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Symmetric embeddings of free lattices into each other

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Dedicated to Ralph Freese and J. B. Nation on their seventieth birthdays

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Abstract. By a 1941 result of Ph. M. Whitman, the free lattice FL(3) on three generators includes a sublattice S that is isomorphic to the lattice $\mathrm{FL}(\omega) = \mathrm{FL}(\aleph_0)$ generated freely by denumerably many elements. The first author has recently "symmetrized" this classical result by constructing a sublattice $S \cong \mathrm{FL}(\omega)$ of FL(3) such that S is *selfdually positioned* in FL(3) in the sense that it is invariant under the natural dual automorphism of FL(3) that keeps each of the three free generators fixed. Now we move to the furthest in terms of symmetry by constructing a selfdually positioned sublattice $S \cong \mathrm{FL}(\omega)$ of FL(3) such that every element of S is fixed by all automorphisms of FL(3). That is, in our terminology, we embed $\mathrm{FL}(\omega)$ into FL(3) in a *totally symmetric* way. Our main result determines all pairs (κ, λ) of cardinals greater than 2 such that $\mathrm{FL}(\kappa)$ is embeddable into $\mathrm{FL}(\lambda)$ in a totally symmetric way. Also, we relax the stipulations on $S \cong \mathrm{FL}(\kappa)$ by requiring only that S is closed with respect to the automorphisms of $\mathrm{FL}(\lambda)$, or S is selfdually positioned and closed with respect to the automorphisms; we determine the corresponding pairs (κ, λ) even in these two cases. We reaffirm some of our calculations with a computer program developed by the first author. This program is for the word problem of free lattices, it runs under Windows, and it is freely available.

1. Introduction and our results

There are many nice and deep results on free lattices of the variety of all lattices. A large part of these results were achieved by Ralph Freese and J. B. Nation, to whom this paper is dedicated. Some of these results are included in [7, 8, 10, 11, 14, 15] and in the monograph Freese, Ježek, and

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Nation [9], but this list is far from being complete. The monograph just mentioned serves as the reference book for the present paper.

By a classical result of Whitman [20], the free lattice $\mathrm{FL}(\omega) = \mathrm{FL}(\aleph_0)$ on denumerably many free generators is isomorphic to a sublattice of the free lattice FL(3) with three free generators. In fact, we know from a deep result of Tschantz [19] that there are many copies of $\mathrm{FL}(\omega)$ in FL(3); namely, every infinite interval of FL(3) includes a sublattice isomorphic to $\mathrm{FL}(\omega)$. For more about free lattices, the reader is referred to Freese, Ježek and Nation [9]. In this paper, we embed free lattices into each other *symmetrically.* For a free lattice F,

$$\begin{array}{l}\delta = \delta_F \text{ will denote the } \textit{natural} \text{ dual automorphism} \\ \text{of } F \text{ that keeps the free generators fixed;}\end{array} \tag{1.1}$$

it is uniquely determined. A subset (or a sublattice) S of F is *selfdually positioned* if $\delta(S) = S$. Selfduality is a sort of symmetry, and a selfdually positioned sublattice is necessarily selfdual. As the main result of [2], the first author proved that

$$\begin{array}{l}\mathrm{FL}(3) \text{ has a selfdually positioned sublattice} \\ \text{that is isomorphic to } \mathrm{FL}(\omega).\end{array} \tag{1.2}$$

Besides selfduality, there is a more general concept of symmetry, which is used even outside algebra; it is based on automorphisms. For a lattice L, let $\mathrm{Aut}(L)$ denote the *automorphism group* of L. We call a subset (or a sublattice) S of L *symmetric* if $\pi(S) = S$ for every $\pi \in \mathrm{Aut}(F)$. Also, an element $u \in L$ is a *symmetric element* of L if $\{u\}$ is a symmetric subset of L. Note that there is no symmetric element in $\mathrm{FL}(\omega)$. If S contains only symmetric elements of L, then S is *element-wise symmetric* in L. Our key concept is the following; Theorem 1.2 and Remark 2.1 will explain why.

Definition 1.1. A lattice embedding $\varphi\colon L \to F$ of a lattice L into a free lattice F is *totally symmetric* if its range $\varphi(L) = \{\varphi(u) : u \in L\}$ is a selfdually positioned and element-wise symmetric sublattice of F.

To expand our notation for all *cardinal* numbers $\kappa \geq 3$, we denote by $\mathrm{FL}(\kappa)$ the free lattice with κ many free generators. If $\kappa = n$ is a natural number, then we often write $\mathrm{FL}(n)$. Following the tradition, we often denote $\mathrm{FL}(\aleph_0)$ by $\mathrm{FL}(\omega)$. In order to avoid ambiguity about natural numbers, we adhere to the notations $\mathbb{N}^+ := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}^+$. The elements of $\mathbb{N}_0$ are also cardinals; namely, the finite cardinal numbers. Our main result is the following.

Theorem 1.2.

(A) *Assuming that* $3 \leq \kappa$ *and* $3 \leq \lambda$ *are cardinal numbers, there exists a totally symmetric embedding* $\mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ *if and only if* $\lambda \in \mathbb{N}^+$ *is a natural number and* $\kappa \in \{2k : k \in \mathbb{N}^+\} \cup \{\aleph_0\}$.

(B) *In particular, there exists a totally symmetric embedding* $\mathrm{FL}(\omega) \to \mathrm{FL}(3)$.

For later reference, we mention the following corollary even if it trivially follows from Theorem 1.2.

Corollary 1.3. *There exists a totally symmetric embedding* $\mathrm{FL}(4) \to \mathrm{FL}(3)$.

In addition to Theorem 1.2 on total symmetry, we have some progress in studying selfdually positioned free sublattices, which is stated as follows.

Theorem 1.4. *Assuming that* $3 \le \kappa$ *and* $3 \le \lambda$ *are cardinal numbers,* $\mathrm{FL}(\lambda)$ *has a selfdually positioned sublattice isomorphic to* $\mathrm{FL}(\kappa)$ *if and only if the inequality* $\max\{\kappa, \aleph_0\} \le \max\{\lambda, \aleph_0\}$ *holds.*

This theorem is stronger than (1.2), the main result of Czédli [2]. Implicitly, the particular case of Theorem 1.4 where $\lambda = 3$ and κ belongs to $\{\aleph_0\} \cup \{2k : k \in \mathbb{N}^+\}$ is also included in [2].

Prerequisites

The reader is expected to have some basic familiarity only with the rudiments of lattice theory. That is, only some preliminary sections of the monographs, say, Grätzer [12] or Nation [16] are assumed. The results on free lattices that we need from the literature, mainly from Freese, Ježek, and Nation [9], are known for most lattice theorists and will be quoted with sufficient details since the paper is intended to be self-contained. In Section 5, we quote some recent achievements from [2]; when reading this section, the reader does not have to but may want to look into Czédli [2] to verify how we quote from it.

Main ideas of the paper

In this subsection, we deal mainly with the totally symmetric embeddability of $\mathrm{FL}(\omega)$ into $\mathrm{FL}(3)$, that is, with Part (B) of Theorem 1.2; the rest of the results are derived from or proved like Theorem 1.2(B), or they are easier. This subsection references several lemmas that will be formulated only in later sections of the paper; the reader may want to postpone understanding some details in these lemmas.

First, we define *symmetric* elements $m_1 < \dots < m_4$ in $\mathrm{FL}(3)$, see (3.2), and $\overline{m_i}$ will be the dual of m_i for $i \in \{1, \dots, 4\}$. With some computation based on Whitman's condition, we can prove that

(plan1) $P = \{m_1, \dots, m_4\} \cup \{\overline{m_1}, \dots, \overline{m_4}\}$ is the cardinal sum of two 4-element chains; see Figure 1 and Lemmas 4.7 and 4.9 for an illustration and for proofs, respectively, and

(plan2) we prove some properties of P implying that the sublattice $[P]_{\mathrm{FL}(3)}$ generated by P in $\mathrm{FL}(3)$ is isomorphic to the completely free lattice $\mathrm{CF}(P; \le)$ generated by the ordered set $(P; \le)$; see Corollary 4.10.

Combining the isomorphism $[P]_{\mathrm{FL}(3)} \cong \mathrm{CF}(P; \le)$ with the main result of Rival and Wille [17], we could immediately obtain an element-wise symmetric sublattice S of $\mathrm{FL}(3)$ such that $S \cong \mathrm{FL}(\omega)$. However, we want more. Hence,

(plan3) we define $a, b \in [P]_{\mathrm{FL}(3)}$ in (3.2), see also Figure 1, such that some computation based on Whitman's condition yields that $\mathrm{FL}(4)$ is isomorphic to $[a, b, \overline{a}, \overline{b}]_{\mathrm{FL}(3)}$, the sublattice of $\mathrm{FL}(3)$ generated by $\{a, b, \overline{a}, \overline{b}\}$, where $\overline{a}$ and $\overline{b}$ are the duals of a and b, respectively; see Lemma 3.1 for a more general statement. Note that the restriction of $\delta_{\mathrm{FL}(3)}$ to

$[a,b,\overline{a},\overline{b}]_{\mathrm{FL}(3)} \cong \mathrm{FL}(4)$ is not the natural dual automorphism of the free lattice $[a,b,\overline{a},\overline{b}]_{\mathrm{FL}(3)}$ since it swaps the free generators of $[a,b,\overline{a},\overline{b}]_{\mathrm{FL}(3)}$.

At this stage, Corollary 1.3 is already proved. Next, let δ_4^{sw} denote the unique dual automorphism of $\mathrm{FL}(4) = \mathrm{FL}(y_0, y_1, y_2, y_3)$ for which we have that $\delta_4^{\mathrm{sw}}(y_0) = y_1$, $\delta_4^{\mathrm{sw}}(y_1) = y_0$, $\delta_4^{\mathrm{sw}}(y_3) = y_4$, $\delta_4^{\mathrm{sw}}(y_4) = y_3$; we call it the *swapping* dual automorphism of FL(4).

(plan4) By the "diagonal method" of Czédli [2], $\mathrm{FL}(\omega)$ is isomorphic to a sublattice of FL(4) closed with respect to δ_4^{sw}.

Finally, a straightforward computation will show that if we combine (plan3) and (plan4), then their "swapping" features neutralize each other and we obtain a totally symmetric embedding $\mathrm{FL}(\omega) \to \mathrm{FL}(3)$, as required; a generalized form of this computation is given in the proof of Lemma 5.1.

If we embed $\mathrm{FL}(\omega)$ or $\mathrm{FL}(\kappa)$ into $\mathrm{FL}(\lambda)$, rather than into FL(3), then some of the above-mentioned computations, most of which can be done by a computer, become longer. Fortunately, we can often rely on the following fact, which deserves separate mentioning here: with two trivial exceptions, every symmetric element of $\mathrm{FL}(\lambda)$ is given by a near-unanimity term; see Lemma 4.4.

Let us note that the isomorphism $[P]_{\mathrm{FL}(3)} \cong \mathrm{CF}(P;\leq)$ and the above-mentioned result of Rival and Wille [17] are only motivating facts and will not be used in the detailed proof. Note also that this subsection will not be used in the rest of the paper; due to elaborated details and many internal references, the proofs are readable without keeping the main ideas in mind. Finally, since we also need to prove (our second) Theorem 1.4, we will prove more on $\{a,b,\overline{a},\overline{b}\}$ than what is required by (plan3); see Lemma 3.1.

Outline

Our main results, Theorems 1.2 and 1.4, and our main ideas have already been presented; the rest of the paper is structured as follows. We add some comments and two corollaries to the main result in Section 2. These corollaries characterize the pairs (κ,λ) of cardinals having the property that there is an embedding $\mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ with symmetric range or with selfdually positioned and symmetric range. The lion's share of our construction and (the Key) Lemma 3.1 stating that this construction works are given in Section 3. The Key Lemma is proved in Section 4. Section 5 combines the construct given in Section 3 with that given in Czédli [2]. Section 6 completes the proofs of our theorems and proves the corollaries. Finally, Section 7 describes our computer program for the word problem of free lattices; note that this program and its source file are freely available and the program proves Corollary 1.3 in less than a millisecond.

