DISTRIBUTIVE LATTICES DETERMINED BY WEIGHTED DOUBLE SKELETONS

GÁBOR CZÉDLI, JOANNA GRYGIEL, AND KATARZYNA GRYGIEL

ABSTRACT. Related to his S-glued sum construction, the skeleton S(L) of a finite lattice L was introduced by C. Herrmann in 1973. Our theorem asserts that if D is a finite distributive lattice and its second skeleton, S(S(D)), is the trivial lattice, then D is characterized by its weighted double skeleton, introduced by the second author in 2006. The assumption on the second skeleton is essential.

1. INTRODUCTION

Let L be a finite modular lattice. Then, according to Herrmann [11], L is the union of its maximal complemented (equivalently, atomistic) intervals, which are glued together along a lattice S = S(L), the *skeleton* of L. His construction of S(L) makes sense even without modularity, so we drop this assumption until otherwise stated. It pointed out somewhat later that S(L) is a factor lattice of L by a tolerance relation in the sense of the first author [3]. Define $S^0(L) := L$ and $S^{i+1}(L) := S(S^i(L))$. Then there is a smallest n such that $|S^n(L)| = 1$, which we will call the Herrmann rank of L. We say that L is H^n -irreducible if its Herrmann rank is at most n. Equivalently, L is H^n -irreducible iff $|S^n(L)| = 1$. H^1 -irreducibility was previously called H-irreducibility by the second author in the monograph [9] and in many of her papers, including [9] and [10].

The skeleton of L does not tell too much on L. Indeed, the second author [9, Corollary 3.2.6] proved that each finite lattice S is the skeleton of infinitely many pairwise non-isomorphic finite distributive lattices. The weighted double skeleton $S^{\text{wd}}(L)$ of L, introduced by the second author in [10] and to be defined below, carries much more information on the initial lattice.

Let \mathcal{K} be a class of finite distributive lattices, and let $L \in \mathcal{K}$. If for any $L' \in \mathcal{K}$ such that $S^{\text{wd}}(L')$ is isomorphic to $S^{\text{wd}}(L)$ the lattice L' is isomorphic to L, then we say that L is *determined* by its weighted double skeleton in the class \mathcal{K} .

As usual, the partially ordered set (in short, the poset, in other words, the order) of all non-zero join-irreducible elements of L is denoted by $\mathsf{Ji}(D) = (\mathsf{Ji}(D), \leq)$. The sets $\{1, 2, 3, \ldots\}$ and $\{0, 1, 2, 3, \ldots\}$ are denoted by \mathbb{N} and \mathbb{N}_0 , respectively. The

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length of a finite poset Q is $length(Q) := max\{n \in \mathbb{N}_0 : Q \text{ has an } n + 1\text{-element chain}\}$. A *nontrivial lattice* is a lattice that has at least two elements. Postponing the rest of definitions to the next section, our main result reads as follows.

Theorem 1.1. Let L be a finite nontrivial lattice.

- (i) If L is modular and H^2 -irreducible, then length($\operatorname{Ji}(L)$) ≤ 1 .
- (ii) If L is distributive and H²-irreducible, then L is determined by its weighted double skeleton in the class of finite distributive lattices.
- (iii) If L is distributive and length(Ji(L)) ≤ 1 , then L is determined by its weighted double skeleton in the class of finite distributive lattices D satisfying the inequality length(Ji(D)) ≤ 1 .

Notice that $S^{\text{wd}}(L)$ determines some properties of a modular L even if H^2 irreducibility is not assumed. Namely, $S^{\text{wd}}(L)$ clearly determines length(L), and
see Lemma 3.1 for further properties in the distributive case. However, we will
soon prove the following remark, which indicates that Theorem 1.1 is optimal in
some sense.

Remark 1.2. There exist H^3 -irreducible finite distributive lattices L_1 and L_2 such that $S^{\text{wd}}(L_1)$ is isomorphic to $S^{\text{wd}}(L_2)$ but L_1 is not isomorphic to L_2 . Also, there is a finite distributive lattice L such that length(Ji(L)) ≤ 1 but L is not H^2 -irreducible.

A well-known economic way of describing a finite distributive lattice D by a little amount of data is to consider $Ji(D) = (Ji(D), \leq)$. The next remark outlines a more economic way for certain distributive lattices.

Remark 1.3. Let *D* be a finite distributive lattice with length($\operatorname{Ji}(D)$) ≤ 1 . Assume that *D* is the union of few maximal boolean intervals but |D| is large. Then $S^{\operatorname{wd}}(D)$ constitutes an economic description of *D*.

2. Basic concepts and statements

For the basic concepts of Lattice Theory the reader is referred to Grätzer [7]. By a *tolerance* of a lattice L, we mean a reflexive, symmetric, compatible relation of L. Equivalently, a tolerance of L is the image of a congruence by a surjective lattice homomorphism onto L, see the first author and Grätzer [4]. Let R be a tolerance of L. If $X \subseteq L$ is a maximal subset with respect to the property $X \times X \subseteq R$, then Xis called a *block* of R. Blocks are convex sublattices by Bandelt [1] and Chajda [2]. Let α and β be blocks of R. As it follows immediately from Zorn's Lemma, there are blocks γ and δ of R such that

(2.1)
$$\{x \lor y : x \in \alpha, \ y \in \beta\} \subseteq \gamma =: \alpha \lor \beta, \\ \{x \land y : x \in \alpha, \ y \in \beta\} \subseteq \delta =: \alpha \land \beta.$$

The first author [3] proved that γ and δ are uniquely determined, and the set L/R of all blocks of R with the join and meet defined by (2.1) is a lattice. This lattice, also denoted by L/R, is called the *factor lattice* (or quotient lattice) of L modulo R. Notice that there is an alternative way, which does not rely on the axiom of choice (and, therefore, on Zorn's Lemma), to define L/R in an order-theoretic way and to prove that it is a lattice, see Grätzer and Wenzel [8].

