Associativity in monoids and categories

Dedicated to Béla Csákány on his seventieth birthday

June 3, 2001

Gábor Czédli, Eszter K. Horváth and Lajos Klukovits

University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, HUNGARY–6720.

E-mail: czedli@math.u-szeged.hu, horeszt@math.u-szeged.hu, klukovits@math.u-szeged.hu

Key words: associativity, isolated triplet, monoid, semigroup, category.

Mathematics Subject Classification 2000: Primary 18A05, secondary 20M99.

Abstract. Given a nonempty set A and we consider the possible groupoids $(A, \cdot)$ with base set A. If there is no proper subset T of A^3 such that the satisfaction of $(xy)z = x(yz)$ for all $(x, y, z) \in T^3$ implies that $(A, \cdot)$ is a semigroup then we say that the associativity conditions are *independent* over the set A. Szász [4, 5] showed that this is the case iff $|A| \geq 4$. In this note the analogous problem is considered for categories and, as particular cases, for monoids. It is proved that if for all objects A and B, $\mathrm{Hom}(A, B)$ is either empty or has at least five elements then the associativity conditions are independent. The bound "five" is shown to be sharp.

This research was partially supported by the NFSR of Hungary (OTKA), grant no. T023186, T022867, T026243 and T034137, and also by the Hungarian Ministry of Education, grant no. FKFP 1259/1997.

Introduction and results

Given a groupoid $(A, \cdot)$, a triplet $(x, y, z) \in A^3$ is said to be an *associative* triplet if $(xy)z = x(yz)$. The groupoid is called a *semigroup* if all triplets belonging to A^3 are associative. Szász [4, 5] investigated the question whether we really have to test if all triplets are associative or there exists a proper subset T of A^3 such that the associativity of triplets in T implies the associativity of $(A, \cdot)$. If there is no such T then we say that the associativity conditions are *independent* over the set A.

Another way to approach the notion of independence is the following one. We call a triplet $(a, b, c) \in A^3$ *isolated* (cf. Szász [4, 5]) if there is a binary operation $*$ on A such that (a, b, c) is not associative in $(A, *)$ but all other triplets in $A^3 \setminus \{(a, b, c)\}$ are associative.

For example, if $A = \{a, b, c\}$ then (a, a, b) is isolated. Indeed, if we define an operation $*$ by the following Cayley table

$*$	a	b	c
a	c	b	c
b	c	c	c
c	c	c	c

then (a, a, b) is the only non-associative triplet of $(A, *)$.

It is not hard to see that if (x, y, z) is a triplet in A^3 then it depends only on $(|\{x, y\}|, |\{x, z\}|, |\{y, z\}|, |A|)$, i.e., on the equations among components and the size of A, whether (x, y, z) is isolated or not.

Clearly, the associativity conditions are independent over A iff all triplets in A^3 are isolated. We recall the following theorem.

Theorem A. *(Szász [4, 5]) If $|A| \geq 4$ then the associativity conditions are independent over A. If $1 \leq |A| \leq 3$ and $a \in A$ then (a, a, a) is not isolated. If $|A| = 3$ then $(a, b, c) \in A^3$ is isolated provided $|\{a, b, c\}| \geq 2$.*