2. Remarks and corollaries

We will often use the convenient notation $\mathrm{FL}(\kappa) = \mathrm{FL}(x_i : i < \kappa)$ in order the specify the free generating set $\{x_i : i < \kappa\}$ of size κ; in this case, i denotes an ordinal number and $i < \kappa$ is understood as $|i| < \kappa$. An element u of a lattice L is *doubly irreducible* if $L \setminus \{u\}$ is closed with respect to both joins and meets, that is, if u is both *join irreducible* and *meet irreducible*. The set of doubly irreducible elements of L will be denoted by $\mathrm{Irr}_\wedge^\vee(L)$. We know from Whitman [20], see also Corollary 1.9 and the first sentence of the proof of Corollary 1.12 in Freese, Ježek, and Nation [9], that

$$\mathrm{Irr}_\wedge^\vee(\mathrm{FL}(\kappa)) = \{x_i : i < \kappa\}, \text{ and} \tag{2.1}$$

$$\text{every element of } \mathrm{FL}(\kappa) \text{ is join or meet irreducible.} \tag{2.2}$$

Note that we consider 0 and 1 join irreducible and meet irreducible, respectively, if these elements exist. Since a dual automorphism maps join-irreducible elements to meet irreducible ones and vice versa, (2.1) and (2.2) imply that for every dual automorphism ψ of $\mathrm{FL}(\kappa)$, we have that

$$\{u : \psi(u) = u\} \subseteq \psi(\{x_i : i < \kappa\}) = \{x_i : i < \kappa\}. \tag{2.3}$$

Remark 2.1. The concept of totally symmetric embeddings might raise the question whether we could consider even stronger embeddings whose ranges are element-wise symmetric and are in *element-wise* selfdual positions. We obtain from (2.3) that the answer is negative, since at most the free generators are in element-wise selfdual positions and they form an antichain. This justifies our terminology to call the embeddings in Theorem 1.2 *totally* symmetric.

Remark 2.2. Assume that $3 \leq \kappa \leq \aleph_0$ and $3 \leq \lambda \leq \aleph_0$; as a comparison between the result of Whitman [20] and Theorem 1.2, note the following. It follows immediately from Whitman' result that $\mathrm{FL}(\kappa)$ is embeddable into $\mathrm{FL}(\lambda)$, because embeddability is a transitive relation and $\gamma_1 \leq \gamma_2$ implies that $\mathrm{FL}(\gamma_1)$ is embeddable into $\mathrm{FL}(\gamma_2)$. However, the analogous implication fails for totally symmetric embeddability, since a symmetric element of $\mathrm{FL}(\gamma_1)$ is not symmetric in $\mathrm{FL}(\gamma_2)$ for $\gamma_2 > \gamma_1$. This explains that, as opposed to Whitman's result, Theorem 1.2 contains two parameters, κ and λ.

For a selfdual lattice L, let $\mathrm{DAut}(L)$ be the set of all automorphisms and dual automorphisms of L. As a consequence of (2.1) and (2.2), note that for $\kappa \leq \omega$,

> $\mathrm{Irr}_\wedge^\vee(\mathrm{FL}(\kappa))$ is closed with respect to every $\pi \in \mathrm{DAut}(\mathrm{FL}(\kappa))$. Furthermore, each $\pi \in \mathrm{DAut}(\mathrm{FL}(\kappa))$ is determined by its restriction to $\mathrm{Irr}_\wedge^\vee(\mathrm{FL}(\kappa))$ if we know whether it is an automorphism or a dual automorphism. (2.4)

With respect to composition, $\mathrm{DAut}(L)$ is a group and $\mathrm{Aut}(L)$ is a normal subgroup in it with index $[\mathrm{DAut}(L) : \mathrm{Aut}(L)] = 2$. Let us call a subset S of L a *DAut-symmetric* subset if $\pi(S) = S$ for all $\pi \in \mathrm{DAut}(L)$. A dually positioned and element-wise symmetric sublattice of $\mathrm{FL}(\lambda)$, like the range of a totally symmetric embedding, is clearly DAut-symmetric but not conversely. This might give some hope that a counterpart of Theorem 1.2 for embeddings with DAut-symmetric ranges would allow the case when κ is an odd natural number. However, the following corollary of Theorem 1.2 shows that this is not so if $\kappa \neq \lambda$. This corollary as well as Corollary 2.4 will be proved in Section 6.

Corollary 2.3. *Assuming that $3 \leq \kappa$ and $3 \leq \lambda$ are cardinal numbers, the following two conditions are equivalent.*

(i) *There exists an embedding* $\mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ *with DAut-symmetric range.*
(ii) *Either $\kappa = \lambda$, or we have that $\lambda \in \mathbb{N}^+$ and $\kappa \in \{2k : k \in \mathbb{N}^+\} \cup \{\aleph_0\}$.*

The situation is different if we deal with embeddings whose ranges are symmetric with respect only to $\mathrm{Aut}\,\mathrm{FL}(\lambda)$.

Corollary 2.4. *Assuming that $3 \leq \kappa$ and $3 \leq \lambda$ are cardinal numbers, there exists an embedding* $\mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ *with symmetric range if and only if either $\lambda \in \mathbb{N}^+$ and $\kappa \leq \aleph_0$, or $\kappa = \lambda \geq \aleph_0$.*

Since our concepts are based on the automorphisms of $\mathrm{FL}(n)$, where $n := \lambda$ is a positive integer, let us have a look at what these automorphisms and the elements of $\mathrm{DAut}(\mathrm{FL}(n))$ are. Let $\mathrm{Sym}_n = \mathrm{Sym}(\{0, 1, \dots, n-1\})$ denote the group of all permutations of $\{0, 1, \dots, n-1\}$ with respect to composition, and let $\mathsf{C}_2 = \{1, -1\}$ be the cyclic group of order 2, considered a subgroup of the group $(\mathbb{R} \setminus \{0\}; \cdot)$ of nonzero real numbers with respect to multiplication. Using that $\mathrm{FL}(n) = \mathrm{FL}(x_i : i < n)$ is freely generated by the set $\{x_i : i < n\}$, the following remark is straightforward and its proof is omitted.

Remark 2.5. For $n \in \mathbb{N}^+$ and the free lattice $\mathrm{FL}(n) = \mathrm{FL}(\{x_i : i < n\})$, the groups $\mathrm{Aut}(\mathrm{FL}(n))$ and $\mathrm{DAut}(\mathrm{FL}(n))$ are isomorphic to Sym_n and $\mathrm{Sym}_n \times \mathsf{C}_2$, respectively. More specifically, for $\sigma \in \mathrm{Sym}_n$, let $\sigma^{\mathrm{aut}} \colon \mathrm{FL}(n) \to \mathrm{FL}(n)$ and $\overline{\sigma^{\mathrm{aut}}} \colon \mathrm{FL}(n) \to \mathrm{FL}(n)$ be the maps defined by

$$\sigma^{\mathrm{aut}}(t(x_0, \dots, x_{n-1})) := t(x_{\sigma(0)}, \dots, x_{\sigma(n-1)}) \text{ and}$$
$$\overline{\sigma^{\mathrm{aut}}}(t(x_0, \dots, x_{n-1})) := \delta_{\mathrm{FL}(n)}(t(x_{\sigma(0)}, \dots, x_{\sigma(n-1)})),$$

respectively, where t denotes an n-ary lattice term and $\delta = \delta_{\mathrm{FL}(n)}$ has been defined in (1.1). Then the maps

$$\mathrm{Sym}_n \to \mathrm{Aut}(\mathrm{FL}(n)), \text{ defined by } \sigma \mapsto \sigma^{\mathrm{aut}}, \text{ and}$$
$$\mathrm{Sym}_n \times \mathsf{C}_2 \to \mathrm{DAut}(\mathrm{FL}(n)), \text{ defined by } (\sigma, k) \mapsto \begin{cases} \overline{\sigma^{\mathrm{aut}}}, & \text{if } k = -1, \\ \sigma^{\mathrm{aut}}, & \text{if } k = 1 \end{cases}$$

are group isomorphisms. (In particular, they are well defined maps.)

3. Construction and the Key Lemma

3.1. Notation

The elements of a free lattice $\mathrm{FL}(\kappa) = \mathrm{FL}(x_i : i < \kappa)$ will be represented by *lattice terms* over the set $\{x_i : i < \kappa\}$ of variables. Although there are many terms representing the same element of $\mathrm{FL}(\kappa)$, it will not cause any confusion that

$$\text{we often treat and call these } \textit{terms as elements} \text{ of the free lattice;} \tag{3.1}$$

see pages 10–11 in Freese, Ježek and Nation [9] for a more rigorous setting. The dual of a term t will always be denoted by $\overline{t}$; the overline reminds us that dualizing at visual level means to reflect Hasse diagrams across horizontal axes. The Symmetric part of the Free Lattice of $\mathrm{FL}(\kappa)$ will be denoted as follows; capitalization explains the acronym:

$$\mathrm{SFL}(\kappa) := \{u \in \mathrm{FL}(\kappa) : \pi(u) = u \text{ for all } \pi \in \mathrm{Aut}\,\mathrm{FL}(\kappa)\}.$$

Clearly, $\mathrm{SFL}(\kappa)$ is a sublattice of $\mathrm{FL}(\kappa)$; this fact will often be used implicitly.

3.2. Constructing some important terms

In this subsection, we give a construction for the particular case $(\kappa, \lambda) = (4, n)$ of Theorem 1.2(A). Let us agree to the following conventions. The set

$$\{0, 1, \dots, n-1\} \text{ will also be denoted by } [n).$$

The inequality $i < n$ is equivalent to $i \in [n)$. Whenever x_i, x_j, etc. refer to a free generator of $\mathrm{FL}(n) = \mathrm{FL}(x_0, \dots, x_{n-1})$, then $i, j, \dots$ will automatically belong to $[n)$; this convention will often save us from indicating, say, that $i < n$ or $i, j \in [n)$ below the $\bigvee$ and $\bigwedge$ operation signs. Also, we frequently abbreviate the conjunction of $i \in [n)$ and $j \in [n) \setminus \{i\}$ by the short form $i \neq j$, and self-explanatory similar other abbreviations may also occur. For the rest of this section, let $3 \leq n = \lambda \in \mathbb{N}^+$ and

$$\mathrm{FL}(\lambda) = \mathrm{FL}(n) = \mathrm{FL}(x_0, \dots, x_{n-1}) = \mathrm{FL}(x_i : i < n).$$

By induction on j, we define the following n-ary lattice terms over the set $\{x_i : i < n\}$ of variables; according to (3.1), they will also be considered elements of $\mathrm{FL}(n)$. Namely, we let

$$\begin{aligned}
p_0^{(i)} &:= x_i \text{ for } i \in \{0, 1, \dots, n-1\} = [n),\\
p_j^{(i)} &:= x_i \vee \bigvee_{\substack{i_1 < i_2\\ i_1, i_2 \in [n) \setminus \{i\}}} \bigl(p_{j-1}^{(i_1)} \wedge p_{j-1}^{(i_2)}\bigr) \ \text{ for } i \in [n) \text{ and } j \in \mathbb{N}^+,\\
m_j &:= \bigvee_{i_1 < i_2,\ i_1, i_2 \in [n)} \bigl(p_j^{(i_1)} \wedge p_j^{(i_2)}\bigr) \ \text{ for } j \in \mathbb{N}_0,\\
a &:= m_1 \vee \overline{m_3}, \quad \text{and} \quad b := m_2 \vee \overline{m_4}.
\end{aligned} \tag{3.2}$$

For later reference, we note that the set

$$\{a, \overline{a}, b, \overline{b}\} \tag{3.3}$$

will play an important role in the paper. We say that a subset X of a lattice *freely generates* if the sublattice S generated by X is a free lattice with X as the set of free generators. Next, we formulate our Key Lemma, which is stronger than asserting that the set in (3.3) freely generates. The proof of the Key Lemma will be postponed to Section 4.