In the rest of the paper, all lattices will be assumed to be finite. Then the blocks of a tolerance R are intervals. So if α is a block of R, then α equals the interval



FIGURE 1. Non-isomorphic lattices with isomorphic double skeletons

 $[0_{\alpha}, 1_{\alpha}]$ of L. It was proved in [3] and [6] that, for all $\alpha, \beta \in L/R$,

(2.2)
$$\begin{aligned} 0_{\alpha} \vee 0_{\beta} &= 0_{\alpha \vee \beta}, \quad 1_{\alpha} \vee 1_{\beta} \leq 1_{\alpha \vee \beta}, \\ 1_{\alpha} \wedge 1_{\beta} &= 1_{\alpha \wedge \beta}, \quad 0_{\alpha} \wedge 0_{\beta} \geq 0_{\alpha \wedge \beta}, \\ \alpha \leq \beta \text{ (in } L/R) \iff 0_{\alpha} \leq 0_{\beta} \iff 1_{\alpha} \leq 1_{\beta}. \end{aligned}$$

The most important particular case of L/R, under the name skeleton, was discovered by Herrmann [11] much earlier; we survey it partly and only for the finite case. A tolerance R of (a finite lattice) L is called a *glued tolerance*, see Reuter [12], if its transitive closure R^* is the total relation L^2 . The (unique) smallest glued tolerance of L is called the *skeleton tolerance* of L, and it is denoted by $\Theta(L)$. There are two easy ways to see that $\Theta(L)$ exists. Firstly, we know from the second author [9], and it is routine to check, that for any tolerance R of a finite lattice L,

(2.3)
$$R$$
 is a glued tolerance $\iff (x, y) \in R$ for all $x \prec y \in L$.

This clearly implies that the intersection of all glued tolerances of L is a glued tolerance again, whence it is the skeleton tolerance of L. Secondly, we know from [5] that the transitive closure of lattice tolerances commutes with their (finitary) intersections, which also implies the existence of $\Theta(L)$.

The factor lattice $L/\Theta(L)$ is called the *skeleton* S(L) of L. We claim that length(S(L)) <length(L). Indeed, assume that k =length(S(L)) and $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_k$ is a maximal chain in $S(L) = L/\Theta(L)$. It follows from (2.2) and (2.3) that $0_{\alpha_0} < 0_{\alpha_1} < \cdots < 0_{\alpha_k} < 1_{\alpha_k}$, showing that length $(S(L)) = k < k + 1 \leq$ length(L). The inequality length(S(L)) <length(L) shows that each finite lattice has a Herrmann rank.

It is clear from (2.2) that both $\{0_{\alpha} : \alpha \in S(L)\}$ and $\{1_{\alpha} : \alpha \in S(L)\}$, as subposets of L, are order isomorphic to S(L). Their union carries a lot of information on L provided we equip it with an appropriate structure. Following the second author [10], a structure $(P, \leq, K, \eta_0, \eta_1, \mathsf{w})$ will be called an *abstract weighted double skeleton* if (P, \leq) is a finite poset, K is a lattice, $\eta_0 : K \to P$ is a join-preserving (and, therefore, order-preserving) embedding, $\eta_1 : K \to P$ is a meet-preserving order-embedding, $P = \eta_0(K) \cup \eta_1(K), \eta_0(x) \leq \eta_1(x)$ holds for all $x \in K$, and w is a mapping of the covering relation $\{(a, b) \in P^2 : a \prec b\}$ into \mathbb{N} . The underlying set of $(P, \leq, K, \eta_0, \eta_1, \mathsf{w})$ is P, and we often denote the structure $(P, \leq, K, \eta_0, \eta_1, \mathsf{w})$ simply by P.



FIGURE 2. length($\operatorname{Ji}(L)$) ≤ 1 does not imply H^2 -irreducibility

Let $(P', \leq', K', \eta'_0, \eta'_1, w')$ be another abstract weighted double skeleton, and let (ψ, κ) be a pair of bijective mappings. We say that

$$(\psi,\kappa)\colon (P,\leq,K,\eta_0,\eta_1,\mathsf{w})\to (P',\leq',K',\eta_0',\eta_1',\mathsf{w}')$$

is an *isomorphism* if $\psi: (P, \leq) \to (P', \leq')$ is an order isomorphism, $\kappa: K \to K'$ is a lattice isomorphism, $\psi(\eta_i(x)) = \eta'_i(\kappa(x))$ for all $x \in K$ and $i \in \{0, 1\}$, and $\mathsf{w}(x, y) = \mathsf{w}'(\psi(x), \psi(x))$ for all $x, y \in P$ such that $x \prec y$. If there is such a (ψ, κ) , then the two abstract weighted double skeletons are called *isomorphic*. By the (concrete) weighted double skeleton of L we mean the structure

(2.4)
$$S^{wd}(L) := (\{0_{\alpha} : \alpha \in S(L)\} \cup \{1_{\alpha} : \alpha \in S(L)\}, \leq, S(L), \eta_0, \eta_1, \mathsf{w})$$

where \leq is the ordering inherited (restricted) from L, $\eta_0(\alpha) := 0_\alpha$ and $\eta_1(\alpha) := 1_\alpha$ for all $\alpha \in S(L)$, and $w(x, y) := \text{length}([x, y]_L)$ for any $x \prec_{S^{\text{wd}}(L)} y$.

For example, consider L_j given in Figure 1 for $j \in \{1, 2\}$. Then $S(L_j)$ is the threeelement chain $\{\zeta_0 \prec \alpha \prec \beta\}$, and $S^{wd}(L_j)$ is depicted in the middle of the figure. For $x \prec y$ in $S^{wd}(L_j)$, the edge $x \prec y$ of the diagram is labeled by w(x, y). Since $L_1 \not\cong L_2$ and $S^n(L_j)$ is the (4 - n)-element chain for $n \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, Figure 1, together with the self-explanatory Figure 2, proves Remark 1.2.

We have defined all the concepts Theorem 1.1 is based on. The rest of the paper is devoted to proofs, including some auxiliary statements.

