A CLASS OF CLONES ON COUNTABLE SETS ARISING FROM IDEALS

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Let X be a non-empty set. By $\mathbb{O}(X)$ we denote the set of all finitary operations on X, i.e. all functions $X^n \to X$ of arbitrary arity n. A *clone* on X is a subset of $\mathbb{O}(X)$ which contains all projections, i.e. all mappings $(x_1, \ldots, x_n) \mapsto x_i$ for arbitrary $1 \le i \le n$, and which is closed under superposition, i.e. together with $f : X^n \to X$ and $g_1, \ldots, g_n : X^m \to X$ the function $(x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$ must belong to the clone. In the sequel this superposition will be written as $f(g_1, \ldots, g_n)$.

It is well-known that the collection $\mathfrak{L}(X)$ of all clones on X forms an algebraic lattice under inclusion. If X is finite, much is known about these lattices, but very little in the infinite case. One of the major open questions is whether or not $\mathfrak{L}(X)$ is always dually atomic. It was known for a while ([1], [3]) that for infinite sets X there is the maximal possible number, namely $2^{2^{|X|}}$, of coatoms (from now on maximal clones) in $\mathfrak{L}(X)$. Until very recently the proof of this fact was rather indirect. In [2] Goldstern and Shelah gave the first explicit construction of that many maximal clones using prime ideals on X. It was their proof that led us to the question what happens when arbitrary ideals are used.

Let I be an ideal on X (more precisely of the power-set Boolean algebra $P(X)$). Then it is very easily checked that

$$\mathbb{C}_I := \{f \in \mathbb{O}(X) : f[A^{n_f}] \in I \text{ for all } A \in I\}$$

is a clone on X. Here n_f means the arity n of f and $f[A^n]$ is the usual image $\{f(a_1, \ldots, a_n) : a_1, \ldots, a_n \in A\}$.

In this note we investigate the position of these clones in the lattice $\mathfrak{L}(X)$ and, in particular, the question whether they are maximal. It turns out that this is often but not always the case.

Date: February 6, 2001.

1991 *Mathematics Subject Classification.* 08A40.

Key words and phrases. Clone, maximal clone, countable set, Heindorf.

The research of the first author was partially supported by the NFSR of Hungary (OTKA), grant no. T023186, T022867 and T026243, and also by the Hungarian Ministry of Education, grant no. FKFP 1259/1997.

We restrict our attention to countably infinite sets X. The more complicated uncountable case will be dealt with in a future paper.

The countability assumption will (always tacitly) be used for choosing functions that map one arbitrarily given infinite subset A of X onto any given non-empty $B \subseteq X$.

1. Some first observations

For the rest of the paper X is a countably infinite set and I and J denote nontrivial ideals on X. The trivial ideal $\{\emptyset\}$ gives rise to the full clone $\mathbb{O}(X)$ and is, therefore, of little interest.

Nontrivial ideals include some, possibly all, singletons. We call the set $\{y \in X : \{y\} \in I\} = \bigcup I$ the *support* of I. If $\bigcup I = X$, we say that I has *full support*. The support of I can also be characterized as the only subset Y of X for which $P_{fin}(Y) \subseteq I \subseteq P(Y)$ holds. Here and in the following $P_{fin}(Y)$ denotes the collection of all finite subsets of Y.

Proposition 1. *Assume $\mathbb{C}_I = \mathbb{C}_J$ for nontrivial ideals $I \neq J$. Then there is a non-empty subset Y of X such that $\{I, J\} = \{P_{fin}(Y), P(Y)\}$.*

Proof. The ideals I and J must have the same support which we denote by Y. This is because $\{y\} \in I$ iff the constant function with value y belongs to $\mathbb{C}_I$. As mentioned above, $P_{fin}(Y) \subseteq I, J \subseteq P(Y)$.

Claim 1. If $I \neq P_{fin}(Y)$, then $J \subseteq I$.