Lemma 3.1 (Key Lemma). *If* $3 \le n \in \mathbb{N}^+$, *then the elements* m_j *and* $\overline{m_j}$ *for* $j \in \mathbb{N}_0$, a, b, $\overline{a}$, *and* $\overline{b}$ *all belong to* $\mathrm{SFL}(n)$. *Furthermore,* $\{a, \overline{a}, b, \overline{b}, x_0\}$ *is a five-element subset of* $\mathrm{SFL}(n)$ *that freely generates.*

For later reference, based on Remark 2.5, note the following trivial lemma.

Lemma 3.2. *For every* $i \in [n)$, $j \in \mathbb{N}_0$ *and* $\sigma \in \mathrm{Sym}([n))$, *we have that* $\sigma^{\mathrm{aut}}(p_j^{(i)}) = p_j^{(\sigma(i))}$ *and* $\sigma^{\mathrm{aut}}(m_j) = m_j$.

Proof of Lemma 3.2. The first equality above follows from the fact that in (3.2), each stipulation of the form $i_1 < i_2$ can be replaced by $i_1 \neq i_2$ by idempotence. The second equality follows from the first one. □

4. The proof of the Key Lemma

From the theory of free lattices, we only use three basic facts, which we recall below as lemmas; all of them can be found in Freese, Ježek and Nation [9]. An element u of a lattice L is *join prime* if for all $k \in \mathbb{N}^+$ and $x_0, \dots, x_{k-1} \in L$, the inequality $u \le x_0 \vee \dots \vee x_{k-1}$ implies that $u \le x_i$ for some $i \in [k)$. Meet prime elements are defined dually. An element is *doubly prime* if it is join prime and meet prime.

Lemma 4.1 (Whitman [20]**; Freese, Ježek and Nation** [9, Corollary 1.5]**).** *In every free lattice* $\mathrm{FL}(X)$, *the free generators are doubly prime elements.*

The following statement says that free lattices satisfy *Whitman's condition* (W), see Whitman [20].

Lemma 4.2 (Whitman [20]**; Freese, Ježek and Nation** [9, Theorem 1.8]**).** *For arbitrary elements* $u_1, \dots, u_r, v_1, \dots, v_s$ *of a free lattice* $\mathrm{FL}(X)$,

(W) *The inequality* $u = u_1 \wedge \dots \wedge u_r \le v_1 \vee \dots \vee v_s = v$ *implies that either* $u_i \le v$ *for some subscript* i, *or* $u \le v_j$ *for some* j.

Next, we describe whether a subset of a free lattice generates freely or not.

Lemma 4.3 (Whitman [20]**; Freese, Ježek and Nation** [9, Corollary 1.13]**).** *A nonempty subset* Y *of* $\mathrm{FL}(X)$ *generates freely if and only if for all* $h \in Y$ *and all finite subsets* $Z \subseteq Y$, *the following condition and its dual hold.*

$$h \notin Z \quad \textit{implies} \quad h \not\le \bigvee_{z \in Z} z.$$

For the rest of this section, assume that $3 \le n \in \mathbb{N}^+$. A (lattice) term t is called a *near-unanimity lattice term* or, shortly, an *NU-term* if it satisfies

$$t(y, x, \dots, x) = t(x, y, x, \dots, x) = \dots = t(x, \dots, x, y) = x.$$

Since the lattice operations are idempotent, it is obvious that

$$\text{the join and the meet of two } n\text{-ary NU-terms are NU-terms.} \tag{4.1}$$

If t_1 and t_2 are n-ary lattice terms such that $t_1 = t_2$ in $\mathrm{FL}(n)$, see (3.1), then $t_1 \in \mathrm{SFL}(n)$ iff $t_2 \in \mathrm{SFL}(n)$. Also, for $t_1 = t_2 \in \mathrm{FL}(n)$, t_1 is an NU-term iff so is t_2. So convention (3.1) still applies.

Lemma 4.4. *If $t \in \mathrm{SFL}(n) = \mathrm{SFL}(x_0, \dots, x_{n-1})$ such that neither of the equalities $t(x_0, \dots, x_{n-1}) = \bigwedge_{i \in [n]} x_i$ and $t(x_0, \dots, x_{n-1}) = \bigvee_{i \in [n]} x_i$ holds in $\mathrm{FL}(n)$, then t is a near-unanimity term.*

Proof. It is straightforward to see that

$$\begin{gathered}\text{if } g \in \mathrm{SFL}(n) \text{ and } x_i \le g \text{ in } \mathrm{FL}(n) \text{ for some } i \in [n], \\ \text{then } g = 1_{\mathrm{SFL}(n)} = x_0 \vee \dots \vee x_{n-1}, \text{ and dually.}\end{gathered} \tag{4.2}$$

Next, assume that t satisfies the assumptions of the lemma. As a binary lattice term, $t(x, \dots, x, y)$ equals one of x, y, $x \wedge y$, and $x \vee y$ in $\mathrm{FL}(x, y)$. If we had that, in $\mathrm{FL}(x, y)$, $t(x, \dots, x, y) = x \wedge y$, then

$$\begin{aligned} t(x_0, \dots, x_{n-1}) &\le t(x_0 \vee \dots \vee x_{n-2}, \dots, x_0 \vee \dots \vee x_{n-2}, x_{n-1}) \\ &\le (x_0 \vee \dots \vee x_{n-2}) \wedge x_{n-1} \le x_{n-1} \end{aligned}$$

together with (4.2) would yield that $t = 0_{\mathrm{SFL}(n)} = x_0 \wedge \dots \wedge x_{n-1}$, a contradiction. Hence, $t(x, \dots, x, y)$ is distinct from $x \wedge y$, and it is distinct also from $x \vee y$ by duality. The case $t(x, x, \dots, x, y) = y$ is impossible, because it would imply that $1 = t(0, \dots, 0, 1) \le t(0, 1, \dots, 1) = t(1, \dots, 1, 0) = 0$ holds in the two-element lattice **2**, which is a contradiction. Hence, $t(x, x, \dots, x, y) = x$, which means that t is an NU-term since it is symmetric. ☐

Lemma 4.5. *There is exactly one atom in* $\mathrm{SFL}(n)$, *and it is*

$$s := \bigvee_{i \in [n]} \bigwedge_{i' \in [n] \setminus \{i\}} x_{i'} .$$

The only coatom of $\mathrm{SFL}(n)$ *is* $\overline{s}$. *Except from its bottom* $0_{\mathrm{SFL}(n)} = \bigwedge_{i<n} x_i$ *and top* $1_{\mathrm{SFL}(n)} = \bigvee_{i<n} x_i$, *every element of* $\mathrm{SFL}(n)$ *is in the interval* $[s, \overline{s}]$.

The statement of this lemma for $3 < n \in \mathbb{N}^+$ is illustrated by Figure 1, where only the two thick edges stand for coverings in $\mathrm{SFL}(n)$; the thin lines indicate comparabilities that need not be coverings. (These comparabilities will be proved later; see Lemma 4.9.) The reflection across the symmetry center point, which is not indicated in the figure, represents the restriction of $\delta_{\mathrm{FL}(n)}$ to $\mathrm{SFL}(n)$. We could obtain a similar figure for $n = 3$ by removing the vertices m_0 and $\overline{m_0}$ and decreasing the subscripts of the remaining vertices by 1; see Lemma 4.9. Note that, as opposed to $\mathrm{SFL}(n)$, the free lattice $\mathrm{FL}(n)$ has exactly n atoms; for more details, the reader is referred to the discussion of the bottom of $\mathrm{FL}(n)$ in Freese, Ježek, and Nation [9, Section III.7].

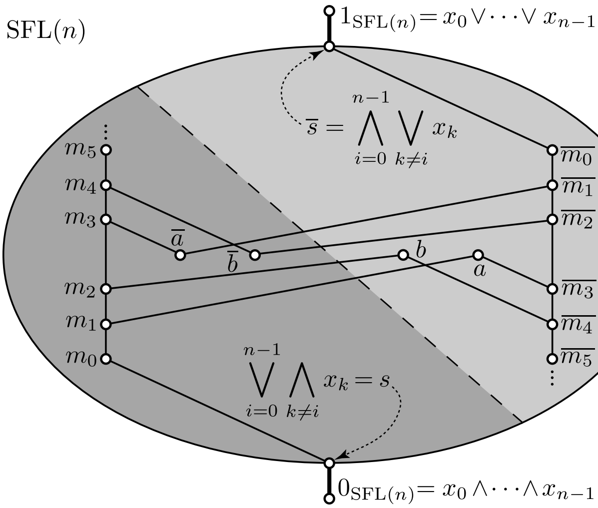

FIGURE 1. The lattice $\mathrm{SFL}(n)$ for $n > 3$

Proof. By Lemma 4.4 and the duality principle, we need to show only that $t \geq s$ holds for every near-unanimity term $t \in \mathrm{SFL}(n)$. This follows from the fact that for each $i \in [n)$,

$$t(x_0, \dots, x_{i-1}, x_i, x_{i+1}, \dots x_{n-1})$$
$$\geq t(\bigwedge_{i' \neq i} x_{i'}, \dots, \bigwedge_{i' \neq i} x_{i'}, x_i, \bigwedge_{i' \neq i} x_{i'}, \dots, \bigwedge_{i' \neq i} x_{i'}) = \bigwedge_{i' \neq i} x_{i'}. \qquad \square$$

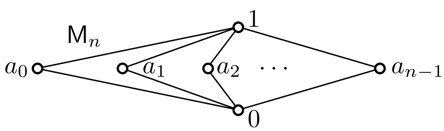

FIGURE 2. The lattice M_n

In order to get some preliminary insight into $\mathrm{SFL}(n)$, note the following. As usual, M_n denotes $(n+2)$-element lattice given in Figure 2. For a permutation $\pi\colon [n) \to [n)$, let π_X and π_A denote the permutation of $\{x_i : i < n\}$ and that of $\{a_i : i < n\}$ defined by $\pi_X(x_i) = x_{\pi(i)}$ and $\pi_A(a_i) = a_{\pi(i)}$ for all $i < n$, respectively. These permutations uniquely extend to automorphisms $\pi_X^{\mathrm{aut}} \in \mathrm{Aut}(\mathrm{SFL}(n))$ and $\pi_A^{\mathrm{aut}} \in \mathrm{Aut}(\mathsf{M}_n)$, respectively. Consider the natural homomorphism

$$\eta\colon\ \mathrm{FL}(n) \to \mathsf{M}_n \text{ defined by } \eta(t(x_0, \dots, x_{n-1})) = t(a_0, \dots, a_{n-1}). \qquad (4.3)$$

Clearly, $\pi_A^{\mathrm{aut}} \circ \eta = \eta \circ \pi_X^{\mathrm{aut}}$. This implies easily that the η-image of a symmetric element of $\mathrm{FL}(n)$ is a symmetric element of M_n. But the symmetric elements of M_n form $\mathbf{2}$ (as a sublattice), and we obtain that $\eta\rceil_{\mathrm{SFL}(n)}\colon\ \mathrm{SFL}(n) \to \mathbf{2}$ is a surjective homomorphism. So the kernel of $\eta\rceil_{\mathrm{SFL}(n)}$ cuts $\mathrm{SFL}(n)$ into a prime ideal, the dark-grey lower part together with the bottom element in Figure 1, and its complementary prime filter, the light-grey upper part together with the top in the figure. Besides that their shades are distinct in the figure, the two parts are separated by a dashed line.

Lemma 4.6. *For every $i < n$, the sequence $\{p_j^{(i)} : j \in \mathbb{N}_0\}$ is strictly increasing, that is, $p_0^{(i)} < p_1^{(i)} < p_2^{(i)} < \dots$ in* $\mathrm{FL}(n)$.

Proof. First, we show by induction on j that

$$p_j^{(r)} \wedge p_j^{(s)} \not\leq x_k \text{ for all } j \in \mathbb{N}^+ \text{ and } k, r, s \in [n). \tag{4.4}$$

Suppose, for a contradiction, that (4.4) fails, and let $j \in \mathbb{N}^+$ be the smallest number violating it. Pick $k, r, s \in [n)$ such that $p_j^{(r)} \wedge p_j^{(s)} \leq x_k$. Since x_k is meet prime by Lemma 4.1, we can assume that $p_j^{(r)} \leq x_k$. By (3.2), $x_r \leq x_k$, whence $r = k$ and we have that $p_j^{(k)} \leq x_k$. Pick $r' < s'$ in $[n) \setminus \{k\}$; this is possible since $n \geq 3$. Since $p_j^{(k)} \leq x_k$, (3.2) gives that $p_{j-1}^{(r')} \wedge p_{j-1}^{(s')} \leq x_k$. Since this inequality does not violate (4.4) by the choice of j, it follows that $j - 1 \notin \mathbb{N}^+$, whence $j = 1$. Then (3.2) turns $p_{j-1}^{(r')} \wedge p_{j-1}^{(s')} \leq x_k$ into $x_{r'} \wedge x_{s'} \leq x_k$, which is a contradiction since $k \notin \{r', s'\}$. This proves (4.4).