Notice that associativity is just one identity, and there have been investigations of independence related to other identities, cf. Szász [6], Wiegandt [7] and Klukovits [1,2,3]. Now if $(A, \cdot, 1)$ is a groupoid with unit element 1 (i.e., $1x = x1 = x$ holds for all $x \in A$) and $1 \in \{x, y, z\} \subseteq A$ then (x, y, z) is necessarily associative. Referring to triplets (x, y, z) with $1 \notin \{x, y, z\}$ as *nontrivial triplets*, the question of isolatedness is interesting only for nontrivial triplets. However, instead of monoids, we carry out this investigation in a more general setting. To formulate our result we have to introduce a simple concept. By an *underlying system* $\mathcal{C}$ we mean a class $\mathbf{Ob}(\mathcal{C})$ of arbitrary objects together with sets $\mathrm{Hom}(A, B)$ for $A, B \in \mathbf{Ob}(\mathcal{C})$ and distinguished elements $1_A \in \mathrm{Hom}(A, A)$ for all $A \in \mathbf{Ob}(\mathcal{C})$. We assume that $\mathrm{Hom}(A_1, B_1) \cap \mathrm{Hom}(A_2, B_2) = \emptyset$ whenever $(A_1, B_1) \neq (A_2, B_2)$. Note that for $A \neq B$, $\mathrm{Hom}(A, B)$ can be empty. By a *categoroid* $(\mathcal{C}, \cdot)$ we mean an underlying system $\mathcal{C}$ together with a multiplication $\cdot$ with the following usual properties: ab is defined iff $a \in \mathrm{Hom}(A, B)$ and $b \in \mathrm{Hom}(B, C)$ for some $A, B, C \in \mathbf{Ob}(\mathcal{C})$, if $a \in \mathrm{Hom}(A, B)$ and $b \in \mathrm{Hom}(B, C)$ then $ab \in \mathrm{Hom}(A, C)$, if $1_A a$ is defined then $1_A a = a$, and if $a1_B$ is defined then $a1_B = a$. The elements of $\mathrm{Hom}(A, B)$ are called *morphisms* (from A to B), and all the 1_A, $A \in \mathbf{Ob}(\mathcal{C})$, are referred to as *identity morphisms*. A triplet (a, b, c) of morphisms is called *admissible* if there exist $A, B, C, D \in \mathbf{Ob}(\mathcal{C})$ such that $a \in \mathrm{Hom}(A, B)$, $b \in \mathrm{Hom}(B, C)$ and $c \in \mathrm{Hom}(C, D)$. By a *category* we mean a categoroid such that $(ab)c = a(bc)$ holds for every admissible triplet of morphisms; in other words, if all admissible triplets are associative. This definition coincides with the usual one. An admissible triplet is called *nontrivial* if none of its components is an identity morphism. If (a, b, c) is an admissible triplet in an underlying system $\mathcal{C}$ and there is a categoroid $(\mathcal{C}, \cdot)$ such that (a, b, c) is not associative but all other admissible triplets are associative then (a, b, c) is called an *isolated* triplet. We say that the associativity conditions for an underlying system are *independent* if all nontrivial admissible triplets are isolated. Now we can formulate our main result.

Theorem 1. *Let $\mathcal{C}$ be an underlying system. If*

(i) for any two distinct $A, B \in \mathbf{Ob}(\mathcal{C})$ either $Hom(A, B)$ is empty or $|Hom(A, B)| \geq 4$ and

(ii) $|Hom(A, A)| \geq 5$ for each $A \in \mathbf{Ob}(\mathcal{C})$,

then the associativity conditions for $\mathcal{C}$ are independent.

A slightly weaker but less complicated statement reads as follows.

Corollary 1. *Let $\mathcal{C}$ be an underlying system. If for each $A, B \in \mathbf{Ob}(\mathcal{C})$ either $Hom(A, B)$ is empty or $|Hom(A, B)| \geq 5$ then the associativity conditions for $\mathcal{C}$ are independent.*

As a particular case in connection with monoids rather than categories, Theorem 1 clearly implies the following statement; the notions in it are self-explaining.

Corollary 2. *If $(A, 1)$ is a pointed set with at least five elements then the associativity conditions for $(A, 1)$ are independent.*

We should notice that, using standard unitary extension, this corollary follows also from Theorem A. Unfortunately, computational difficulties have prevented us from reaching the converse of Theorem 1. We present only the following assertion, and raise the question if it is true with three instead of two. (In virtue of Theorem 1, it is definitely false with four instead of two.)

Assertion 1. *Let $\mathcal{C}$ be an underlying system and suppose that there exist $A, B \in \mathbf{Ob}(\mathcal{C})$ such that $Hom(A, B)$ is non-empty and has at most two elements. Then the associativity conditions for $\mathcal{C}$ are not independent.*

Even if we could not achieve the converse of Theorem 1, we know that it will not be valid with 4 in (ii) instead of 5, not even for the particular case when we are dealing with monoids rather than categories. This fact is formulated in the following statement, which is not a consequence of Theorem A.

Proposition 1. *If $(A, 1) = (\{a, b, c, 1\}, 1)$ is a pointed set of four elements then (a, a, a) is not isolated.*

Proofs

Proof of Theorem 1. We have to show that all nontrivial admissible triplets are isolated. Several cases will be distinguished.