Take some infinite $A \in I$. Let $B \in J$ be arbitrary. We prove $B \in I$. This is clear if $B = \emptyset$. Otherwise we can take a function $f : X \to B$ such that $f[A] = B$. Being in $\mathbb{C}_J$ (all values are in a set from J!), f must be in $\mathbb{C}_I$, too. This forces B into I and proves the claim.

It implies that one of the ideals, say I, must be $P_{fin}(Y)$. Because if both were different from $P_{fin}(Y)$, they would be subsets of each other, i.e. equal.

It remains to see that J is $P(Y)$ or, equivalently, that $Y \in J$. Being not equal to $I = P_{fin}(Y)$, J contains an infinite set B. Fix a function $g : X \to Y$ such that $f[B] = Y$. Then $f \in \mathbb{C}_{P_{fin}(Y)} = \mathbb{C}_J$, which implies $Y = f[B] \in J$. ☐

Call a function $f : X^n \to X$ *almost conservative* iff there is a finite set E_f such that $f(x_1, \dots, x_n) \in \{x_1, \dots, x_n\} \cup E_f$ for all $x_1, \dots, x_n \in X$. If $E_f = \emptyset$ works, then f will be called *conservative*.

It is quite easy to see that a function is conservative iff it belongs to all $\mathbb{C}_I$. A little more interesting is the following

Proposition 2. *A function is almost conservative iff it belongs to all $\mathbb{C}_I$ with $P_{fin}(X) \subseteq I$.*

Proof. It should be clear that all almost conservative functions belong to all those $\mathbb{C}_I$.

To establish the converse we need a simple combinatorial fact. As it will be used a second time much later, we honour it with a name (but skip the obvious proof).

Lemma 1 (Thinning-out Lemma). *Every sequence $(a_n^1, a_n^2, \ldots, a_n^m)_{n=0}^\infty$ of m tuples has a subsequence $(a_{n_k}^1, a_{n_k}^2, \ldots, a_{n_k}^m)_{k=0}^\infty$ such that for each $i = 1, 2, \ldots, m$ the coordinate sequence $(a_{n_k}^i)_{k=0}^\infty$ is either constant or injective.*

Now consider a function f which is not almost conservative. For simplicity of notation we assume it binary. The aim is to construct an ideal such that $f \notin \mathbb{C}_I$. More precisely, we find some $A \subseteq X$ such that $f[A^2] \setminus A$ is infinite. Then the ideal $I := \{B \subseteq X : B \setminus A \text{ is finite}\}$ does the required job.

First, there are x_0, y_0 such that $f(x_0, y_0) \notin \{x_0, y_0\}$. Then choose recursively x_{n+1}, y_{n+1} in such a way that

$$f(x_{n+1}, y_{n+1}) \notin \{x_{n+1}, y_{n+1}\} \cup \{x_0, \ldots, x_n, y_0, \ldots, y_n, f(x_0, y_0), \ldots, f(x_n, y_n)\}$$

This is always possible, for otherwise f were almost conservative.

By Thinning-out Lemma we can further assume that one of three cases takes place: either both sequences $(x_n), (y_n)$ are injective or one is injective and the other is constant. They cannot both be constant, for all the values $f(x_n, y_n)$ are pairwise distinct, by construction.

Put $C := \bigcup_{n=0}^\infty \{x_n, y_n\}$. If $f[C^2] \setminus C$ is infinite, then we are done. Otherwise, there is some n_0 such that for all $n > n_0$ there is some m such that $f(x_n, y_n) \in \{x_m, y_m\}$. In all three of the possible cases, there can be at least two such m (but $f(x_n, y_n) = x_{m_1} = y_{m_2}$ is possible). By construction, we must have $n < m$. This makes it possible to choose another subsequence by recursion: Take n_0 as given above and choose n_{k+1} in such a way that $n_{k+1} > n_k$ and $\{x_{n_{k+1}}, y_{n_{k+1}}\} \cap \{f(x_{n_0}, y_{n_0}), \ldots, f(x_{n_k}, y_{n_k})\} = \emptyset$. Putting $A := \bigcup_{k=0}^\infty \{x_{n_k}, y_{n_k}\}$ now we have forced $f[A^2] \setminus A$ to be infinite, which completes the proof of Proposition 2. □

