factor congruence pair of A.*

Proof. (i) Since the variety $\mathcal{V}(A)$ is congruence distributive, in view of Proposition 3.6, $\mathrm{Tol}A$ is pseudocomplemented.

In order to prove that $\mathrm{Tol}A$ is 0-modular, by the way of contradiction let us assume that $\{\triangle, \alpha, \beta, \gamma, \nu\}$ is an N_5 sublattice in $\mathrm{Tol}A$ with $\triangle < \alpha < \gamma < \nu$, $\triangle < \beta < \nu$ and $\alpha \sqcup \beta = \gamma \sqcup \beta = \nu$, $\alpha \wedge \beta = \gamma \wedge \beta = \triangle$. Take any $a, c \in A$ with $(a,c) \in \gamma$. As by Lemma 3.8(ii) we have $\nu = \alpha \sqcup \beta = (\alpha \circ \beta) \cap (\beta \circ \alpha)$ and since $\gamma < \nu$, we obtain $(a,c) \in (\alpha \circ \beta) \cap (\beta \circ \alpha)$. Then there exist $c, d \in A$ such that $(a,b) \in \alpha$, $(b,c) \in \beta$ and $(a,d) \in \beta$, $(d,c) \in \alpha$, i.e. such that (a,b,c,d) is an (α,β)-circle.

From $(a, c) \in \gamma$ and (1) we obtain $b = m(a, b, c)\ \gamma\ m(c, b, c) = c$. Thus we obtain $(b, c) \in \gamma \wedge \beta = \triangle$, i.e. $b = c$. Hence we conclude $(a, c) = (a, b) \in \alpha$. We have shown $\gamma \leq \alpha$, a contradiction. Therefore Tol$A$ is 0-modular.

(ii) If α and β are complements of each other then they form a factor congruence pair in virtue of Proposition 3.2(ii). Conversely, suppose that $\alpha, \beta \in$ Con$A$ form a factor congruence pair. Then $\alpha\circ\beta = \beta\circ\alpha = \nabla$ and $\alpha \wedge \beta = \triangle$, whence we conclude from Lemma 3.8(ii) that $\alpha \sqcup \beta = (\alpha \circ \beta) \cap (\beta \circ \alpha) = \nabla$. Thus α and β are complements of each other in Tol$A$. *Q.E.D.*

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