3. Proofs and auxiliary statements

3.1. The number of join-irreducible elements in a block. Unless otherwise stated, by a *block* of a lattice *L* we mean a block of its skeleton tolerance $\Theta(L)$, that is, a member of the skeleton S(L). Throughout this subsection, *L* denotes a finite modular lattice. We are going to extend the weight function w of $S^{wd}(L)$, see (2.4), to a function $w^* : S^{wd}(L) \times S^{wd}(L) \to \mathbb{N}_0$. If $x, y \in S^{wd}(L)$ and $x \not< y$, then we let $w^*(x, y) := 0$. If $x, y \in S^{wd}(L)$ and x < y, then take a maximal chain $z_0 = x \prec_{S^{wd}(L)} z_1 \prec_{S^{wd}(L)} \cdots \prec_{S^{wd}(L)} z_t = y$ in $S^{wd}(L)$, and define

$$w^*(x,y) := \sum_{i=0}^{t-1} w(z_i, z_{i+1}).$$

The lattice theoretical Jordan-Hölder theorem applies on L and we conclude that $w(x, y) := \text{length}([x, y]_L)$ for any $x \leq_{S^{\text{wd}}(L)} y$. This guarantees that $w^*(x, y)$ does not depend on the maximal chain chosen. Given a poset (Q, \leq) , the *Möbius function*

 $\mu_Q \colon Q^2 \to \mathbb{Z}$ is defined recursively as follows:

$$\mu_Q(x, y) = \begin{cases} 1, & \text{if } x = y; \\ -\sum_{x \le z < y} \mu_Q(x, z), & \text{if } x < y; \\ 0, & \text{if } x \le y. \end{cases}$$

If L is a finite modular lattice, then every $\alpha \in S(L)$ is an atomistic lattice by Herrmann [11]. In this case, $\operatorname{Ji}(\alpha)$ stands for the set of join-irreducible elements of α that are distinct from 0_{α} . The set of atoms of α is denoted by $\operatorname{At}(\alpha)$. We let

(3.1)
$$J_{\alpha} := \mathsf{Ji}(L) \cap \mathsf{Ji}(\alpha) = \mathsf{Ji}(L) \cap \mathsf{At}(\alpha) = \mathsf{Ji}(L) \cap (\alpha \setminus \{0_{\alpha}\}).$$

Since $S^{\text{wd}}(L)$ determines S(L), the next lemma, based on the notation above, implies that $|J_{\alpha}|$ is determined by $S^{\text{wd}}(L)$, provided L is distributive.

Lemma 3.1. Let L be a finite distributive lattice. Then for each $\alpha \in S(L)$,

(3.2)
$$|J_{\alpha}| = \sum_{\beta \le \alpha, \ \beta \in S(L)} \mu_{S(L)}(\beta, \alpha) \cdot \mathsf{w}^{*}(0_{\alpha}, 1_{\beta}).$$

Proof. For $k \in \mathbb{N}$ and a poset K, let $\mathsf{Ji}_k(K)$ denote the set of elements of K with exactly k lower covers. Notice that $\mathsf{Ji}(L) = \mathsf{Ji}_1(L)$. Reuter [12, Corollary 3] asserts, even when L is only modular, that for each $\alpha \in S(L)$ and $k \in \mathbb{N}$,

(3.3)
$$|\mathsf{Ji}_k(L) \cap \mathsf{Ji}_k(\alpha)| = \sum_{\beta \le \alpha, \ \beta \in S(L)} \mu_{S(L)}(\beta, \alpha) \cdot |\mathsf{Ji}_k(\alpha \cap \beta)|.$$

For k = 1, the lefthand side of (3.2) equals that of (3.3). Hence, it suffices to show that $\mathsf{w}^*(0_\alpha, 1_\beta) = |\mathsf{Ji}_1(\alpha \cap \beta)|$ holds for $\beta \leq \alpha$. This is obvious if $\alpha \cap \beta = \emptyset$ since then $0_\alpha \not\leq 1_\beta$ follows from (2.2). Hence, we assume that $\alpha \cap \beta \neq \emptyset$. Then, again by (2.2), $\alpha \cap \beta = [0_\alpha, 1_\beta]$ is a principal ideal of α , whence $\alpha \cap \beta$ is a boolean sublattice of the boolean interval α . Thus,

$$|\mathsf{Ji}_1(\alpha \cap \beta)| = |\mathsf{Ji}([0_\alpha, 1_\beta])| = |\mathsf{At}([0_\alpha, 1_\beta])| = \mathrm{length}([0_\alpha, 1_\beta]) = \mathsf{w}^*(0_\alpha, 1_\beta). \quad \Box$$

3.2. More about blocks. Although the following lemma requires a proof in the present setting, it is a part of the original definition of S(L) given by Herrmann [11]. For the reader's convenience, we present an easy proof.

Lemma 3.2 (Wille [13, Proposition 9]). Let L be a finite lattice. If $\alpha \prec_{S(L)} \beta$, then $\alpha \cap \beta \neq \emptyset$.

Proof. Assume that $\alpha \prec_{S(L)} \beta$, and let $a := 1_{\alpha}$. Since $1_{\alpha} < 1_{\beta}$ by (2.2), we can take an element $b \in L$ such that $a = 1_{\alpha} \prec b \leq 1_{\beta}$. Since $L = \bigcup_{\nu \in S(L)} \nu$, there is a $\gamma_1 \in S(L)$ containing b. By (2.1), $\gamma_2 := \alpha \lor \gamma_1$ contains $b = a \lor b$, and $\gamma_3 := \beta \land \gamma_2$ also contains $b = 1_{\beta} \land b$. Clearly, $\alpha \leq \gamma_3 \leq \beta$. Taking $b \in \gamma_3 \setminus \alpha$ and $\alpha \prec_{S(L)} \beta$ into account, we conclude that $b \in \gamma_3 = \beta$.

Next, $\{a, b\}^2 \subseteq \Theta(L)$ since $a \prec b$. Hence, there is a block $\gamma \in S(L)$ such that $\{a, b\} \subseteq \gamma$. Using (2.1) repeatedly, we obtain that $a = a \lor a \in \alpha \lor \gamma$, $b = a \lor b \in \alpha \lor \gamma$, $a = a \land b \in (\alpha \lor \gamma) \land \beta$ and $b = b \land b \in (\alpha \lor \gamma) \land \beta$. That is, $\{a, b\} \subseteq (\alpha \lor \gamma) \land \beta$. On the other hand, $\alpha \leq (\alpha \lor \gamma) \land \beta \leq \beta$, together with $\alpha \prec_{S(L)} \beta$, yields that $\alpha = (\alpha \lor \gamma) \land \beta$ or $\beta = (\alpha \lor \gamma) \land \beta$. Hence, $\{a, b\} \subseteq \alpha$ or $\{a, b\} \subseteq \beta$, and we conclude that $\alpha \cap \beta \neq \emptyset$.

For the reader's convenience, we also prove the following lemma. Due to the forthcoming formula (3.4), the present approach is slightly simpler than the original one of the second author [9].