2. A maximality test and some applications

Theorem 1. *Assume that I is a nontrivial ideal on the countably infinite set X and let Y denote its support.*

(1) *If Y is a proper subset of X, then the clone $\mathbb{C}_I$ is maximal (in $\mathbb{O}(X)$) iff $I \in \{P_{fin}(Y), P(Y)\}$.*

(2) *If I has full support, then $\mathbb{C}_I$ is maximal iff $P_{fin}(X) \subset I \subset P(X)$ and for each set $B \notin I$ there exists some $f \in \mathbb{C}_I$ such that $f[B^{n_f}] = X$.*

(3) *$\mathbb{C}_I$ is maximal in $\mathbb{C}_{P(Y)}$ iff $P_{fin}(Y) \subset I \subset P(Y)$ and for each set $B \subseteq Y$ such that $B \notin I$ there exists some $f \in \mathbb{C}_I$ such that $f[B^{n_f}] = Y$.*

Proof. (1) It is easily checked that

$$\mathbb{C}_I \subseteq \{f \in \mathbb{O}(X) : f[Y^{n_f}] \subseteq Y\} = \mathbb{C}_{P(Y)}.$$

The latter clone is known (e.g. 6.1 in [4]), to be maximal (for $\emptyset \neq Y \neq X$). So $\mathbb{C}_I$ will be maximal iff it coincides with that clone. By Proposition 1 this takes place iff $I \in \{P_{fin}(Y), P(Y)\}$.

(2) being a special case we go over to (3). Assume first that $\mathbb{C}_I$ is maximal in $\mathbb{C}_{P(Y)}$, which, by Proposition 1, is equal to $\mathbb{C}_{P_{fin}(Y)}$. It follows that the inclusions $P_{fin}(Y) \subset I \subset P(Y)$ must be strict.

Next we assume that the second condition in (3) is not satisfied and show that $\mathbb{C}_I$ can be strictly extended to some $\mathbb{C}_J \subset \mathbb{C}_{P(Y)}$.

Let $B \subseteq Y, B \notin I$ be such that $f[B^{n_f}] \neq Y$ for any $f \in \mathbb{C}_I$ and put

$$J := \{A \subseteq X : A \subseteq f[B^{n_f}] \text{ for some } f \in \mathbb{C}_I\}.$$

Claim 2. J is an ideal such that $P_{fin}(Y) \subset J \subset P(Y)$.

By its very definition, J is downward closed. To see that it is closed under unions, assume that $A_1 \subseteq f_1[B^n], A_2 \subseteq f_2[B^m]$ with $f_1, f_2 \in \mathbb{C}_I$. Given that the switching function s with

$$s(x, y, u, v) = \begin{cases} u, & \text{if } x = y \\ v, & \text{if } x \neq y \end{cases}$$

is conservative, it belongs to $\mathbb{C}_I$, as does the $2 + m + n$-ary function

$$g(x, y, u_1, \ldots, u_n, v_1, \ldots, v_m) := s(x, y, f_1(u_1, \ldots, u_n), f_2(v_1, \ldots, v_m)).$$

It follows that $A_1 \cup A_2 \subseteq f_1[B^n] \cup f_2[B^m] = g[B^{2+n+m}]$ is in J, as desired.

Constant functions show that all singletons $\{y\}$ are in J. $\mathbb{C}_I$-functions map $B \subseteq Y$ into Y. So, we have $P_{fin}(Y) \subseteq J \subseteq P(Y)$. The identity function puts the (infinite because not in I) set B in J and, by assumption, $Y \notin J$. Therefore, the inclusions are strict. This proves Claim 2.

Claim 3. $\mathbb{C}_I \subseteq \mathbb{C}_J \subseteq \mathbb{C}_{P(Y)}$.