Next, based on (3.2), a trivial induction on j shows that

$$\text{for all } j \in \mathbb{N}_0 \text{ and } i \in [n),\ \ p_j^{(i)}(a_0, \dots, a_{n-1}) = a_i \text{ holds in } \mathsf{M}_n. \tag{4.5}$$

This implies that, for any $j, j' \in \mathbb{N}_0$ and $i, i' \in [n)$,

$$\text{if } i \neq i', \text{ then the terms } p_j^{(i)} \text{ and } p_{j'}^{(i')} \text{ are incomparable in } \mathrm{FL}(n). \tag{4.6}$$

A straightforward induction yields that the sequence is increasing, that is,

$$p_0^{(i)} \leq p_1^{(i)} \leq p_2^{(i)} \leq \dots \text{ holds in } \mathrm{FL}(n). \tag{4.7}$$

Armed with the preparations above, it suffices to prove the strict inequalities in the lemma only for $i = 0$, since then the case $i > 0$ will follow by symmetry. For the sake of contradiction, suppose that there exists a $j \in \mathbb{N}_0$ such that

$$p_{j+1}^{(0)} \overset{(3.2)}{=} x_0 \vee \bigvee_{i_1 < i_2,\ i_1, i_2 \in [n) \setminus \{0\}} \left(p_j^{(i_1)} \wedge p_j^{(i_2)}\right) \leq p_j^{(0)}. \tag{4.8}$$

Let j be minimal with this property. By symmetry, the superscript 0 does not play a distinguished role here. That is, until the end of the proof,

$$\text{for each } i \in [n),\ j \in \mathbb{N}_0 \text{ is minimal such that } p_{j+1}^{(i)} \leq p_j^{(i)}. \tag{4.9}$$

We are going to derive a contradiction from (4.8) by infinite descent. Since $p_1^{(0)} = x_0 \vee (x_1 \wedge x_2) \vee \dots \leq x_0 = p_0^{(0)}$ would lead to $x_1 \wedge x_2 \leq x_0$, which fails even in **2**, we obtain that $j > 0$. So, as the first step of the descent, we conclude that $j-1 \in \mathbb{N}_0$. Thus, (4.8) and $j-1 \in \mathbb{N}_0$ imply that

$$p_j^{(1)} \wedge p_j^{(2)} \leq p_j^{(0)} \overset{(3.2)}{=} x_0 \vee \bigvee_{i_1<i_2,\ i_1,i_2\in[n)\setminus\{0\}} \bigl(p_{j-1}^{(i_1)} \wedge p_{j-1}^{(i_2)}\bigr). \tag{4.10}$$

By (4.6), this inequality would fail if one of the two meetands on the left was omitted. Hence, (W) and (4.4) yield that $p_j^{(1)} \wedge p_j^{(2)} \leq p_{j-1}^{(u_1)} \wedge p_{j-1}^{(v_1)}$ for some $v_1, u_1 \in [n) \setminus \{0\}$. In particular, $p_j^{(1)} \wedge p_j^{(2)} \leq p_{j-1}^{(u_1)}$. We formulate this inequality together with $j-1 \in \mathbb{N}_0$ also in the following way: the condition

$$j-m \in \mathbb{N}_0 \text{ and there is a } u_m \in [n) \text{ such that } p_j^{(1)} \wedge p_j^{(2)} \leq p_{j-m}^{(u_m)} \tag{4.11}$$

holds for $m = 1$. In order to continue the descent in infinitely many steps, we assert the following.

$$\text{Assume that } p_{j-1}^{(1)} < p_j^{(1)}. \text{ Then, for every } m \in \mathbb{N}^+, \text{ if } (4.11) \text{ holds for } m, \text{ then it holds also for } m+1. \tag{4.12}$$

(The first sentence in (4.12) is included for later reference.) In order to prove (4.12), assume (4.11) for m. The case $j-m = 0$ is ruled out by (4.4), whence $j-(m+1) \in \mathbb{N}_0$. Hence, the inequality in (4.11) gives that

$$p_j^{(1)} \wedge p_j^{(2)} \leq p_{j-m}^{(u_m)} \overset{(3.2)}{=} x_{u_m} \vee \bigvee_{\substack{i_1<i_2\\ i_1,i_2\in[n)\setminus\{u_m\}}} \bigl(p_{j-(m+1)}^{(i_1)} \wedge p_{j-(m+1)}^{(i_2)}\bigr). \tag{4.13}$$

As before, we are going to apply (W) to (4.13); however, the argument for the meetands on the left is a bit longer. If $u_m \notin \{1, 2\}$, then we obtain from (4.6) that none of the meetands on the left of (4.13) can be omitted from the inequality. If $u_m = 1$, then $p_j^{(1)}$ still cannot be omitted by (4.6), but we need the same fact for the other meetand, $p_j^{(2)}$. Observe that if $p_j^{(2)}$ was omitted and u_m equaled 1, then we would have by (4.7) that $p_j^{(1)} \leq p_{j-m}^{(1)} \leq p_{j-1}^{(1)}$, contradicting the first sentence of (4.12). So none of the two meetands in question can be omitted if $u_m = 1$, and the same is true for $u_m = 2$ since 1 and 2 play symmetric roles. This shows that no matter what is u_m, none of the two meetands on the left of (4.13) can be omitted. Therefore, (4.13), (W) and (4.4) imply that $p_j^{(1)} \wedge p_j^{(2)} \leq p_{j-(m+1)}^{(u_{m+1})} \wedge p_{j-(m+1)}^{(v_{m+1})}$ for some $u_{m+1}, v_{m+1} \in [n) \setminus \{u_m\}$. In particular, $p_j^{(1)} \wedge p_j^{(2)} \leq p_{j-(m+1)}^{(u_{m+1})}$. Thus, we conclude that (4.11) holds for $m+1$, completing the proof of (4.12).

Finally, (4.7) and (4.9) yield the validity of the first sentence of (4.12). Consequently, it follows from (4.12) that (4.11) holds for all $m \in \mathbb{N}^+$, which contradicts the finiteness of $j \in \mathbb{N}_0$ and completes the proof of Lemma 4.6. □

The following lemma is visualized by a part of Figure 1.

Lemma 4.7. *The sequence* $\{m_j : j \in \mathbb{N}_0\}$ *is strictly increasing, that is, we have that* $m_0 < m_1 < m_2 < \dots$ *in* $\mathrm{FL}(n)$. *Also,* $\{\overline{m_j} : j \in \mathbb{N}_0\}$ *is strictly decreasing.*

Proof. It suffices to deal only with $\{m_j : j \in \mathbb{N}_0\}$. The sequence in question is increasing by its definition, see (3.2), and Lemma 4.6. For the sake of contradiction, suppose the sequence is not strictly increasing, so that $m_j \le m_{j-1}$ holds for some $j \in \mathbb{N}^+$. Then, since all joinands of m_j are less than or equal to m_{j-1}, we have that, in particular,

$$p_j^{(1)} \wedge p_j^{(2)} \le m_j = \bigvee_{i_1<i_2,\ i_1,i_2\in[n)} \bigl(p_{j-1}^{(i_1)} \wedge p_{j-1}^{(i_2)}\bigr). \tag{4.14}$$

It follows from (4.5) that, with $\vec{a} = (a_0, \dots, a_{n-1}) \in \mathsf{M}_n \times \dots \times \mathsf{M}_n$, we have that $m_j(\vec{a}) = 0$ while $p_j^{(1)}(\vec{a}) = a_1$ and $p_j^{(2)}(\vec{a}) = a_2$. Hence, neither of the meetands on the left of (4.14) can be omitted without breaking the inequality. Thus, applying (W) to (4.14), it follows that $p_j^{(1)} \wedge p_j^{(2)} \le p_{j-1}^{(u_1)} \wedge p_{j-1}^{(v_1)}$ for some $u_1, v_1 \in [m)$. This yields that $p_j^{(1)} \wedge p_j^{(2)} \le p_{j-1}^{(u_1)}$. Therefore, (4.11) holds for $m = 1$. By Lemma 4.6, so does the assumption in the first sentence of (4.12). Consequently, it follows from (4.12) that (4.11) holds for all $m \in \mathbb{N}^+$, which is a contradiction since $j - m \notin \mathbb{N}_0$ for $m > j$. □

The following lemma states something on $\mathrm{SFL}(n)$, not on $\mathrm{FL}(n)$.

Lemma 4.8. *For all $j \in \mathbb{N}_0$, m_j and $\overline{m_j}$ are doubly prime elements of* $\mathrm{SFL}(n)$.

Proof. By duality, it suffices to deal only with m_j. In order to show that m_j is join prime, assume that $m_j \le h_1 \vee h_2$ where $h_1, h_2 \in \mathrm{SFL}(n)$. Remember that the containment here means that h_1 and h_2 are fixed points of every automorphism of $\mathrm{FL}(n)$. We have to show that $m_j \le h_i$ for some $i \in \{1, 2\}$. There are two cases to consider. First, assume that

$$\text{there exists an } i \in \{1,2\} \text{ such that } p_j^{(0)} \wedge p_j^{(1)} \le h_i. \tag{4.15}$$

In this case, for each $(u_1, u_2) \in [n) \times [n)$ with $u_1 < u_2$, pick a permutation $\sigma \in \mathrm{Sym}([n))$ such that $\sigma(0) = u_1$ and $\sigma(1) = u_2$. By Remark 2.5 and Lemma 3.2,

$$\begin{aligned} p_j^{(u_1)} \wedge p_j^{(u_2)} &= p_j^{(\sigma(0))} \wedge p_j^{(\sigma(1))} \\ &= \sigma^{\mathrm{aut}}(p_j^{(0)} \wedge p_j^{(1)}) \le \sigma^{\mathrm{aut}}(h_i) = h_i. \end{aligned} \tag{4.16}$$

Forming the join of these inequalities for all meaningful pairs (u_1, u_2), we obtain that $m_j \le h_i$, as required.

Second, assume that (4.15) fails. Then (W) applied to the (side terms of the) inequality $p_j^{(0)} \wedge p_j^{(1)} \le m_j \le h_1 \vee h_2$ gives that $p_j^{(0)} \le h_1 \vee h_2$ or $p_j^{(1)} \le h_1 \vee h_2$; we can assume that $p_j^{(0)} \le h_1 \vee h_2$. By (3.2) and Lemma 4.6, we have that $x_0 \le h_1 \vee h_2$. Since x_0 is join prime by Lemma 4.1, $x_0 \le h_i$ holds for some $i \in \{1, 2\}$. Applying (4.2), we obtain that $h_i = 1_{\mathrm{SFL}(n)}$. Hence, $m_j \le 1_{\mathrm{SFL}(n)} = h_i$, as required. Now that both the validity and the failure of (4.15) have been considered, we conclude that m_j is a join prime element of $\mathrm{SFL}(n)$.