Lemma 3.3 ([9, Theorem 2.2.10]). Let L be a finite lattice. Assume that $\alpha, \beta \in S(L)$ such that $(\alpha, \beta) \in \Theta(S(L))$. Then $\alpha \cap \beta \neq \emptyset$.

Proof. A straightforward induction based on (2.1) shows that, for any lattice term $p(x_1, \ldots, x_n)$ and for any $\nu_1, \ldots, \nu_n \in S := S(L)$,

(3.4)
$$\{p(x_1, \dots, x_n) : x_1 \in \nu_1, \dots, x_n \in \nu_n\} \subseteq p(\nu_1, \dots, \nu_n).$$

Next, let $S = \{\sigma_1, \ldots, \sigma_t\}$, and let $\kappa_1 \prec_S \lambda_1, \ldots, \kappa_k \prec_S \lambda_k$ be a list of all covering pairs of S. Since the skeleton tolerance $\Theta(S)$ of S is generated by $\{(\kappa_1, \lambda_1), \ldots, (\kappa_k, \lambda_k)\}$, it coincides with the subalgebra of S^2 generated by

$$\{(\kappa_1,\lambda_1),\ldots,(\kappa_k,\lambda_k),(\lambda_1,\kappa_1),\ldots,(\lambda_k,\kappa_k),(\sigma_1,\sigma_1),\ldots,(\sigma_t,\sigma_t)\}.$$

Hence, there exists a (2k + t)-ary lattice term p such that

$$(\alpha, \beta) = p((\kappa_1, \lambda_1), \dots, (\kappa_k, \lambda_k), (\lambda_1, \kappa_1), \dots, (\lambda_k, \kappa_k), (\sigma_1, \sigma_1), \dots, (\sigma_t, \sigma_t)))$$

= $(p(\kappa_1, \dots, \kappa_k, \lambda_1, \dots, \lambda_k, \sigma_1, \dots, \sigma_t),$
 $p(\lambda_1, \dots, \lambda_k, \kappa_1, \dots, \kappa_k, \sigma_1, \dots, \sigma_t)).$

It follows from Lemma 3.2 that there are $x_1, \ldots, x_k, y_1, \ldots, y_t \in L$ such that $x_i \in \kappa_i \cap \lambda_i = \lambda_i \cap \kappa_i$ for $i = 1, \ldots, k$ and $y_j \in \sigma_j$ for $j = 1, \ldots, t$. Hence, the above expression for (α, β) , together with (3.4), yields that

$$p(x_1, \dots, x_k, x_1, \dots, x_k, y_1, \dots, y_t) \in \alpha \cap \beta.$$

3.3. More about join-irreducible elements in blocks.

Notation 3.4. Let $\operatorname{Ji}^+(\mathsf{L}) := \operatorname{Ji}(L) \setminus \operatorname{At}(L)$. Let $\zeta_0 = [0, z_0]$ be the least element of S(L), and let $S^+(L) := S(L) \setminus \{\zeta_0\}$. For $x \in L$, the set of atoms *dominated* by x is domin $(x) := \{y \in \operatorname{At}(L) : y \leq x\}$. Similarly, for $\alpha \in S^+(L)$, domin $(\alpha) := \{y \in \operatorname{At}(L) : y \leq 0_\alpha\}$ is the set of atoms dominated by α .

The next lemma is easy. Having no reference at hand, we will give a proof.

Lemma 3.5. Let *L* be a finite modular lattice. Then $[0, \bigvee_{x \in At(L)} x] = \zeta_0$. Furthermore, if $\alpha \in S(L)$ such that $J_{\alpha} \cap At(L) \neq \emptyset$, then $\alpha = \zeta_0$.

Proof. Let $z_0 := \bigvee_{x \in \mathsf{At}(L)} x$. Since $(0, x) \in \Theta(L)$ for all $x \in \mathsf{At}(L)$, we obtain that $(0, z_0) \in \Theta(L)$. Hence, we can extend $\{0, z_0\}$ to a block $\alpha = [0, y]$ of $\Theta(L)$. Obviously, $z_0 \leq y$. We know from Herrmann [11] that α is an atomistic lattice. Hence, $y = \bigvee_{x \in \mathsf{At}(\alpha)} x \leq \bigvee_{x \in \mathsf{At}(L)} x = z_0$, and we conclude that $[0, z_0] = [0, y] = \alpha \in S(L)$. It is the smallest element of S(L), that is ζ_0 , by (2.2). Finally, if an atom $a \in \mathsf{At}(L)$ belongs to J_{α} , then $0_{\alpha} \prec a$ implies $0_{\alpha} = 0 = 0_{\zeta_0}$, whence $\alpha = \zeta_0$ by (2.2).

Lemma 3.6. Let L be a finite modular lattice. Then

- (i) $\operatorname{Ji}(L) = \bigcup \{ J_{\alpha} : \alpha \in S(L) \};$
- (ii) for all $\alpha, \beta \in S(L)$, if $\alpha \neq \beta$, then $J_{\alpha} \cap J_{\beta} = \emptyset$;
- (iii) $\operatorname{Ji}^+(\mathsf{L}) = \bigcup \{ J_\alpha : \alpha \in S^+(L) \}$ and $\operatorname{At}(L) = J_{\zeta_0} = \operatorname{At}(\zeta_0).$

Proof. Assume that $a \in \text{Ji}(L)$, and let a^- stand for its unique lower cover. Then $\{a^-, a\} \subseteq \Theta(L)$ since $\Theta(L)$ is a glued tolerance. We can extend $\{a^-, a\}$ to a block $\alpha \in S(L)$. Then $a \in \text{Ji}(L) \cap (\alpha \setminus \{0_\alpha\}) = J_\alpha$. This proves that $\text{Ji}(L) \subseteq \bigcup \{J_\alpha : \alpha \in S(L)\}$. The reverse inclusion in part (i) is trivial.

Assume that $x \in J_{\alpha} \cap J_{\beta}$. Then, by (3.1), $x \succ 0_{\alpha}$ and $x \succ 0_{\beta}$. Hence, $x \in \mathsf{Ji}(L)$ implies that $0_{\alpha} = 0_{\beta}$. This, together with (2.2), yields $\alpha = \beta$, which proves part (ii). Finally, parts (i) and (ii), together with Lemma 3.5, imply part (iii).