The first inclusion is trivial, because if $A \subseteq f[B^n]$ and $g : X^m \to X$ with $f, g \in \mathbb{C}_I$, then $g[A^m] \subseteq g[f[B^n]^m] = h[B^{mn}]$ for the $\mathbb{C}_I$-function

$$h(x_1^1, \ldots, x_n^1, \ldots, x_1^m, \ldots, x_n^m) := g(f(x_1^1, \ldots, x_n^1), \ldots, f(x_1^m, \ldots, x_n^m)).$$

The second inclusion is clear from $J \subseteq P(Y)$ and definitions. This proves Claim 3.

Now it remains to apply Proposition 1. From $B \in J \setminus I$ and $J \neq \mathbb{C}_{P(Y)}$ we get $\mathbb{C}_I \neq \mathbb{C}_J \neq \mathbb{C}_{P(Y)}$.

Finally we prove the sufficiency of condition (3). This needs the following lemma.

Lemma 2. *Suppose I has support Y and $f : X^n \to X$ is a function in $\mathbb{C}_{P(Y)}$ such that $f[A^n] = Y$ for some $A \in I$. Then f together with $\mathbb{C}_I$ generates $\mathbb{C}_{P(Y)}$.*

Proof. Let $h : X^m \to X$ be such that $h[Y^m] \subseteq Y$. We have to represent it as a superposition of f and functions from $\mathbb{C}_I$. To achieve this we choose functions $g_1, \dots, g_n : X \to A$ such that $f(g_1(y), \dots, g_n(y)) = y$ for all $y \in Y$. Then $h(\bar{y}) = f(g_1(h(\bar{y})), \dots, g_n(h(\bar{y})))$ for all $\bar{y} \in Y^m$. To check that all $g_i(h)$ belong to $\mathbb{C}_I$, let $B \in I$ be arbitrary. Then $g_i(h)[B^m] \subseteq g_i[X] \subseteq A \in I$.

Next we consider the $m+1$-ary function $\tilde{h}$ defined by

$$\tilde{h}(\bar{x}, u) := \begin{cases} u, & \text{if } \bar{x} \in Y^m \text{ and } u \in Y \\ h(\bar{x}), & \text{otherwise.} \end{cases}$$

Assuming $B \in I$ we have $B \subseteq Y$, hence $\tilde{h}[B^{m+1}] = B \in I$. It follows that $\tilde{h} \in \mathbb{C}_I$. It remains to observe that $h(\bar{x}) = \tilde{h}(\bar{x}, f(g_1(h(\bar{x})), \dots, g_n(h(\bar{x}))))$. The lemma is proved. □

Assume now that I satisfies condition (3) and consider some, say m-ary, function $g \in \mathbb{C}_{P(Y)} \setminus \mathbb{C}_I$. We have to show that $\mathbb{C}_I$ together with g generates every function preserving Y. The lemma reduces this task to generating a function h that maps some power of some $A \in I$ to the whole of Y.

The function g not being in $\mathbb{C}_I$, there exists some $A \in I$ such that $B := g[A^m] \notin I$. As $A \subseteq Y$, we have $B = g[A^m] \subseteq g[Y^m] \subseteq Y$. Therefore, the condition yields some $f \in \mathbb{C}_I$ such that $f[B^n] = Y$. It follows that the mn-ary function

$$h(x_1^1, \dots, x_m^1, \dots, x_1^n, \dots, x_m^n) := f(g(x_1^1, \dots, x_m^1), \dots, g(x_1^n, \dots, x_m^n))$$

maps A^{mn} to Y. □

From the above proof and notably Lemma 2 we get a

Corollary 1. *Except for $I \in \{\{\emptyset\}, P_{fin}(X), P(X)\}$, each clone of the form $\mathbb{C}_I$ can be extended to a maximal clone.*

In the exceptional cases $\mathbb{C}_I$ equals $\mathbb{O}(X)$, so there is no chance.

Proof. If I does not have full support, then part (1) in the proof of Theorem 1 shows that $\mathbb{C}_I \subseteq \mathbb{C}_{P(Y)}$, which is maximal.