Next, we are going to show that m_j is meet prime in $\mathrm{SFL}(n)$. Suppose the contrary, and pick $h_1, h_2 \in \mathrm{SFL}(n)$ such that $h_1 \wedge h_2 \le m_j$ but $h_1 \not\le m_j$

and $h_2 \not\leq m_j$. We are going to obtain a contradiction by infinite descent. In order to do so, it suffices to show that the condition

$$j - r \in \mathbb{N}_0 \text{ and there is a } u_r \in [n) \text{ such that } h_1 \wedge h_2 \leq p_{j-r}^{(u_r)} \tag{4.17}$$

holds for $r = 0$, and to show that

$$\text{for every } r \in \mathbb{N}_0, \text{ if (4.17) holds for } r, \text{ then it also holds for } r+1. \tag{4.18}$$

Applying (W) to $h_1 \wedge h_2 \leq m_j = \bigvee_{u_0<v_0}(p_j^{(u_0)} \wedge p_j^{(v_0)})$, we obtain $u_0, v_0 \in [n)$ such that $h_1 \wedge h_2 \leq p_j^{(u_0)} \wedge p_j^{(v_0)} \leq p_j^{(u_0)} = p_{j-0}^{(u_0)}$. Hence, (4.17) holds for $r = 0$. Next, in order to show (4.18), assume that $r \in \mathbb{N}_0$ satisfies condition (4.17). We cannot have that $h_1 \wedge h_2 \leq x_{u_r}$, because otherwise $h_i \leq x_{u_r}$ for some $i \in \{1, 2\}$ by the meet primeness of x_{u_r}, see Lemma 4.1, and so (4.2) would give that $h_i = 0_{\mathrm{SFL}(n)} \leq m_j$, contradicting our assumption. Combining $h_1 \wedge h_2 \not\leq x_{u_r}$ with (4.17) and (3.2), we obtain that $j - r \neq 0$. Hence, $j - (r + 1) \in \mathbb{N}_0$ and

$$h_1 \wedge h_2 \leq p_{j-r}^{(u_r)} = x_{u_r} \vee \bigvee_{u_{r+1}<v_{r+1}} \big(p_{j-(r+1)}^{(u_{r+1})} \wedge p_{j-(r+1)}^{(v_{r+1})}\big). \tag{4.19}$$

In order to exclude that $h_1 \leq p_{j-r}^{(u_r)}$, suppose the contrary. Then, by Lemma 4.6, $h_1 \leq p_{j-r}^{(u_r)} \leq p_j^{(u_r)}$. Using a permutation of $[n)$ with $u_r \mapsto v_r \in [n) \setminus \{u_r\}$ as in (4.16), we obtain that $h_1 \leq p_j^{(v_r)}$. Hence, $h_1 \leq p_j^{(u_r)} \wedge p_j^{(v_r)} \leq m_j$ is a contradiction, proving that $h_1 \not\leq p_{j-r}^{(u_r)}$. Similarly, $h_2 \not\leq p_{j-r}^{(u_r)}$. Now that we have seen that $h_1 \wedge h_2 \not\leq x_{u_r}$, $h_1 \not\leq p_{j-r}^{(u_r)}$, and $h_2 \not\leq p_{j-r}^{(u_r)}$, we are in the position to apply (W) to (4.19). So we obtain that $h_1 \wedge h_2 \leq p_{j-(r+1)}^{(u_{r+1})} \wedge p_{j-(r+1)}^{(v_{r+1})}$ for some $u_{r+1}, v_{r+1} \in [n)$. Hence, $h_1 \wedge h_2 \leq p_{j-(r+1)}^{(u_{r+1})}$ and (4.17) holds for $r + 1$. We have verified (4.18). Thus, it follows that (4.17) holds for all $r \in \mathbb{N}_0$. This is the required contradiction proving that m_j is meet prime in $\mathrm{SFL}(n)$, completing the proof of Lemma 4.8. ☐

For $3 \leq n \in \mathbb{N}^+$, in connection with (3.2) and Lemma 4.5, let

$$P_n := \begin{cases} \{m_j \in \mathrm{SFL}(n) : j \in \mathbb{N}^+\} \cup \{\overline{m_j} \in \mathrm{SFL}(n) : j \in \mathbb{N}^+\}, & \text{if } n = 3 \\ \{m_j \in \mathrm{SFL}(n) : j \in \mathbb{N}_0\} \cup \{\overline{m_j} \in \mathrm{SFL}(n) : j \in \mathbb{N}_0\}, & \text{if } n > 3. \end{cases}$$

With the ordering of $\mathrm{SFL}(n)$ restricted to P_n, $\mathbf{P}_n = (P_n; \leq)$ is an ordered set, which is described by the following lemma; see also Figure 1 for $3 < n \in \mathbb{N}^+$. This lemma explains why the case $n = 3$ differs slightly from the case $n > 3$.

Lemma 4.9. *The following four assertions hold.*

(i) *If $n > 3$, then $s < m_0 \not\leq \overline{m_0}$.*
(ii) *If $n = 3$, then $m_1 \not\leq \overline{m_1}$. However, $m_0 = s < \overline{s} = \overline{m_0}$, so m_0 and $\overline{m_0}$ are the unique atom and the unique coatom of* SFL(3), *respectively.*
(iii) *For $n \geq 3$ and $i, j \in \mathbb{N}^+$, $m_i \not\leq \overline{m_j}$.*
(iv) *For $n \geq 3$ and $i, j \in \mathbb{N}_0$, $\overline{m_j} \not\leq m_i$.*

Proof. If $n > 3$, then letting $\vec{w} := (1, 1, 0, \dots, 0)$, we have that $s(\vec{w}) = 0$, $m_0(\vec{w}) = 1$, and $\overline{m_0}(\vec{w}) = 0$ hold in the two-element lattice **2**. This proves (i), because $s \le m_0$ follows from Lemma 4.5.

In addition to a straightforward calculation, which will be omitted, (ii) has also been checked by the computer program of Section 7.

Next, to deal with (ii), we assume that $n = 3$. Clearly, $m_0 = s$ and $\overline{m_0} = \overline{s}$. Lemma 4.5 gives that $s \le \overline{s}$ while M_3 witnesses that $s \ne \overline{s}$. We have seen that $m_0 = s < \overline{s} = \overline{m_0}$. For the sake of contradiction, suppose that $m_1 \le \overline{m_1}$. Then each of the joinands of m_1 is less than or equal to every meetand of $\overline{m_1}$. In particular, we have that $p_1^{(0)} \wedge p_1^{(1)}$ is less than or equal to its dual, that is,

$$\begin{aligned} &\big(x_0 \vee (x_1 \wedge x_2)\big) \wedge \underline{\big(x_1 \vee (x_0 \wedge x_2)\big)} \\ &\le \big(x_0 \wedge (x_1 \vee x_2)\big) \vee \big(x_1 \wedge (x_0 \vee x_2)\big). \end{aligned} \tag{4.20}$$

By (W), duality, and since x_0 and x_1 play symmetric roles, we can assume that (4.20) holds after omitting its underlined meetand. Hence, x_0 is less than or equal to the right hand side of (4.20). Using that x_0 is join prime by Lemma 4.1, we obtain that either $x_0 \le x_0 \wedge (x_1 \vee x_2) \le x_1 \vee x_2$, or $x_0 \le x_1 \wedge (x_0 \vee x_2) \le x_1$, so we have obtained a contradiction, proving (ii).

Next, we turn our attention to (iii). Suppose, for a contradiction, that $m_i \le \overline{m_j}$ for some $i, j \in \mathbb{N}^+$. We obtain from Lemma 4.7 that

$$m_0 < m_1 \le m_i \le \overline{m_j} \le \overline{m_1} < \overline{m_0}.$$

In particular, $m_0 < \overline{m_0}$ and $m_1 \le \overline{m_1}$. The first of these two inequalities contradicts part (i) if $n > 3$, while the second one contradicts part (ii) if $n = 3$. So we obtain a contradiction for all $n \ge 3$, whereby (iii) holds.

Finally, we obtain (iv) basically from $\eta(\overline{m_j}) = 1$ and $\eta(m_i) = 0$, see (4.3), as follows. Using (4.5), its dual, and (3.2) defining our terms, we obtain that $\overline{m_j}(a_0, \dots, a_{n-1}) = 1_{\mathsf{M}_n}$ and $m_i(a_0, \dots, a_{n-1}) = 0_{\mathsf{M}_n}$. This implies (iv) and completes the proof of Lemma 4.9. □

Now, to indicate that we are progressing in the desired direction, we are going to formulate a corollary. Note, however, that neither this corollary, nor its proof, nor the concept defined in the present paragraph will be used in this paper, so the reader can skip over them. Following Dean [3] and Dilworth [5], an ordered set $P = (P; \le_P)$ *completely freely generates* a lattice K if P is a subset of K, $\le_P$ is the restriction of the lattice order $\le_K$ to P, and for every lattice L and every order-preserving map $\varphi\colon (P; \le_P) \to L$, there exists a lattice homomorphism $K \to L$ that extends φ. If so, then we denote K by $\mathrm{CF}(P; \le)$. The ordered set $(P_n; \le)$ was defined right before (and in) Lemma 4.9; let $[P_n]_{\mathrm{FL}(n)} = [P_n]_{\mathrm{SFL}(n)}$ denote the sublattice generated by it.

Corollary 4.10.

(i) *As an ordered set,* $(P_n; \le)$ *is described by Lemmas* 4.7 *and* 4.9*; note that for* $3 < n \in \mathbb{N}^+$, $(P_n; \le)$ *is given also by Figure* 1.

(ii) *The sublattice* $[P_n]_{\mathrm{FL}(n)} = [P_n]_{\mathrm{SFL}(n)}$ *is completely freely generated by* $(P_n; \le)$.
(iii) *Furthermore,* $\mathrm{SFL}(n)$ *has a sublattice isomorphic to* $\mathrm{FL}(\omega)$.

Part (iii) is a consequence of Theorem 1.2(B); the point is that we can easily conclude Corollary 4.10(iii) from known results and the previous lemmas.

Proof of Corollary 4.10. As opposed to other proofs in the paper, the present argument relies on some outer references that are not quoted with full details. Part (i) is clear. For the validity of Part (ii), we need to show that for arbitrary $(k+k)$-ary lattice terms t_1 and t_2, the inequality

$$t_1(m_i : i < k, \overline{m_i} : i < k) \le t_2(m_i : i < k, \overline{m_i} : i < k) \tag{4.21}$$

holds in $\mathrm{FL}(n)$ iff it holds in the completely free lattice $\mathrm{CF}(P_n; \le)$. The satisfaction of (4.21) in $\mathrm{CF}(P_n; \le)$ can be tested by Dean's algorithm, which is a generalization of Whitman's algorithm; see Dean [3] or see Freese, Ježek, and Nation [9, Theorem 5.19]. This is a recursive algorithm that uses only the following three properties of $\mathrm{CF}(P_n; \le)$ and $(P_n; \le)$:

(D1) $\mathrm{CF}(P_n; \le)$ satisfies (W),
(D2) the elements of P_n are doubly prime, and
(D3) the description of the ordering of P_n.

It follows from Lemmas 4.2, 4.7, 4.8, and 4.9 that these properties hold for P_n as a subset of $\mathrm{FL}(n)$. Therefore, Dean's algorithm gives the same result in $\mathrm{CF}(P_n; \le)$ as (D1)–(D3) give in $\mathrm{FL}(n)$. Hence, we conclude that $[P_n]_{\mathrm{FL}(n)}$ is completely freely generated by $(P_n; \le)$, proving Part (ii). Thus, $\mathrm{CF}(P_n; \le)$ can be embedded into $\mathrm{SFL}(n)$. Using that $\mathrm{FL}(\omega)$ can be embedded into $\mathrm{FL}(3)$ by Whitman [20] and $\mathrm{FL}(3)$ can be embedded into $\mathrm{CF}(P_n; \le)$ by the main result of Rival and Wille [17], we conclude by transitivity that $\mathrm{SFL}(n)$ has a sublattice isomorphic to $\mathrm{FL}(\omega)$. This proves Part (iii). □

Now, we are ready to prove our Key Lemma.

Proof of Lemma 3.1. It is clear by (3.2) that

$$\{m_j : j \in \mathbb{N}_0\} \cup \{\overline{m_j} : j \in \mathbb{N}_0\} \cup \{a, b, \overline{a}, \overline{b}\} \subseteq \mathrm{SFL}(n). \tag{4.22}$$

In order to apply Lemma 4.3 and complete the proof in this way, it suffices to show that none of the inequalities

(ineq1) $x_0 \le a \vee \overline{a} \vee b \vee \overline{b}$,
(ineq2) $a \le x_0 \vee \overline{a} \vee b \vee \overline{b}$,
(ineq3) $b \le x_0 \vee a \vee \overline{a} \vee \overline{b}$,
(ineq4) $\overline{a} \le x_0 \vee a \vee b \vee \overline{b}$, and
(ineq5) $\overline{b} \le x_0 \vee a \vee \overline{a} \vee b$

holds in $\mathrm{FL}(n)$, because then the same will be true for their duals. For example, if $a \ge x_0 \wedge \overline{a} \wedge b \wedge \overline{b}$ held, then we could apply $\delta = \delta_{\mathrm{FL}(n)}$ from (1.1) to this inequality to obtain that (ineq4) holds.