Lemma 3.7. Let α , β and ζ be distinct blocks of a finite modular lattice L such that $\zeta < \alpha$, $\zeta < \beta$ and $\{\alpha, \beta, \zeta\}^2 \subseteq \Theta(S(L))$. Then $a \parallel b$ holds for all $a \in J_{\alpha}$ and $b \in J_{\beta}$.

Proof. Assume that *a* is comparable with *b*. It follows from Lemma 3.6(ii) that $a \neq b$. Hence, we can assume that a < b. We infer $\alpha \cap \zeta \neq \emptyset$ and $\beta \cap \zeta \neq \emptyset$ by Lemma 3.3. This, together with $\zeta < \alpha$ and $\zeta < \beta$, yields that $0_{\zeta} < 0_{\alpha} \leq 1_{\zeta}$ and $0_{\zeta} < 0_{\beta} \leq 1_{\zeta}$. Since $a \in J_{\alpha} \subseteq \operatorname{At}(\alpha)$ by (3.1), we have that $0_{\alpha} \prec a$. Similarly, $0_{\beta} \prec b$. Since $b \in J_{\beta} \subseteq \operatorname{Ji}(L)$, 0_{β} is the only lower cover of *b*. This, together with a < b, implies that $a \leq 0_{\beta}$. Therefore, $0_{\zeta} < 0_{\alpha} \prec a \leq 0_{\beta} \leq 1_{\zeta}$. This means that $a \in \zeta$ but, since $\operatorname{Ji}(\zeta) = \operatorname{At}(\zeta)$, *a* is join-reducible in ζ , whence it is also join-reducible in *L*. This contradicts $a \in J_{\alpha} \subseteq \operatorname{Ji}(L)$.

Proof of Theorem 1.1(i). Let us assume for a contradiction that there exist a_1, a_2, a_3 in Ji(L) such that $a_1 < a_2 < a_3$. By Lemma 3.6(i), we can choose $\alpha_1, \alpha_2, \alpha_3 \in S(L)$ such that $a_i \in J_{\alpha_i}$, for $i \in \{1, 2, 3\}$. Since $J_{\alpha_i} \subseteq \operatorname{At}(\alpha_i)$ is an antichain for i = 1, 2, 3, the blocks $\alpha_1, \alpha_2, \alpha_3$ are pairwise distinct. Hence, at least two of them, say α_j and α_k , are distinct from the smallest element ζ_0 of S(L). Since $S^2(L) = S(L)/\Theta(S(L))$ is the singleton lattice, $\Theta(S(L))$ is the full relation on S(L). Therefore, applying Lemma 3.7 to α_j, α_k and ζ_0 , we obtain that $a_j \parallel a_k$, a contradiction.

Clearly, $|\mathsf{Ji}^+(\mathsf{L})| \neq \emptyset$ iff length $(\mathsf{Ji}(L)) \ge 1$. Hence, the next lemma would (vacuously) also hold if length $(\mathsf{Ji}(L))$ were 0. Notation 3.4 will be in effect.

Lemma 3.8. Let *L* be a finite distributive lattice such that length(Ji(L)) = 1. Let $\alpha, \alpha_1, \ldots, \alpha_n \in S^+(L)$ such that none of $J_\alpha, J_{\alpha_1}, \ldots, J_{\alpha_n}$ is empty. Then

(i) domin(α) $\neq \emptyset$;

(ii) $\alpha_1 \leq \alpha_2$ iff $\operatorname{domin}(\alpha_1) \subseteq \operatorname{domin}(\alpha_2)$;

(iii) $|\operatorname{domin}(\alpha_1) \cup \ldots \cup \operatorname{domin}(\alpha_n)| = \mathsf{w}^*(0, 0_{\alpha_1 \vee \cdots \vee \alpha_n}).$

Proof. Since $0 \neq 0_{\alpha}$, part (i) is trivial. If $\alpha_1 \leq \alpha_2$, then $0_{\alpha_1} \leq 0_{\alpha_2}$ by (2.2), whence domin $(\alpha_1) \subseteq$ domin (α_2) . To prove the reverse implication of (ii), assume that $\alpha_1 \not\leq \alpha_2$. Then (2.2) implies $0_{\alpha_1} \not\leq 0_{\alpha_2}$, which yields an $x \in \operatorname{Ji}(L)$ such that $x \leq 0_{\alpha_1}$ but $x \not\leq 0_{\alpha_2}$. Since $J_{\alpha_1} \neq \emptyset$ by the assumption, there is a $y \in \operatorname{Ji}(L)$ such that $x \leq 0_{\alpha_1} < y$. This, together with length $(\operatorname{Ji}(L)) = 1$, shows that $x \in \operatorname{At}(L)$. Hence, $x \in \operatorname{domin}(\alpha_1) \setminus \operatorname{domin}(\alpha_2)$, proving part (ii).

Next, we claim that

(3.5) $\operatorname{domin}(\alpha_1) \cup \ldots \cup \operatorname{domin}(\alpha_n) = \{ x \in \mathsf{At}(L) : x \le 0_{\alpha_1 \vee \cdots \vee \alpha_n} \}.$

The " \subseteq " inclusion is an evident consequence of (2.2). To prove the converse inclusion, assume that x belongs to the right-hand side of (3.5). Then, by (2.2) and distributivity,

$$x = x \wedge 0_{\alpha_1 \vee \cdots \vee \alpha_n} = x \wedge (0_{\alpha_1} \vee \cdots \vee 0_{\alpha_n}) = (x \wedge 0_{\alpha_1}) \vee \cdots \vee (x \wedge 0_{\alpha_n}).$$

By the join-irreducibility of x, there exists an $i \in \{1, ..., n\}$ such that $x = x \wedge 0_{\alpha_i}$. Hence, $x \leq 0_{\alpha_i}$ implies that $x \in \text{domin}(\alpha_i)$, proving (3.5).

For $i \in \{1, \ldots, n\}$, 0_{α_i} is the join of some (possibly only one) join-irreducible elements of L. These elements are necessarily atoms since $\operatorname{length}(\operatorname{Ji}(L)) = 1$ and $J_{\alpha_i} \neq \emptyset$. Therefore, all the 0_{α_i} belong to ζ_0 , and so does their join, which is $0_{\alpha_1 \vee \cdots \vee \alpha_n}$ by (2.2). Since ζ_0 is a boolean lattice, the number of atoms below $0_{\alpha_1 \vee \cdots \vee \alpha_n}$, that is, the size of the set given in (3.5), is $\operatorname{length}([0, 0_{\alpha_1 \vee \cdots \vee \alpha_n}]) =$ $w^*(0, 0_{\alpha_1 \vee \cdots \vee \alpha_n})$. This, together with (3.5), proves part (iii).