If I has full support, then $P_{fin}(X) \subseteq I$, but there must be an infinite $A \in I$, because $I \neq P_{fin}(X)$. Take a mapping $f : X \to X$ such that $f[A] = X$. It cannot be in $\mathbb{C}_I$, because $X \notin I$. By the lemma, $\mathbb{C}_I$ together with f generates everything.

The rest of the argument is standard. Apply Zorn's Lemma to the collection of all clones $\mathbb{A}$ such that $\mathbb{C}_I \subseteq \mathbb{A} \not\ni f$. A maximal element in that collection is easily seen to be a maximal clone above $\mathbb{C}_I$. □

Problem 1. Can every $\mathbb{C}_I$ be extended to a maximal clone, which is of the form $\mathbb{C}_J$?

As our first **application** of the theorem we reprove (the countable version of) the Goldstern-Shelah result.

Theorem 2 (Goldstern and Shelah [2]). *If Q is a prime ideal on a countable set X, then $\mathbb{C}_Q$ is a maximal clone.*

Proof. If Q is principal, i.e. $Q = P(X \setminus \{x_0\})$ for some $x_0 \in X$, then $\mathbb{C}_Q$ is maximal, because condition (1) of the theorem is satisfied. So assume that Q is non-principal, then $P_{fin} \subset Q \subset P(X)$ and we have to map an arbitrary $A \notin Q$ by a $\mathbb{C}_Q$-function to the whole of X. As Q is non-principal, A is infinite and we can split it into two disjoint infinite parts: $A = A_1 \cup A_2$. Exactly one of them is in Q (this is what prime means), say $A_2 \in Q \not\ni A_1$. Notice that $X \setminus A_1$ also belongs to Q. Choose a function $f : X \to X$ that is identical on A_1, maps A_2 onto $X \setminus A_1$ and $X \setminus A$ (if it is non-empty) somehow into $X \setminus A_1$.

By construction, this f maps A onto X. To see that it belongs to $\mathbb{C}_Q$ consider any $B \in Q$. Then

$$f[B] = f[B \cap A_1] \cup f[B \cap A_2] \cup f[B \setminus A] \subseteq B \cup (X \setminus A_1) \in Q$$

proving the result. □

Next we establish a 'metaexample' which will have some interesting special cases.

Lemma 3. *Assume that $P_{fin}(X) \subset I$ and that each $A \notin I$ contains an infinite B such that no infinite subset of B is in I. Then $\mathbb{C}_I$ is maximal.*

Proof. Let $A \notin I$ be given and choose $B \subseteq A$ according to the assumption. Choose $f : X \to X$ in such a way that $f[B] = X$ and $f[X \setminus B]$ is a singleton. Then $f[A] \supseteq f[B] = X$ and for each $C \in I$, the intersection $B \cap C$ is finite, hence $f[C] = f[B \cap C] \cup f[C \setminus B]$ is finite, too. So $f \in \mathbb{C}_I$ as desired. □

Corollary 2. *If $P_{fin}(X) \subset I$ and I is countably generated, then $\mathbb{C}_I$ is maximal.*

Proof. Indeed, if I is countably generated, then there is some increasing chain $C_0 \subseteq C_1 \subseteq C_2 \subseteq \dots$ of infinite sets such that

$$\begin{aligned} I &= \{A \subseteq X : A \subseteq C_n \text{ for some } n\} = \\ &= \{A \subseteq X : A \setminus C_n \text{ is finite for some } n\}. \end{aligned}$$

Assume that $A \notin I$. Take $a_0 \in A$ arbitrarily and choose recursively $a_{n+1} \in A \setminus [C_n \cup \{a_0, \dots, a_n\}]$. Then $B := \{a_0, a_1, \dots\}$ has the properties assumed in Lemma 2. □

It is well-known that for a finite set X and an arbitrary total order on it the set of all monotone functions is a maximal clone. For infinite X this is wrong, as easily follows from

Corollary 3. *Let $\leq$ be a total order on X which is not a well-ordering. Let I be the ideal of all well-ordered subsets of X. Then $\mathbb{C}_I$ is maximal.*

Proof. Indeed, if A is not well ordered, it contains an infinite strictly decreasing sequence. Its members form the desired B. □

3. A non-maximal example

Having so many maximal clones of the form $\mathbb{C}_I$ with $I \supset P_{fin}(X)$ one is tempted to conjecture that $\mathbb{C}_I$ is always maximal. Here we give an example showing that this is not true.