First, we consider (ineq1). Suppose, for a contradiction, that it holds. Using (4.22), the elements we are going to deal with are in $\mathrm{SFL}(n)$. For $i \in \mathbb{N}^+$, we have that $m_i < m_{i+1}$ by Lemma 4.7, whereby Lemma 4.5 gives that $m_i \leq \overline{s}$; see Figure 1 for the meaning of $\overline{s}$ (and that of s). Since $m_0 < m_i$ by Lemma 4.7, Lemma 4.5 gives that $s \leq m_i$, whence $\overline{m_i} \leq \overline{s}$. Since $m_i \leq \overline{s}$ and $\overline{m_i} \leq \overline{s}$ for all $i \in \mathbb{N}^+$, (3.2) gives that $a \vee \overline{a} \vee b \vee \overline{b} \leq \overline{s}$. This is a contradiction, because (4.2) and (ineq1) imply that $a \vee \overline{a} \vee b \vee \overline{b} = 1_{\mathrm{SFL}(n)} > \overline{s}$.

Second, for the sake of contradiction, suppose that (ineq2) holds. Since $\overline{m_3} \leq a$, we obtain that

$$\overline{m_3} \leq x_0 \vee (\overline{m_1} \wedge m_3) \vee m_2 \vee \overline{m_4} \vee (\overline{m_2} \wedge m_4). \tag{4.23}$$

By Lemma 4.7, $\overline{m_3} \not\leq \overline{m_4}$. None of the inequalities $\overline{m_3} \leq \overline{m_1} \wedge m_3 \leq m_3$, $\overline{m_3} \leq m_2$, and $\overline{m_3} \leq \overline{m_2} \wedge m_4 \leq m_4$ holds by Lemma 4.9(iv). Thus, since $\overline{m_3}$ is join prime by Lemma 4.8, it follows from (4.23) that $\overline{m_3} \leq x_0$. Hence, (4.2) yields that $\overline{m_3} = 0_{\mathrm{SFL}(n)}$. So $0_{\mathrm{SFL}(n)} = \overline{m_3} > \overline{m_4} \in \mathrm{SFL}(n)$ by Lemma 4.7, and this is a contradiction. Thus, (ineq2) fails, as required.

Third, for the sake of contradiction, we suppose that (ineq3) holds. Since $m_2 \leq b$, we obtain that

$$m_2 \leq x_0 \vee m_1 \vee \overline{m_3} \vee (\overline{m_1} \wedge m_3) \vee (\overline{m_2} \wedge m_4). \tag{4.24}$$

By Lemma 4.7, $m_2 \not\leq m_1$. None of the inequalities $m_2 \leq \overline{m_1} \wedge m_3 \leq \overline{m_1}$, $m_2 \leq \overline{m_3}$, and $m_2 \leq \overline{m_2} \wedge m_4 \leq \overline{m_2}$ holds by Lemma 4.9(iii). Therefore, since m_2 is join prime by Lemma 4.8, (4.24) gives that $m_2 \leq x_0$. Hence, (4.2) and Lemma 4.7 yield that $m_1 < m_2 = 0_{\mathrm{SFL}(n)}$, contradicting $m_1 \in \mathrm{SFL}(n)$. Therefore, (ineq3) fails, as required.

Clearly, there is a lot of similarity between the treatment for (ineq2) and that for (ineq3). Namely, both arguments rely on (4.2), Lemmas 4.7, 4.8, and 4.9, and some comparabilities among the subscripts. In an analogous way, the argument for (ineq4) and that for (ineq5) are also very similar; this justifies that only the first of them will be detailed.

For the sake of contradiction, suppose that (ineq4) is satisfied, that is,

$$\overline{m_1} \wedge m_3 \leq x_0 \vee m_1 \vee \overline{m_3} \vee m_2 \vee \overline{m_4} \vee \underline{(\overline{m_2} \wedge m_4)}. \tag{4.25}$$

We are going to parse this inequality by (W), taking into account that, according to Lemmas 4.1 and 4.8, both meetands on the left and the five non-underlined joinands on the right of (4.25) are doubly prime elements. Therefore, either one of the two meetands is less than or equal to one of the five non-underlined joinands, or $\overline{m_1} \wedge m_3 \leq \overline{m_2} \wedge m_4$; so we need to consider only these possibilities. If we had that $\overline{m_1} \leq x_0$ or $m_3 \leq x_0$, then (4.2) and Lemma 4.7 would lead to $\overline{m_2} < \overline{m_1} = 0_{\mathrm{SFL}(n)}$ or $m_2 < m_3 = 0_{\mathrm{SFL}(n)}$, which are contradictions. If one of the two meetands was less than or equal to another non-underlined joinand, then Lemma 4.7 or Lemma 4.9 would prompt give a contradiction. We are left with the case $\overline{m_1} \wedge m_3 \leq \overline{m_2} \wedge m_4$, but then $\overline{m_1} \wedge m_3 \leq \overline{m_2}$, so the meet primeness of $\overline{m_2}$ gives that $\overline{m_1} \leq \overline{m_2}$, contradicting Lemma 4.7, or $m_3 \leq \overline{m_2}$, contradicting Lemma 4.9(iii). Therefore, (ineq4) fails, as required. Finally, as we have already mentioned, (ineq5) fails by an analogous argument. This completes the proof of Lemma 3.1 □

5. From the Key Lemma to a stronger statement

If a subset X of a lattice freely generates, then so do the subsets of X. Thus, (the Key) Lemma 3.1 implies that, for every $3 \le \lambda \in \mathbb{N}^+$, there is a totally symmetric embedding $\mathrm{FL}(4) \to \mathrm{FL}(\lambda)$. In particular, Lemma 3.1 implies Corollary 1.3. In this section, with the extensive help of Czédli [2], we lift the rank 4 of FL(4) to all even natural numbers $\kappa \ge 4$ and even to $\aleph_0$. That is, we are going to prove the following lemma. Remember that a, $\overline{a}$, b and $\overline{b}$ have been defined in (3.2).

Lemma 5.1. *If $\kappa = \aleph_0$ or $\kappa \ge 4$ is an even integer, then for every integer $\lambda = n \ge 3$, there exists a totally symmetric embedding $\tau_{\kappa\lambda}\colon \mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ with the additional property that $\tau_{\kappa\lambda}(\mathrm{FL}(\kappa))$ is included in the sublattice generated by $\{a, \overline{a}, b, \overline{b}\}$.*

Proof. First, in order to make our references to Czédli [2] convenient, we need to deal with the notation. Let $(y_1, y_2, y_3, y_4) := (a, \overline{a}, b, \overline{b}) \in \mathrm{SFL}(n)^4$. It follows from (the Key) Lemma 3.1 that $\{y_1, \dots, y_4\}$ freely generates. This allows us to write $\mathrm{FL}(4) = \mathrm{FL}(y_1, \dots, y_4)$ in the present proof, so FL(4) is a sublattice of $\mathrm{SFL}(n)$. Since $\{y_1, \dots, y_4\} = \{a, \overline{a}, b, \overline{b}\}$ is closed with respect to $\delta_{\mathrm{FL}(n)}$ defined in (1.1), $\mathrm{FL}(4) = \mathrm{FL}(y_1, \dots, y_4)$ is selfdually positioned in $\mathrm{FL}(n)$. Hence,

$$\text{the restriction } \delta_4^{\text{sw}} := \delta_{\mathrm{FL}(n)}\rceil_{\mathrm{FL}(4)} \text{ of } \delta_{\mathrm{FL}(n)} \text{ to } \mathrm{FL}(4), \tag{5.1}$$

is a dual automorphism of FL(4). Note the rule that

$$\delta_4^{\text{sw}}(y_1) = y_2, \quad \delta_4^{\text{sw}}(y_2) = y_1, \quad \delta_4^{\text{sw}}(y_3) = y_4, \quad \delta_4^{\text{sw}}(y_4) = y_3. \tag{5.2}$$

We do not need the exact definition of the lattice terms x_1^{1+i} and x_2^{1+i} given in [2, Section 4], but we have to recall some of their properties. For $i \in \mathbb{N}_0$, x_1^{1+i} and x_2^{1+i} are lattice terms over $\{y_1, \dots, y_4\}$, that is, they belong to FL(4). For brevity, we denote $\{x_1^{1+i} : 2i < \kappa\} \cup \{x_2^{1+i} : 2i < \kappa\}$ by $\{x_1^{1+i}, x_2^{1+i} : 2i < \kappa\}$. By [2, Lemma 4.1], $\{x_1^{1+i}, x_2^{1+i} : 2i < \kappa\}$ freely generates a sublattice of FL(4). Hence, in the present proof, we can write that

$$\mathrm{FL}(\kappa) = \mathrm{FL}(x_1^{1+i}, x_2^{1+i} : 2i < \kappa) = [x_1^{1+i}, x_2^{1+i} : 2i < \kappa]_{\mathrm{FL}(4)}.$$

For example, $\mathrm{FL}(6) = \mathrm{FL}(x_1^{1+i}, x_2^{1+i} : 2i < 6) = \mathrm{FL}(x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_2^3)$. It is important that 6 and, in general, κ is even or $\aleph_0$, because for an odd integer $\kappa \in \mathbb{N}^+$, $\mathrm{FL}(x_1^{1+i}, x_2^{1+i} : 2i < \kappa)$ would be $\mathrm{FL}(\kappa + 1)$ rather than $\mathrm{FL}(\kappa)$.

This paragraph is to tailor the second half of [2, Lemma 4.1] to the present situation; the reader may want to skip over it. It is irrelevant for us what a and b denote in [2]; they are not the same as here. It is also irrelevant that $\mathrm{FL}(4) = \mathrm{FL}(y_1, \dots, y_4)$ is embedded into $\mathrm{FL}(x, y, z)$ in [2] but into $\mathrm{FL}(n)$ here; these two embeddings are different even for $n = 3$. Using the first page, Lemma 2.1(B), and the second line of Section 4 of [2], we obtain from [2] that

$\delta = \delta_{\mathrm{FL}(x,y,z)}\rceil_{\mathrm{FL}(y_1,\dots,y_4)}$ in [2, Lemma 4.1] denotes a dual automorphism of $\mathrm{FL}(4) = \mathrm{FL}(y_1,\dots,y_4)$ such that (5.2) holds also for δ. Therefore, δ in [2, Lemma 4.1] is the same as δ_4^{sw} here, and the second half of [2, Lemma 4.1] asserts that $\delta_4^{\mathrm{sw}}(x_1^i) = x_2^i$ and $\delta_4^{\mathrm{sw}}(x_2^i) = x_1^i$ for all meaningful i.