3.4. A lemma on bipartite graphs. In order to formulate a statement that we need in the proof of Theorem 1.1(ii)-(iii), we have to associate a number-valued function with bipartite graphs. By a finite directed bipartite graph we shall mean a structure G = (U, X, E) where U and X are finite nonempty sets, referred to as upper and lower vertex sets, and $E \subseteq U \times X$ is an arbitrary relation. The power set, that is the set of all subsets, of U is denoted by P(U), and P(X) has the analogous meaning. Let $P^{\infty}(U) := P(U) \cup \{\infty\}$ where ∞ is a symbol not in P(U). For $V \in P(U)$, we let $d_G(V) := \{x \in X : \text{there is a } v \in V \text{ such that } (v, x) \in E\}$. This set is called the set of (lower) vertices dominated by V. (We shall not use the word "covered" in this context since we want to avoid any confusion with the order-theoretic covering relation.) We define $d_G(\infty) := X$. Let $d_G^{\sharp}(V)$ stand for $|d_G(V)|$; if $V = \{v\}$, then we write $d_G^{\sharp}(v)$ rather than $d_G^{\sharp}(\{v\})$. This way d_G^{\sharp} , called the domination function associated with G, is a $P^{\infty}(U) \to \mathbb{N}_0$ mapping. If $\varphi : U \to U'$ is a bijection and $V \in P^{\infty}(U)$, then $\varphi(V) := \{\varphi(v) : v \in V\}$ for $V \in P(U)$ while $\varphi(\infty) := \infty \in P^{\infty}(U')$.

Lemma 3.9. Let G = (U, X, E) and G' = (U', X', E') be finite directed bipartite graphs. Then these two graphs are isomorphic iff there is a bijection $\varphi : U \to U'$ that preserves the domination function, that is, $d_{G'}^{\sharp}(\varphi(V)) = d_{G}^{\sharp}(V)$ holds for all $V \in P^{\infty}(U)$.

Proof. In order to prove the non-trivial direction of the lemma, assume that $\varphi \colon U \to U'$ is a bijection that preserves the domination function. We associate two additional mappings with G as follows:

$$s_G \colon P(U) \to P(X), \quad V \mapsto \{x \in X : (v, x) \in E \text{ for all } v \in V\},\\ e_G \colon P(U) \to P(X), \quad V \mapsto \{x \in X : (v, x) \in E \text{ for all } v \in V \text{ and} \\ (u, x) \notin E \text{ for all } u \in U \setminus V\}.$$

The corresponding number-valued functions are denoted by s_G^{\sharp} and e_G^{\sharp} , that is, $s_G^{\sharp}(V) := |s_G(V)|$ and $e_G^{\sharp}(V) := |e_G(V)|$. These functions will be called the *strong domination function* and the *exact domination function*, respectively. Replacing G by G', we obtain the definition for $s_{G'}^{\sharp}$ and $e_{G'}^{\sharp}$. Usually, we will elaborate our formulas only for G since, sometimes implicitly, we will rely on the fact that the analogous formulas hold for G' as well.

Firstly, we prove that φ preserves the strong domination function. Let $V \in P(U)$; we show $s_{G'}^{\sharp}(\varphi(V)) = s_{G}^{\sharp}(V)$ by induction on |V|. If |V| = 0, then $s_{G}^{\sharp}(\emptyset) = |X| = d_{G}^{\sharp}(\infty)$, and the desired equality follows easily. The case |V| = 1, say $V = \{v\}$, is even easier since $s_{G}^{\sharp}(v) = d_{G}^{\sharp}(v)$. Next, assume that $1 < n \in \mathbb{N}, V = \{v_1, \ldots, v_n\}$ is an *n*-element subset of U, and the desired equality holds for all subsets of U

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with less than n elements. Based on the inclusion-exclusion principle, also called (logical) sieve formula,

$$|T_1 \cup \dots \cup T_n| = \sum_{i=1}^n (-1)^{i-1} \sum_{\substack{I \subseteq \{1,\dots,n\} \ |I|=i}} \left| \bigcap_{j \in I} T_j \right|,$$

and using that s_G agrees with d_G on singleton sets and satisfies the identity $s_G(V_1) \cap s_G(V_2) = s_G(V_1 \cup V_2)$, we can compute as follows.

$$\begin{aligned} d_G^{\sharp}(V) &= |d_G(V)| = |d_G(v_1) \cup \dots \cup d_G(v_n)| = |s_G(v_1) \cup \dots \cup s_G(v_n)| \\ &= (-1)^{n-1} \Big| \bigcap_{j=1}^n s_G(v_j) \Big| + \sum_{i=1}^{n-1} (-1)^{i-1} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=i}} \Big| \bigcap_{j \in I} s_G(v_j) \Big| \\ &= (-1)^{n-1} s_G^{\sharp}(V) + \sum_{i=1}^{n-1} (-1)^{i-1} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=i}} s_G^{\sharp}(\{v_j: j \in I\}). \end{aligned}$$

Now, φ preserves all summands but $(-1)^{n-1}s_G^{\sharp}(V)$ in the previous line by the induction hypothesis. Since $d_G^{\sharp}(V)$ is also preserved, we conclude that $s_G^{\sharp}(V)$ is preserved, completing the induction. Thus, φ preserves the strong domination function.