We fix a decomposition $X = \bigcup_{\varphi \in \Phi} X_\varphi$ of X into infinitely many pairwise disjoint infinite subsets. Here Φ is some index set and we denote the unique φ such that $x \in X_\varphi$ by $\varphi(x)$. Next we define two ideals

$$\begin{aligned} I &:= \{A \subseteq X : \text{ there is some natural number } n \\ &\qquad\qquad\qquad \text{ such that } |A \cap X_\varphi| \leq n \text{ for all } \varphi\} \\ J &:= \{A \subseteq X : A \cap X_\varphi \text{ is finite for all } \varphi\} \end{aligned}$$

Clearly, $P_{fin}(X) \subset I \subset J \subset P(X)$. We show that $\mathbb{C}_I \subseteq \mathbb{C}_J$. From Proposition 1 it then follows that $\mathbb{C}_I \subset \mathbb{C}_J \subset \mathbb{C}_{P(X)} = \mathbb{O}(X)$, so I is the promised counterexample.

Let some k-ary $f \in \mathbb{C}_I$ and some $B \in J$ be given. We have to show that $f[B^k] \in J$. Striving for a contradiction, we assume that this is not true. Then there is some $\beta \in \Phi$ and an infinite sequence $(b_n^1, \dots, b_n^k)_{n=0}^\infty$ of elements of B^k such that the images $f(b_n^1, \dots, b_n^k)$ are pairwise distinct elements of X_β.

An application of Thinning-out Lemma to the sequence $(\varphi(b_n^1), \dots, \varphi(b_n^k))_{n=0}^\infty$ allows us to assume that for $i = 1, \dots, k$ all sequences

$\varphi(b_n^i)_{n=0}^{\infty}$ are either constant or injective. For notational simplicity, we assume that the former takes place for $i = 1, \dots m$ and the latter for $i = m+1, \dots k$. Let $\alpha_1, \dots, \alpha_m$ be the corresponding constant values.

As $B \in J$, the intersections $B \cap X_{\alpha_i}$ are finite, with n_i elements, say. We put $n := n_1 + \dots + n_m + (k - m)$ and $D := \bigcup_{n=0}^{\infty} \{b_n^1, \dots, b_n^k\}$.

Claim 4. $|D \cap X_\varphi| \le n$ for all φ.

Indeed, if φ equals α_i, then from $D \subseteq B$ we get $|D \cap X_\varphi| \le |B \cap X_{\alpha_i}| = n_i \le n$.

Now, let φ be distinct from all α_i. For b_n^i to belong to X_φ means $\varphi(b_n^i) = \varphi$ and this is possible only if $i > m$. Moreover, for each i there can be at most one n such that $\varphi(b_n^i) = \varphi$. This shows that $\varphi(b_n^i) = \varphi$ is possible for at most $k - m \le n$ elements, which proves our claim.

It shows that $D \in I$. On the other hand $f[D^k]$ is not even in J, because it contains the infinitely many $f(b_n^1, \dots, b_n^k)$ of X_β. This contradicts the assumption $f \in \mathbb{C}_I$ and ends our proof.

Remark 1. Using Lemma 2 it is easily seen that $\mathbb{C}_J$ is maximal.

Remark 2. It would be possible to establish the non-maximality of $\mathbb{C}_I$ by referring to our theorem. Essentially the same argument as above shows that no B that has finite intersection with all X_φ can be mapped onto X by a $\mathbb{C}_I$-function. The ideal J that was constructed in the proof of the theorem is identical with our J above; it does not depend on B.

Problem 2. We do not know if the constructed $\mathbb{C}_I \subset \mathbb{C}_J$ is a covering in $\mathfrak{L}(X)$.

Problem 3. Is every clone of the form $\mathbb{C}_I$ covered by a clone of the same type?

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