So, [2, Lemma 4.1] yields that $\delta_4^{\mathrm{sw}}(x_1^i) = x_2^i$ and $\delta_4^{\mathrm{sw}}(x_2^i) = x_1^i$ hold for all i such that $2i < \kappa$. Therefore, since δ_4^{sw} is the restriction of $\delta_{\mathrm{FL}(n)}$ by (5.1), the set $\{(x_1^{1+i}, x_2^{1+i} : 2i < \kappa\}$ is selfdually positioned in $\mathrm{FL}(n)$. Consequently, so is the sublattice $\mathrm{FL}(\kappa) = [x_1^{1+i}, x_2^{1+i} : 2i < \kappa]_{\mathrm{FL}(n)}$. Finally, $\mathrm{FL}(\kappa) \subseteq [y_1,\dots,y_4]_{\mathrm{FL}(n)} = [a,\overline{a},b,\overline{b}]_{\mathrm{FL}(n)} \subseteq \mathrm{SFL}(n)$. Hence, the inclusion map $\tau_{\kappa\lambda}\colon \mathrm{FL}(\kappa) \to \mathrm{FL}(n) = \mathrm{FL}(\lambda)$ is a totally symmetric embedding and $\tau_{\kappa\lambda}(\mathrm{FL}(\kappa)) \subseteq [a,\overline{a},b,\overline{b}]$. This completes the proof of Lemma 5.1. $\square$

6. The rest of the proofs

Proof of Theorem 1.4. If $\mathrm{FL}(\lambda)$ has a selfdually positioned sublattice isomorphic to $\mathrm{FL}(\kappa)$, then $\max\{\kappa,\aleph_0\} = |\,\mathrm{FL}(\kappa)| \le |\,\mathrm{FL}(\lambda)| = \max\{\lambda,\aleph_0\}$, as required. Conversely, assume that $\max\{\kappa,\aleph_0\} \le \max\{\lambda,\aleph_0\}$. To specify the free generators, we let $\mathrm{FL}(\kappa) = \mathrm{FL}(y_i : i < \kappa)$ and $\mathrm{FL}(\lambda) = \mathrm{FL}(x_i : i < \lambda)$. We can assume that $\kappa > \lambda$, since otherwise the sublattice $[x_i : i < \kappa]$ generated by $\{x_i : i < \kappa\}$ in $\mathrm{FL}(\lambda)$ is selfdually positioned and it is isomorphic to $\mathrm{FL}(\kappa)$. The inequality $\kappa > \lambda$ together with $\max\{\kappa,\aleph_0\} \le \max\{\lambda,\aleph_0\}$ and $3 \le \kappa$ give that $4 \le \kappa \le \aleph_0$ and $\lambda \in \mathbb{N}^+$. We can assume that $\kappa = 2k+1 \ge 4$ is an odd integer, since otherwise Lemma 5.1 gives a totally symmetric embedding $\varphi\colon\ \mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ and $\varphi(\mathrm{FL}(\kappa))$ does the job. Again by Lemma 5.1, we can pick a $2k$-element subset $C = \{c_i : i < 2k\}$ of the sublattice $[a,\overline{a},b,\overline{b}]$ of $\mathrm{FL}(\lambda)$ such that the sublattice $[C]$ is selfdually positioned in $\mathrm{FL}(\lambda)$ and $[C]$ is freely generated by $C = \{c_i : i < 2k\}$. Since the natural dual automorphism $\delta_{\mathrm{FL}(\lambda)}$ from (1.1) preserves double irreducibility in $[C]$, it follows, after slight notational changes, from (2.1) and (2.3) that the set C itself is selfdually positioned, that is, $\delta_{\mathrm{FL}(\lambda)}(C) = C$. Let $D = C \cup \{x_0\}$. Since $\delta_{\mathrm{FL}(\lambda)}(x_0) = x_0$ by definition, $\delta_{\mathrm{FL}(\lambda)}(D) = D$. This yields that $[D] = [D]_{\mathrm{FL}(\lambda)}$, the sublattice generated by D, is selfdually positioned, that is, $\delta_{\mathrm{FL}(\lambda)}([D]) = [D]$. Therefore, since $|D| = 2k+1 = \kappa$, we need to show only that D freely generates. By duality and Lemma 4.3, it suffices to exclude that

$$x_0 \le c_0 \vee c_1 \vee \dots \vee c_{2k-1}, \qquad \text{or} \tag{6.1}$$

$$c_j \le x_0 \vee \bigvee_{i\in[2k)\setminus\{j\}} c_i \qquad \text{for some } j \in [2k). \tag{6.2}$$

We know from Lemma 3.1 that the sublattice $S := [a,\overline{a},b,\overline{b},x_0]$ of $\mathrm{FL}(n)$ is freely generated by the set $\{a,\overline{a},b,\overline{b},x_0\}$. Therefore, the self-maps

$$\xi_1 := \begin{pmatrix} a & \overline{a} & b & \overline{b} & x_0 \\ 0_S & 0_S & 0_S & 0_S & 1_S \end{pmatrix} \quad \text{and} \quad \xi_2 := \begin{pmatrix} a & \overline{a} & b & \overline{b} & x_0 \\ a & \overline{a} & b & \overline{b} & 0_S \end{pmatrix}$$

extend to endomorphisms $\widehat{\xi_1}\colon S \to S$ and $\widehat{\xi_2}\colon S \to S$, respectively. Using the inclusion $C \subseteq [a, \overline{a}, b, \overline{b}]$, we obtain that $\xi_1([a, \overline{a}, b, \overline{b}]) = \{0_S\}$. Taking the equality $\xi_1(x_0) = 1_S$ also into account, we obtain that the endomorphism $\widehat{\xi_1}$ does not preserve inequality (6.1). Hence, (6.1) fails, as required. Next, suppose that (6.2) holds. Using $C \subseteq [a, \overline{a}, b, \overline{b}]$, it follows that the restriction of $\widehat{\xi_2}$ to $[a, \overline{a}, b, \overline{b}]$ is the identity map. Therefore, since $\widehat{\xi_2}$ is order-preserving, its application to (6.2) yields that $c_j \le \bigvee_{i\in[2k)\setminus\{j\}} c_i$. But this is a contradiction since C freely generates, and we conclude that (6.2) fails, as required. $\square$

Proof of Corollary 2.3. In order to prove the implication (ii) ⇒ (i), assume that (ii) holds. We can also assume that $\kappa \ne \lambda$ since otherwise the identity map $\mathrm{FL}(\kappa) \to \mathrm{FL}(\kappa) = \mathrm{FL}(\lambda)$ does the job. As it is pointed out right after (2.4), $[\mathrm{DAut}(\mathrm{FL}(\lambda)) : \mathrm{Aut}(\mathrm{FL}(\lambda))] = 2$. Hence, with $\delta_{\mathrm{FL}(\lambda)}$ from (1.1),

$$\mathrm{DAut}(\mathrm{FL}(\lambda)) = \mathrm{Aut}(\mathrm{FL}(\lambda)) \cup \{\delta_{\mathrm{FL}(\lambda)} \circ \varphi : \varphi \in \mathrm{Aut}(\mathrm{FL}(\lambda))\}. \tag{6.3}$$

Thus, the embedding given by Lemma 5.1 has a DAut-symmetric range. Hence, (ii) ⇒ (i).

Before proving the converse implications, we formulate and verify some observations, some of which will be useful also in the proof of Corollary 2.4 below. This is why instead of assuming DAut-symmetry, we often assume less, the usual symmetry (with respect to automorphisms). Since $\mathrm{Aut}(\mathrm{FL}(X))$ acts transitively on the set X of free generators, it follows trivially that

$$\text{if } S \text{ is a symmetric sublattice of } \mathrm{FL}(X) \text{ such that } S \cap X \ne \emptyset, \text{ then } S = \mathrm{FL}(\lambda). \tag{6.4}$$

As a straightforward consequence of (2.1), observe that

$$\text{if } \varphi\colon \mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda) \text{ is an arbitrary embedding and } S := \varphi(\mathrm{FL}(\kappa)), \text{ then } |\mathrm{Irr}^{\vee}_{\wedge}(S)| = \kappa. \tag{6.5}$$

Since automorphisms and dual automorphisms preserve double primeness, we obtain the following observation.

Let S be a symmetric sublattice of $\mathrm{FL}(\lambda)$; then $\tau\rceil_{\mathrm{Irr}^{\vee}_{\wedge}(S)}\colon \mathrm{Irr}^{\vee}_{\wedge}(S) \to \mathrm{Irr}^{\vee}_{\wedge}(S)$ is a bijective map for every $\tau \in \mathrm{Aut}(\mathrm{FL}(\lambda))$. If, in addition, S is DAut-symmetric, then the same holds even for every $\tau \in \mathrm{DAut}(\mathrm{FL}(\lambda))$. (6.6)

We are going to prove the following property of orbits $\{\tau(u) : \tau \in \mathrm{Aut}(\mathrm{FL}(\lambda))\}$ of elements $u \in \mathrm{FL}(\lambda)$.

$$\text{If } \lambda \ge \aleph_0,\ S \text{ is a symmetric sublattice of } \mathrm{FL}(\lambda), \text{ and } u \in S, \text{ then } |S| = \lambda = |\{\tau(u) : \tau \in \mathrm{Aut}(\mathrm{FL}(\lambda))\}|. \tag{6.7}$$

In order to show (6.7), let S be a symmetric sublattice of $\mathrm{FL}(\lambda) = \mathrm{FL}(X)$, where $|X| = \lambda$, and let $u \in S$. Obviously, $|S| \le |\,\mathrm{FL}(\lambda)| = \lambda$. It is clear by (3.1) that there is a finite subset $Y \subseteq X$ such that u is in the sublattice $[Y]$ generated by Y. By the rudiments of cardinal arithmetics, there is a family $\{\pi_i : i < \lambda\}$ of permutations of X such that $\pi_i(Y) \cap \pi_j(Y) = \emptyset$ for $i \ne j$.

Each of these π_i extends to an automorphism π_i^{aut} of $\mathrm{FL}(\lambda)$. If $i \neq j$, then the map

$$X \to \mathbf{2}, \quad x \mapsto \begin{cases} 1, & \text{if } x \in \pi_i(Y), \\ 0, & \text{if } x \notin \pi_i(Y), \text{ in particular, if } x \in \pi_j(Y) \end{cases}$$

extends to a lattice homomorphism $\mathrm{FL}(\lambda) \to \mathbf{2}$. Since this homomorphism maps $\pi_i^{\mathrm{aut}}(u) \in [\pi_i(Y)]$ and $\pi_j^{\mathrm{aut}}(u) \in [\pi_j(Y)]$ to 1 and 0, respectively, we obtain that $\pi_i^{\mathrm{aut}}(u) \neq \pi_j^{\mathrm{aut}}(u)$. Furthermore, $\{\tau(u) : \tau \in \mathrm{Aut}(\mathrm{FL}(\lambda))\} \subseteq S$ since S is a symmetric sublattice. Hence,

$$\lambda = |\{\pi_i(u) : i < \lambda\}| \leq |\{\tau(u) : \tau \in \mathrm{Aut}(\mathrm{FL}(\lambda))\}| \leq |S| \leq \lambda,$$

which proves (6.7). We also need the following consequence of (6.7).

$$\text{If } \lambda \geq \aleph_0,\ S \text{ is a symmetric sublattice of } \mathrm{FL}(\lambda), \text{ and } \mathrm{Irr}_\wedge^\vee(S) \neq \emptyset, \text{ then } |\mathrm{Irr}_\wedge^\vee(S)| = \lambda. \tag{6.8}$$

In order to show this, let $u \in \mathrm{Irr}_\wedge^\vee(S)$. Since S is symmetric, the restriction $\tau\rceil_S$ of an automorphism $\tau \in \mathrm{Aut}(\mathrm{FL}(\lambda))$ to S is an automorphism of S. Hence, $\tau(u) = \tau\rceil_S(u)$ also belongs to $\mathrm{Irr}_\wedge^\vee(S)$. Thus, we conclude from (6.7) that $\lambda \leq |\mathrm{Irr}_\wedge^\vee(S)| \leq |\mathrm{FL}(\lambda)| = \lambda$, implying the validity of (6.8). In the observation below, $\delta = \delta_{\mathrm{FL}(\lambda)}$ is the natural dual automorphism introduced in (1.1). An *involution* on a set Y is a map $Y \to Y$ whose square is the identity map on Y.

$$\text{If } S \text{ is a DAut-symmetric sublattice of } \mathrm{FL}(\lambda), \text{ then the restriction } \delta\rceil_{\mathrm{Irr}_\wedge^\vee(S)} \text{ of } \delta \text{ to } \mathrm{Irr}_\wedge^\vee(S) \text{ is an involution on } \mathrm{Irr}_\wedge^\vee(S). \tag{6.9}$$

Since every restriction of an involution is again an involution, (6.9) follows immediately from (6.6). Next, we are going to prove that

$$\text{if } \kappa \neq \lambda \text{ and there is an embedding } \mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda) \text{ with DAut-symmetric range, then } \kappa \text{ is not an odd integer.} \tag{6.10}$$

Suppose the contrary, and for an odd $\kappa \in \mathbb{N}^+$, let $\varphi\colon \mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ with range $S := \varphi(\mathrm{FL}(\kappa))$ witness the failure of (6.10). We know from (6.9) that $\delta\rceil_{\mathrm{Irr}_\wedge^\vee(S)}$ is an involution on $\mathrm{Irr}_\wedge^\vee(S)$. If $\delta\rceil_{\mathrm{Irr}_\wedge^\vee(S)}$ has a fixed point $u \in \mathrm{Irr}_\wedge^\vee(S)$, then $u = \delta\rceil_{\mathrm{Irr}_\wedge^\vee(S)}(u) = \delta(u)$ is one of the free generators of $\mathrm{FL}(\lambda)$ by (2.3), whereby (6.4) gives the equality in $\mathrm{FL}(\kappa) \cong S = \mathrm{FL}(\lambda)$, which implies $\kappa = \lambda$ by (2.1), contradicting $\kappa \neq \lambda$. Hence, $\delta\rceil_{\mathrm{Irr}_\wedge^\vee(S)}$ has no fixed point. By (6.5), this fixed-point-free involution acts on a κ-element set. Thus, κ is not an odd integer, proving (6.10).