Next, to show that φ preserves the exact domination function, let $V \in P(U)$. We want to show that $e_{G'}^{\sharp}(\varphi(V)) = e_G^{\sharp}(V)$. This is clear if V = U since $e_G^{\sharp}(U) = s_G^{\sharp}(U)$ and the strong domination function is preserved. Hence, we can assume that $V \neq U$. Let $W := U \setminus V$. It is a k-element set for some $k \in \mathbb{N}$, so we can write $W = \{w_1, \ldots, w_k\}$. For $i = 1, \ldots, k$, let $A_i := \{x \in s_G(V) : (w_i, x) \in E\}$. Notice that $A_i = s_G(V \cup \{w_i\})$. Using the inclusion-exclusion principle again, we obtain that

$$\begin{split} e_{G}^{\sharp}(V) &= |e_{G}(V)| = \left| s_{G}(V) \setminus \bigcup_{i=1}^{k} A_{i} \right| = |s_{G}(V)| - \left| \bigcup_{i=1}^{k} A_{i} \right| \\ &= |s_{G}(V)| + \sum_{i=1}^{k} (-1)^{i} \sum_{\substack{I \subseteq \{1, \dots, k\} \\ |I| = i}} \left| \bigcap_{j \in I} A_{j} \right| \\ &= s_{G}^{\sharp}(V) + \sum_{i=1}^{k} (-1)^{i} \sum_{\substack{I \subseteq \{1, \dots, k\} \\ |I| = i}} \left| s_{G} \left(V \cup \{w_{j} : j \in I\} \right) \right| \\ &= s_{G}^{\sharp}(V) + \sum_{i=1}^{k} (-1)^{i} \sum_{\substack{I \subseteq \{1, \dots, k\} \\ |I| = i}} s_{G}^{\sharp}(V) \cup \{w_{j} : j \in I\}). \end{split}$$

Therefore, since φ preserves the strong domination function, it preserves the exact domination function also.

Now, we are ready to define an isomorphism $(\varphi, \xi): G \to G'$. That is, φ was originally given, and we intend to define a bijection $\xi: X \to X'$ such that E' =

 $\{(\varphi(u),\xi(x)):(u,x)\in E\}.$ Clearly,

(3.6)
$$e_G(V_1) \cap e_G(V_2) = \emptyset$$
 for all $V_1 \neq V_2 \in P(U)$, and $X = \bigcup_{V \in P(U)} e_G(V)$;

and the analogous assertion holds for G'. Notice at this point that the elements of X with degree 0 belong to $e_G(\emptyset)$. For each $V \in P(U)$, let us fix a bijection $\xi_V : e_G(V) \to e_{G'}(\varphi(V))$; this is possible since $e_G^{\sharp}(V) = e_{G'}^{\sharp}(\varphi(V))$. Then $\xi := \bigcup_{V \in P(U)} \xi_V$ is an $X \to X'$ bijection by (3.6).

Observe that the role of (G, φ, ξ) and that of $(G', \varphi^{-1}, \xi^{-1})$ can be interchanged. Hence, in order to prove that (φ, ξ) is an isomorphism, it suffices to show that (φ, ξ) sends edges to edges. To do so, assume that $(u, x) \in E$. Let $V := \{v \in U : (v, x) \in E\}$. Then $u \in V$ and $x \in e_G(V)$. Hence, $\varphi(u) \in \varphi(V)$ and $\xi(x) = \xi_V(x) \in e_{G'}(\varphi(V))$. By the definition of $e_{G'}$, this implies that $(\varphi(u), \xi(x))$ belongs to E', as desired.

3.5. The end of the proof. Based on the auxiliary statements given so far, we are now in the position to complete the proof of the main result.

Proof of Theorem 1.1 (ii) and (iii). Assume that L and L' are finite distributive lattices and

 $(\psi,\kappa)\colon \left(S^{\mathrm{wd}}(L),\leq,S(L),\eta_0,\eta_1,\mathsf{w}\right)\to \left(S^{\mathrm{wd}}(L'),\leq',S(L'),\eta_0',\eta_1',\mathsf{w}'\right)$

is an isomorphism between their weighted double skeletons. Remember that η_0 and η_1 were defined right after (2.4), the meaning of η'_0 and η'_1 is analogous, and the diagram

$$(3.7) \qquad \begin{array}{c} S(L) & \xrightarrow{\eta_i} & S^{\text{wd}}(L) \\ \kappa & & \psi \\ S(L') & \xrightarrow{\eta'_i} & S^{\text{wd}}(L') \end{array}$$

commutes for $i \in \{0, 1\}$. Observe that

(3.8)
$$\operatorname{length}(\operatorname{Ji}(L)) \le 1 \quad \text{and} \quad \operatorname{length}(\operatorname{Ji}(L')) \le 1$$

Indeed, in case of part (iii) this is assumed. In case of part (ii), the assumption, together with the meaning of κ , implies that L' is also H^2 -irreducible, whence length($\mathsf{Ji}(L)$) ≤ 1 and length($\mathsf{Ji}(L')$) ≤ 1 follow from part (i).

Firstly, assume that |S(L)| = 1. Then $|S(L')| = |\kappa(S(L))| = 1$. It is well known, and follows from Lemmas 3.5 and 3.6(iii), that L is boolean iff $1_L = \bigvee \operatorname{At}(L)$. Therefore, we obtain from Lemma 3.5 that |S(L)| = 1 iff L is boolean iff length $(\operatorname{Ji}(L)) = 0$, and the same holds for L'. Therefore,

$$length(L) = w^*(0_L, 1_L) = w^{*'}(\psi(0_L), \psi(1_L)) = w^{*'}(0_{L'}, 1_{L'}) = length(L'),$$

and we conclude that $L \cong L'$. Observe that the role of L and that of L' in the above argument can be interchanged, whence we also conclude that length($\mathsf{Ji}(L)$) = 0 iff length($\mathsf{Ji}(L')$) = 0.

In the rest of the proof, we assume that length($\operatorname{Ji}(L)$) = 1. Then, by the previous paragraph and (3.8), length($\operatorname{Ji}(L')$) = 1 also holds. We are going to define some auxiliary sets and structures associated with $S^{\operatorname{wd}}(L)$; their "primed" counterparts associated with $S^{\operatorname{wd}}(L')$ are understood analogously.

Let $U := \mathsf{Ji}^+(\mathsf{L})$, $X := \mathsf{At}(L)$, and $E := \{(u, x) \in U \times X : u > x\}$. Notice that none of U, X and E is empty since $\operatorname{length}(\mathsf{Ji}(L)) = 1$. Obviously, the directed bipartite graph G := (U, X, E) determines the poset $\mathsf{Ji}(L)$. Therefore, G and G'determine L and L', respectively, up to isomorphism. Consequently, by Lemma 3.9, it suffices to find a bijection $\varphi : U \to U'$ such that

(3.9)
$$d_{G'}^{\sharp}(\varphi(V)) = d_G^{\sharp}(V) \text{ holds for all } V \in P^{\infty}(U).$$

It follows from the commutativity of (3.7) that, for $\gamma \in S(L)$,

(3.10)
$$\psi(0_{\gamma}) = \psi(\eta_0(\gamma)) = \eta'_0(\kappa(\gamma)) = 0_{\kappa(\gamma)}$$
 and, similarly, $\psi(1_{\gamma}) = 1_{\kappa(\gamma)}$.