First, let $a, b, c \in \mathrm{Hom}(A, A) \setminus \{1_A\}$, $A \in \mathbf{Ob}(\mathcal{C})$, such that $|\{a, b, c\}| \geq 2$. We are going to show that (a, b, c) is isolated We pick an element (i.e., a morphism) from $\mathrm{Hom}(A, A) \setminus \{1_A, a, b, c\}$ which will be denoted by $0 = 0_A$. In virtue of Theorem A, (a, b, c) is isolated in $S = \mathrm{Hom}(A, A) \setminus \{0_A, 1_A\}$. So we can define a multiplication $\cdot$ on S such that (a, b, c) is the only non-associative triplet of $(S, \cdot)$. Now we extend the multiplication first to $S \cup \{1_A\}$ such that 1_A be the unit element, and then from $S \cup \{1_A\}$ to $S \cup \{1_A, 0_A\} = \mathrm{Hom}(A, A)$ such that 0_A be the zero of the groupoid $(\mathrm{Hom}(A, A), \cdot)$. Then $(\mathrm{Hom}(A, A), \cdot)$ is a groupoid with zero and unit. Since any triplet in $\mathrm{Hom}(A, A)^3$ containing 0_A or 1_A is clearly associative, (a, b, c) is the only non-associative triplet in

$(\mathrm{Hom}(A,A),\cdot)$. Now we extend the multiplication to the whole underlying system. For each $X, Y \in \mathbf{Ob}(\mathcal{C})$ such that $\mathrm{Hom}(X,Y) \neq \emptyset$ let us fix a morphism $0_{XY} \in \mathrm{Hom}(X,Y)$. Of course, 0_{AA} is chosen to be 0_A, and, for $X = Y$, $0_{XX} \neq 1_X$. We may write 0_X instead of 0_{XX}. We refer to 0_{XY} resp. 1_X as a zero resp. unit. Now, for $u \in \mathrm{Hom}(X,Y)$ and $v \in \mathrm{Hom}(Y,Z)$ we define the product uv as follows. If $X = Y = Z = A$ then uv is already defined. If u or v is zero then their product is zero. (More precisely, if $u = 0_{XY}$ or $v = 0_{YZ}$ then $uv = 0_{XZ}$.) If u resp. v is a unit, which is possible only if $X = Y$ resp. $Y = Z$, then uv is v resp. u. In the rest of cases, let $uv = 0_{XZ}$. It is easy to check that we have defined a categoroid $(\mathcal{C},\cdot)$ in which (a,b,c) is the only non-associative admissible triplet.

The second case is when $a \in \mathrm{Hom}(A,A)$, and we intend to show that the triplet (a,a,a) is isolated. Here, in effect, we can tailor a fragment of [4] to our situation. Since $|\mathrm{Hom}(A,A)| \geq 5$, we can fix one of its elements as $0_A = 0_{AA}$, distinct from a and 1_A, and two further morphisms $b, c \in \mathrm{Hom}(A,A) \setminus \{a, 0_A, 1_A\}$, $b \neq c$. Let us define a multiplication within $\mathrm{Hom}(A,A)$ by the rules $aa = b$, $ab = c$, $1_A x = x 1_A = x$, and let $xy = 0_A$ otherwise. Then $(aa)a = ba = 0_A$ and $a(aa) = ab = c$, so (a,a,a) is non-associative. Now let $(x,y,z) \in \mathrm{Hom}(A,A)^3$ be a non-associative admissible triplet. It is necessarily a nontrivial one. Let us call a morphism u *reducible* if it has a decomposition as a product of two non-unit factors, i.e., if $u = vw$ such that none of v and w is a unit. If v and w are uniquely determined, then they will be referred to as the *unique factors* of u. Since any product of non-unit factors belongs to $\{b, c, 0_A\}$, if (x,y,z) is non-associative then $\{(xy)z, x(yz)\} \cap \{b,c\}$ is non-empty. The unique factors of b, both being a, are irreducible, therefore $b \notin \{(xy)z, x(yz)\}$. The first unique factor of c is irreducible again, so necessarily $x(yz) = c$, $x = a$ and $yz = b$. Thus $x = y = z = a$. This shows that (a,a,a) is the only non-associative triplet in the groupoid $(\mathrm{Hom}(A,A),\cdot)$, and we can conclude this case exactly the same way as the first case.