Now, armed with (6.7), (6.8), and (6.10), we are in the position to prove that (i) implies (ii). Assume that (i) holds, and let $\varphi\colon \mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ be an embedding with DAut-symmetric range $S := \varphi(\mathrm{FL}(\kappa))$. We can also assume that $\kappa \neq \lambda$ since there is nothing to prove otherwise. Since φ is an embedding, $|\mathrm{FL}(\kappa)| \leq |\mathrm{FL}(\lambda)|$. There are two cases, depending on λ. First, if $\lambda < \aleph_0$, then κ is not an odd integer by (6.10) and, furthermore, $|\mathrm{FL}(\kappa)| \leq |\mathrm{FL}(\lambda)| = \aleph_0$ gives that $\kappa \leq \aleph_0$. Hence, (ii) holds in this case. Second, if $\lambda \geq \aleph_0$, then

$$\kappa \overset{(2.1)}{=} |\mathrm{Irr}_\wedge^\vee(\mathrm{FL}(\kappa))| = |\mathrm{Irr}_\wedge^\vee(S)| \overset{(6.8)}{=} \lambda, \tag{6.11}$$

and (ii) holds again. The proof of Corollary 2.3 is complete. □

Proof of Theorem 1.2. It suffices to prove part (A), since it implies part (B). Let $3 \le \lambda \in \mathbb{N}^+$, and assume that $\kappa = \aleph_0$ or $3 \le \kappa \in \mathbb{N}^+$ is even. Then there exists a totally symmetric embedding from $\mathrm{FL}(\kappa)$ to $\mathrm{FL}(\lambda)$ by Lemma 5.1.

Conversely, assume that $3 \le \kappa$, $3 \le \lambda$, and there exists a totally symmetric embedding $\varphi\colon \mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$. Let $S = \varphi(\mathrm{FL}(\kappa))$ denote the range of φ; it consists of some symmetric elements of $\mathrm{FL}(\lambda)$. It follows that $\lambda < \aleph_0$, because otherwise there would be no symmetric element in $\mathrm{FL}(\lambda)$. Thus, we have also that $\kappa \le \aleph_0$, because $\kappa \le |\mathrm{FL}(\kappa)| \le |\mathrm{FL}(\lambda)| = \aleph_0$. Since S is invariant under the natural dual automorphism $\delta := \delta_{\mathrm{FL}(\lambda)}$ and it is symmetric, even element-wise symmetric, (6.3) shows that S is DAut-symmetric. By (6.9), $\delta\rceil_{\mathrm{Irr}^\vee_\wedge(S)}$ is an involution on $\mathrm{Irr}^\vee_\wedge(S)$. No free generator of $\mathrm{FL}(\lambda)$ is a symmetric element of $\mathrm{FL}(\lambda)$, whereby S is disjoint from the set of free generators of $\mathrm{FL}(\lambda)$. If $\delta\rceil_{\mathrm{Irr}^\vee_\wedge(S)}$ had a fixed point u, then u would be a fixed point of δ, so (2.3) would imply that $u \in \mathrm{Irr}^\vee_\wedge(S) \subseteq S$ is a free generator of $\mathrm{FL}(\lambda)$, contradicting the above-mentioned disjointness. Thus, $\delta\rceil_{\mathrm{Irr}^\vee_\wedge(S)}$ has no fixed point. By (6.5), the fixed-point-free involution $\delta\rceil_{\mathrm{Irr}^\vee_\wedge(S)}$ acts on the κ-element set $\mathrm{Irr}^\vee_\wedge(S)$, and we conclude that κ is not an odd integer. That is, $\kappa = \aleph_0$ or $\kappa \in \mathbb{N}^+$ is even, completing the proof. □

Proof of Corollary 2.4. In order to prove the "if" part, we can assume that $\kappa \ne \lambda$ since otherwise the identity map of $\mathrm{FL}(\kappa)$ is a required embedding. So $3 \le \lambda \in \mathbb{N}^+$ and $3 \le \kappa \le \aleph_0$. By Theorem 1.2(A), $\mathrm{FL}(\lambda)$ has an element-wise symmetric sublattice S such that $S \cong \mathrm{FL}(\aleph_0)$. Since $\kappa \le \aleph_0$, $\mathrm{FL}(\lambda)$ has also a sublattice S' such that $\mathrm{FL}(\kappa) \cong S'$. Clearly, any isomorphism from $\mathrm{FL}(\kappa) \to S'$ is an embedding of $\mathrm{FL}(\kappa)$ into $\mathrm{FL}(\lambda)$ with symmetric range; in fact, with element-wise symmetric range. This proves the "if" part.

In order to prove the "only if" part, assume that there is an embedding $\varphi\colon \mathrm{FL}(\kappa) \to \mathrm{FL}(\lambda)$ with symmetric range S. Clearly, $|\mathrm{FL}(\kappa)| \le |\mathrm{FL}(\lambda)|$. Depending on λ, there are two cases to consider. First, if $\lambda < \aleph_0$, then $|\mathrm{FL}(\kappa)| \le |\mathrm{FL}(\lambda)| = \aleph_0$ yields that $\kappa \le \aleph_0$, as required. Second, if $\lambda \ge \aleph_0$, then (6.11) applies and $\kappa = \lambda$, again as required. The proof is complete. □

7. A computer program and its background

Historical background

There are various known algorithms to solve the word problem of free lattices and that of finitely presented lattices. They are discussed in Freese and Nation [11] and in Sections 8 and 9 of Chapter XI of the monograph Freese, Ježek, and Nation [9]; see also Dean [4], Evans [6], McKinsey [13], and Skolem [18] for the original papers. In addition to this list, there is an additional algorithm given in Czédli [1]. We know from [9] that the algorithms given by Skolem, Freese, and Herrmann run in polynomial time; so does the one given in [1]. However, it is only Whitman's algorithm with the modifications explained in [9] that is fast enough for our purposes.

A new computer program

The first author has developed a Dev-Pascal 1.9.2 (Freepascal) program for the word problem of free lattices. This problem is based on the Freese-Whitman algorithm, as it is given in Freese, Ježek, and Nation [9]. The program runs in Windows environment (tested only under Windows 10), and it can be downloaded from the author's website. The program takes its input from a text file; several sample input files are also donwloadable. We used this program on our personal computer with IntelCore i5-4440 CPU, 3.10 GHz, and 8.00 GB RAM.

Results achieved with the computer program

First, we used the program to give alternative proofs. In particular, we used it

to prove the second part of the (Key) Lemma 3.1 asserting that $\{a, \overline{a}, b, \overline{b}, x_0\}$ freely generates. (7.1)

Also, we used the program to prove that

for $n = 3$, the Key Lemma remains valid if we replace m_1, m_2, m_3 and m_4 by m_5, m_7, m_8, and m_9, respectively; (7.2)

this gives an alternative proof of Corollary 1.3. By the paragraph preceding (4.25), it would not be difficult to show that the stipulation $n = 3$ can be omitted from (7.2), but this or a similar strengthening of (7.2) is not pursued.

In addition to reaffirming some results from the previous sections, we could use the program to find an entirely new construction to prove Corollary 1.3. In order to describe it, we use the notation introduced in Remark 2.5 to define a join-homomorphism $\nu^{(\vee)}\colon \mathrm{FL}(3) \to \mathrm{SFL}(3)$ and a meet-homomorphism $\mu^{(\wedge)}\colon \mathrm{FL}(3) \to \mathrm{SFL}(3)$ by the rules

$$\nu^{(\vee)}(u) := \bigvee_{\sigma\in\mathrm{Sym}_3} \sigma^{\mathrm{aut}}(u) \quad\text{and}\quad \mu^{(\wedge)}(u) := \bigwedge_{\sigma\in\mathrm{Sym}_3} \sigma^{\mathrm{aut}}(u). \tag{7.3}$$

In order to ease the notation, we will write x, y, and z instead of x_0, x_1, and x_2, respectively. Note that the program recognizes (appropriate commands for) $\nu^{(\vee)}$ and $\mu^{(\wedge)}$ in input files. Take the following ternary terms, that is, elements of $\mathrm{FL}(3) = \mathrm{FL}(x, y, z)$.

$$\begin{aligned} a_0 &= \nu^{(\vee)}\Big(\big(((x\vee y)\wedge z)\vee y\big)\wedge\big(((y\vee x)\wedge z)\vee x\big)\Big),\\ a' &= \mu^{(\wedge)}\Big(\big(((a_0\wedge x)\vee y)\wedge z\big)\vee\big(((z\wedge x)\vee y)\wedge a_0\big)\Big), \quad\text{and} \qquad (7.4)\\ b' &= \mu^{(\wedge)}\Big(\big((((x\vee y)\wedge(x\vee z))\vee a')\wedge x\big)\vee\big(((x\wedge a_0)\vee y)\wedge z\big)\Big). \qquad (7.5)\end{aligned}$$

With a' and b' from (7.4) and (7.5) and their duals, $\overline{a'}$ and $\overline{b'}$, the program proved that

$$\{a', \overline{a'}, b, \overline{b'}\} \text{ freely generates a sublattice of } \mathrm{FL}(x, y, z), \tag{7.6}$$

which obviously implies Corollary 1.3. Note that for each of (7.1), (7.2), and (7.6), the program ran less than a millisecond on our computer.

	1st generator	2nd generator	N_{var}(generating set)
$(3.3)_{n=3}$	$N_{\text{var}}(a) = 108$	$N_{\text{var}}(b) = 228$	$N_{\text{var}}(\{a, \overline{a}, b, \overline{b}\}) = 672$
(7.6)	$N_{\text{var}}(a') = 612$	$N_{\text{var}}(b') = 4008$	$N_{\text{var}}(\{a', \overline{a'}, b', \overline{b'}\}) = 9240$

(7.7)

Finally, for a lattice term t, we define the *total number* $N_{\text{var}}(t)$ of *variables* of t by induction as follows: $N_{\text{var}}(t) = 1$ if t is a variable and

$$N_{\text{var}}(t_1 \vee t_2) = N_{\text{var}}(t_1 \wedge t_2) = N_{\text{var}}(t_1) + N_{\text{var}}(t_2).$$

Note that, say, $x = x \wedge (x \vee y)$ in $\mathrm{FL}(x, y, z)$ but $N_{\text{var}}(x) = 1$ is distinct from $N_{\text{var}}(x \wedge (x \vee y)) = 3$. Hence, as opposed to what (3.1) suggests, we do not define N_{var} for the elements of $\mathrm{FL}(x, y, z)$. For a set $\{t_1, \dots, t_k\}$ of terms, let $N_{\text{var}}(\{t_1, \dots, t_k\}) = N_{\text{var}}(t_1) + \cdots + N_{\text{var}}(t_k)$. Table 7.7 shows how the function N_{var} compares the *terms* describing the free generating set given in (3.3) for $n = 3$ and those given in (7.4) and (7.6). Another difference between (3.3) and (7.6) is that, as opposed to the set $\{a, \overline{a}, b, \overline{b}, x_0\}$ from (the Key) Lemma 3.1, the program shows that $\{a', \overline{a'}, b, \overline{b'}, x\}$ does *not* generate freely.

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