For $\alpha \in S^+(L)$, Lemma 3.1 and (3.10) imply that

$$\begin{aligned} |J_{\alpha}| &= \sum_{\beta \leq \alpha, \ \beta \in S(L)} \mu_{S(L)}(\beta, \alpha) \cdot \mathsf{w}^{*}(0_{\alpha}, 1_{\beta}) \\ &= \sum_{\beta \leq \alpha, \ \beta \in S(L)} \mu_{S(L')}(\kappa(\beta), \kappa(\alpha)) \cdot \mathsf{w}^{*'}(\psi(0_{\alpha}), \psi(1_{\beta})) \\ &= \sum_{\beta \leq \alpha, \ \beta \in S(L)} \mu_{S(L')}(\kappa(\beta), \kappa(\alpha)) \cdot \mathsf{w}^{*'}(0_{\kappa(\alpha)}, 1_{\kappa(\beta)}) \\ &= \sum_{\beta' \leq \kappa(\alpha), \ \beta' \in S(L')} \mu_{S(L')}(\beta', \kappa(\alpha)) \cdot \mathsf{w}^{*'}(0_{\kappa(\alpha)}, 1_{\beta'}) = |J_{\kappa(\alpha)}| \end{aligned}$$

This allows us to fix a bijection $\varphi_{\alpha} : J_{\alpha} \to J_{\kappa(\alpha)}$. (Notice that if J_{α} happens to be empty, then $\varphi_{\alpha} = \emptyset$ is the empty mapping.) Let φ be the union of all these φ_{α} , $\alpha \in S^+(L)$. It follows from Lemma 3.6(ii) that φ is a mapping. Since the union of the corresponding J_{α} , $\alpha \in S^+(L)$, is $\mathsf{Ji}^+(\mathsf{L}) = U$ by Lemma 3.6(ii), and the analogous assertion holds for $U', \varphi : U \to U'$ is a bijective mapping.

It follows from Lemma 3.6(ii)-(iii) that for each $u \in U$, there is a unique $\alpha(u) \in S^+(L)$ such that $u \in J_{\alpha(u)}$. Similarly, for each $u' \in U'$, there is a unique $\alpha'(u') \in S^+(L')$ such that $u' \in J_{\alpha'(u')}$. The definition of φ implies that

(3.11)
$$\alpha'(\varphi(u)) = \kappa(\alpha(u)), \text{ for all } u \in U.$$

Assume that $u \in U$. Then u is not an atom of L, and its only lower cover in L is $0_{\alpha(u)}$. Hence, for any $a \in \operatorname{At}(L)$, we have that $u > a \iff 0_{\alpha(u)} \ge a \iff a \in \operatorname{domin}(\alpha(u))$. This yields that, for any $V \in P(U)$, $d_G(V) = \bigcup_{u \in V} \operatorname{domin}(\alpha(u))$. Therefore, taking the meaning of η_0 into account and using Lemma 3.8,

(3.12)
$$d_G^{\sharp}(V) = \mathsf{w}^* \Big(\eta_0(\zeta_0), \eta_0\Big(\bigvee_{u \in V} \alpha(u)\Big) \Big).$$

Indicating the referenced formulas or their "primed version" at the equation signs and using that (ψ, κ) preserves the extended weight function, we obtain that

$$\begin{split} d^{\sharp}_{G'}(\varphi(V)) &=^{(3.12)'} \mathsf{w}^{*'}\Big(\eta'_0(\zeta'_0), \eta'_0\Big(\bigvee_{u \in V} \alpha'(\varphi(u))\Big)\Big) \\ &=^{(3.11)} \mathsf{w}^{*'}\Big(\eta'_0(\kappa(\zeta_0)), \eta'_0\Big(\bigvee_{u \in V} \kappa(\alpha(u))\Big)\Big) \\ &= \mathsf{w}^{*'}\Big(\eta'_0(\kappa(\zeta_0)), \eta'_0\Big(\kappa\Big(\bigvee_{u \in V} \alpha(u)\Big)\Big)\Big) \end{split}$$

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$$=^{(3.7)} \mathsf{w}^{*'} \Big(\psi(\eta_0(\zeta_0)), \psi\Big(\eta_0\Big(\bigvee_{u \in V} \alpha(u)\Big)\Big) \Big)$$
$$= \mathsf{w}^{*} \Big(\eta_0(\zeta_0), \eta_0\Big(\bigvee_{u \in V} \alpha(u)\Big)\Big) =^{(3.12)} d_G^{\sharp}(V).$$

This proves (3.9) for $V \in P(U)$.

We are left with the case $V = \infty \in P^{\infty}(U)$. Here, using Lemma 3.6(iii), the validity of (3.9) is obtained as follows:

$$\begin{aligned} d_{G}^{\sharp}(\infty) &= |\mathsf{At}(L)| = |\mathsf{At}(\zeta_{0})| = \mathsf{w}^{*}(0_{\zeta_{0}}, 1_{\zeta_{0}}) = \mathsf{w}^{*'}(\psi(0_{\zeta_{0}}), \psi(1_{\zeta_{0}})) \\ &= ^{(3.10)} \mathsf{w}^{*'}(0_{\kappa(\zeta_{0})}, 1_{\kappa(\zeta_{0})}) = \mathsf{w}^{*'}(0_{\zeta_{0}'}, 1_{\zeta_{0}'}) \\ &= |\mathsf{At}(\zeta_{0}')| = |\mathsf{At}(L')| = d_{G'}^{\sharp}(\infty) = d_{G'}^{\sharp}(\varphi(\infty)). \end{aligned}$$

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URL: http://www.math.u-szeged.hu/~czedli/

UNIVERSITY OF SZEGED, HUNGARY 6720

E-mail address: j.grygiel@ajd.czest.pl URL: http://www.j.grygiel.eu/

Institute of Mathematics and Computer Science, Jan Długosz University of Częstochowa, Poland

E-mail address: grygiel@tcs.uj.edu.pl *URL:* http://tcs.uj.edu.pl/Grygiel

THEORETICAL COMPUTER SCIENCE DEPARTMENT, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, JAGIELLONIAN UNIVERSITY, UL. PROF. ŁOJASIEWICZA 6, 30-348 KRAKÓW, POLAND