The third case is when we consider a nontrivial admissible triplet (a,b,c) such that $a \in \mathrm{Hom}(A,B)$, $b \in \mathrm{Hom}(B,C)$, $c \in \mathrm{Hom}(C,D)$ and $|\{A,B,C,D\}| \geq 2$. As previously, we fix a morphism $0_{XY} \in \mathrm{Hom}(X,Y)$ for each choice of $X, Y \in \mathbf{Ob}(\mathcal{C})$ provided $\mathrm{Hom}(X,Y)$ is non-empty; these 0_{XY} will be called zero morphisms. When $X = Y$ then 0_{XY} and 1_X must be distinct. Let us choose nonzero morphisms $d \in \mathrm{Hom}(A,C)$ and $e \in \mathrm{Hom}(A,D)$ such that a, b, c, d, e are pairwise distinct and none of them is a unit or a zero morphism. It is not quite obvious that this choice is possible, but we can use the fact that distinct "Hom sets" are disjoint, $\mathrm{Hom}(X,X)$ and $\mathrm{Hom}(Y,Y)$ have at least five elements, and $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(Y,X)$ have at least four elements. For example, if $(A,B,C,D) = (X,X,Y,Y)$ with $X \neq Y$ then $\{0_{XY}, b, d, e\}$ is only a four element subset of $\mathrm{Hom}(X,Y)$. There are other cases when $|\{A,B,C,D\}| = 2$; up to duality and notations all of them are listed in Table 1 where $X \neq Y$. The table contains the "necessary" subsets of $\mathrm{Hom}(X,X)$ and $\mathrm{Hom}(Y,Y)$ resp. $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(Y,X)$, and it appears that none of them has more than five resp. four elements.

(A,B,C,D)	$\mathrm{Hom}(X,X)$	$\mathrm{Hom}(X,Y)$	$\mathrm{Hom}(Y,X)$	$\mathrm{Hom}(Y,Y)$
(X,X,X,Y)	$\{0_X,1_X,a,b,d\}$	$\{0_{XY},c,e\}$	$\{0_{YX}\}$	$\{0_Y,1_Y\}$
(X,X,Y,X)	$\{0_X,1_X,a,e\}$	$\{0_{XY},b,d\}$	$\{0_{YX},c\}$	$\{0_Y,1_Y\}$
(X,X,Y,Y)	$\{0_X,1_X,a\}$	$\{0_{XY},b,d,e\}$	$\{0_{YX}\}$	$\{0_Y,1_Y,c\}$
(X,Y,X,Y)	$\{0_X,1_X,d\}$	$\{0_{XY},a,c,e\}$	$\{0_{YX},b\}$	$\{0_Y,1_Y\}$
(X,Y,Y,X)	$\{0_X,1_X,e\}$	$\{0_{XY},a,d\}$	$\{0_{YX},c\}$	$\{0_Y,1_Y,b\}$

Table 1

Now that we have settled the case $|\{A,B,C,D\}|=2$; it is obvious that the choice of our morphisms is possible when $|\{A,B,C,D\}|>2$.

We let $ab=d$, $dc=e$, and we define any other product of non-unit factors as zero. This way we obtain a categoroid $(\mathcal{C},\cdot)$. Notice that d and e are the only nonzero reducible morphisms, and each of them has unique factors. Now let (u,v,w) be a nontrivial non-associative admissible triplet. Then at least one of the morphisms $(uv)w$ and $u(vw)$ is nonzero. Since this morphism is reducible, it is in $\{d,e\}$. We can exclude d, whose unique factors are irreducible, so $e\in\{(uv)w,\ u(vw)\}$. Since the second unique factor of e is irreducible again, $e=(uv)w$, and we easily obtain that $(u,v,w)=(a,b,c)$. Hence (a,b,c) is the only non-associative admissible triplet. $\diamond$

Proof of Assertion 1. We may assume that $|\mathrm{Hom}(A,A)|\geq 5$ and $|\mathrm{Hom}(B,B)|\geq 5$, for otherwise Proposition 1 applied for $\mathrm{Hom}(A,A)$ or $\mathrm{Hom}(B,B)$ would produce a non-isolated triplet. Hence we can pick two morphisms: $a\in\mathrm{Hom}(A,A)\setminus\{1_A\}$ and $b\in\mathrm{Hom}(B,B)\setminus\{1_B\}$. If $\mathrm{Hom}(A,B)=\{c\}$ is a singleton then $(ac)b,\ a(cb)\in\mathrm{Hom}(A,B)$ gives that (a,c,b) is an associative triplet, so it cannot be isolated.

Now let $\mathrm{Hom}(A,B)=\{c,d\}$, and we claim that the triplet (a,c,b) is not isolated. Suppose the contrary, i.e., let (a,c,b) be the only non-associative triplet in an appropriate categoroid $(\mathcal{C},\cdot)$. Then $(ac)b\neq a(cb)$, so $\{(ac)b,a(cb)\}=\{c,d\}$. By duality, we can assume that $(ac)b=c$ and $a(cb)=d$. We will often use the fact that all triplets but (a,c,b) are associative. If we had $ac=c$ then $cb=(ac)b=c$ would yield $c=ac=a(cb)=d$, a contradiction. Therefore $ac=d$, whence $db=c$. Since $ad=d$ would imply $c=db=(ad)b=a(db)=ac=d$, which is impossible, it follows that $ad=c$. From $cb=d$ we would obtain $d=a(cb)=ad=c$, a contradiction. Thus $cb=c$. Now the previously established equations lead to $c=cb=(ad)b=a(db)=ac=d$, a final contradiction. $\diamond$

Proof of Proposition 1. Suppose (a,a,a) is isolated. Then there is a groupoid $(A,\cdot,1)$ with unit element 1 such that $(a,a,a)\in A^3=\{a,b,c,1\}^3$ is the only non-associative triplet; we are looking for a contradiction. Clearly, $aa\neq a$ and $aa\neq 1$. In the forthcoming computations we will often use the fact that all triplets distinct from (a,a,a) are associative. Several cases have to be distinguished

Case 1: $(aa)a=a$. Then $u:=a(aa)\in\{b,c,1\}$.

Subcase 1.1: $aa=u$. Then $ua=a$, $au=u$, so

$$(au)a=ua=a\neq u=aa=a(ua)$$

contradicts the fact that (a,u,a) is associative.

Subcase 1.2: $aa = v \in A \setminus \{1,a,u\}$. Then $va = a,$, $av = u$, so

$$(au)a = \big(a(av)\big)a = \big((aa)v\big)a = (vv)a = v(va) = va = a \neq$$
$$\neq u = av = a(aa) = a\big(a(va)\big) = a\big((av)a\big) = a(ua)$$

contradicts the associativity of (a,u,a).

Now, by duality, we have settled the case when $a \in \{(aa)a, a(aa)\}$.

Case 2: $(aa)a = 1$ and, say, $a(aa) = b$.

Subcase 2.1: $aa = b$. Then $ba = 1$ and $ab = b$, so

$$(ab)a = ba = 1 \neq a = a1 = a(ba)$$

contradicts the associativity of (a,b,a).

Subcase 2.2: $aa = c$. Then $ca = 1$, $ac = b$, so

$$(ca)b = 1b = b \neq 1 = ca = c(1a) = c\big((ca)a\big) = c\big(c(aa)\big) = c(cc) =$$
$$c\big((aa)c\big) = c\big(a(ac)\big) = c(ab)$$

contradicts the associativity of (c,a,b).

We have settled the case when $1 \in \{(aa)a, a(aa)\}$. It remains, up to b — c symmetry, the following case.

Case 3: $(aa)a = b$ and $a(aa) = c$.

Subcase 3.1: $aa = b$. Then $ba = b$, $ab = c$, so

$$(aa)c = bc = b(ab) = (ba)b = bb = b(aa) = (ba)a = ba = b \neq$$
$$\neq c = ab = a(ba) = a\big((ba)a\big) = a\big(b(aa)\big) =$$
$$= a(bb) = a\big((aa)b\big) = a\big(a(ab)\big) = a(ac)$$

contradicts the associativity of (a,a,c).

Subcase 3.2: $aa = c$. Then $ca = b$, $ac = c$, so

$$(ba)a = \big((ca)a\big)a = \big(c(aa)\big)a = (cc)a =$$
$$\big((aa)c\big)a = \big(a(ac)\big)a = (ac)a = ca = b \neq$$
$$\neq c = ac = a(ac) = (aa)c = cc = c(ac) = (ca)c = bc = b(aa)$$

contradicts the associativity of (b,a,a). ◇

References

[1] L. Klukovits , On the independence of the mediality conditions , Contributions to general algebra, 2 (Klagenfurt, 1982), Hölder-Pichler-Tempsky, Vienna , 1983 , 209–214 .

[2] L. Klukovits , The independence of the distributivity conditions in groupoids , Acta Sci. Math. (Szeged) , 49 , 1985 , 93–100 .

[3] L. Klukovits , The independence of regular identities in groupoids , Kobe J. Math. , 6 , 1989 , 189–200 .

[4] G. Szász , The independence of associative conditions (Hungarian) , Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei , 3 , 1953 , 569–577 .

[5] G. Szász , Die Unabhängigkeit der Assoziativitätsbedingungen (German) , Acta Sci. Math. (Szeged) , 15 , 1953 , 20–28 .

[6] G. Szász , Über die Unabhängigkeit der Assoziativitätsbedingungen kommutativer multiplikativer Strukturen (German) , Acta Scientiarum Mathematicarum (Szeged) , 15 , 1953 , 130–142 .

[7] R. Wiegandt and J. Wiesenbauer , Unabhängigkeit der Distributivitätsbedinungen , Acta Accad. Naz. Lincei , 58 , 1975 , 819–822 .