NOTES ON PLANAR SEMIMODULAR LATTICES. VII. RESECTIONS OF PLANAR SEMIMODULAR LATTICES

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Abstract. A recent result of G. Czédli and E. T. Schmidt gives a construction of slim (planar) semimodular lattices from planar distributive lattices by adding elements, adding "forks". We give a construction that accomplishes the same by deleting elements, by "resections".

1. Introduction

Planar semimodular lattices started to play an important role in G. Grätzer, H. Lakser, and E. T. Schmidt [11]. Proving that every finite distributive lattice D can be represented as the congruence lattice of a finite semimodular lattice L, they, in fact, proved that there is such a planar semimodular lattice $L \in O(n^2)$. G. Grätzer and E. Knapp tried to prove that in this result $L \in O(n^2)$ is optimal. They studied planar semimodular lattices in [12]–[15]; their conclusion was that $L \in O(n^2)$ is, indeed, optimal for a class of planar semimodular lattices, they called, rectangular. (The general problem is still unresolved.)

These papers were followed by further studies of planar semimodular lattices. G. Grätzer and J. B. Nation [16] and G. Czédli and E. T. Schmidt [6] proved a generalization of the Jordan-Hölder theorem, new even for groups.

A lattice L is *slim* if it is finite and $\mathrm{Ji}\, L$, the set of non-zero join-irreducible elements of L, contains no three-element antichain. Slim lattices are *planar*, so we will consider *planar diagrams* of slim semimodular lattices, *slim semimodular diagrams*.

G. Grätzer and E. Knapp [12] observed that slim semimodular lattices easily describe all planar semimodular lattices. Indeed, every planar semimodular lattice can be obtained from a slim semimodular lattice by replacing covering squares with covering M_3-s (adding eyes).

Slim semimodular lattices play an essential role in G. Grätzer and E. Knapp [13] and [15], G. Grätzer and T. Wares [17], G. Czédli and E. T. Schmidt [7], [8], and [9], G. Czédli [1], [2], and [3], and G. Czédli, L. Ozsvárt, and B. Udvari [5]. A survey of these results is presented in [4], a chapter of a forthcoming book.

In this paper, we present a construction of slim semimodular lattices. We construct slim semimodular lattices from planar distributive lattices by a series of *resections*. A resection starts with a cover-preserving C_3^2 (the dark gray square of the three-element chain in Figure 1), and it deletes two elements to get an N_7 (see Figure 3) from C_3^2, and then deletes some more elements (all the black-filled ones),

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going up and down to the left and to the right, to preserve semimodularity; see Figure 2 for the result of the resection.

For the basic concepts and notation, we refer the reader to G. Grätzer [10] and G. Czédli and G. Grätzer [4]. In particular, $\mathrm{Mi}\, L$ denotes the set of meet-irreducible elements of L disctinct from 1.

Outline. Section 2 introduces resections. Section 3 states the main result. Section 4 recalls some known results on slim semimodular lattices and proves some facts on (the inverse of) resection schemes. Section 5 contains the proof of the main result.

2. The construction

Let D be a slim semimodular diagram. Two prime intervals of D are *consecutive* if they are opposite sides of a 4-cell (see Section 4). As in G. Czédli and E. T. Schmidt [6], maximal sequences of consecutive prime intervals form a C_2*-trajectory.* So a C_2-trajectory is an equivalence class of the transitive reflexive closure of the "consecutive" relation.

Similarly, let A and B be two cover-preserving C_3-chains of D. If they are opposite sides of a cover-preserving $\mathsf{C}_3 \times \mathsf{C}_2$, then A and B are called *consecutive.* An equivalence class of the transitive reflexive closure of this "consecutive" relation is called a C_3*-trajectory.*

We recall the basic properties of C_2-trajectories from [6] and [8]; they also hold for C_3-trajectories. For $i \in \{2, 3\}$, a C_i-trajectory goes from left to right (unless otherwise stated); they do not branch out. A C_i-trajectory is of two types: an *up-trajectory,* which goes up (possibly, in zero steps) and a *hat-trajectory,* which goes up (possibly in zero steps), then turns to the lower right, and finally it goes down (possibly, in zero steps).

Note that the left and right ends of a C_2-trajectory are on the boundary of L; this may fail for a C_3-trajectory.

The *elements* of a C_i-trajectory are the elements of the C_i-chains forming it. Let A be a cover-preserving C_i-chain in D. By planarity, there is a unique C_i-trajectory through A. The C_i-chains of this trajectory to the left of A and including A form the *left wing of* A. The *right wing of* A is defined analogously.

Next, let B be a cover-preserving $\mathsf{C}_3^2 = \mathsf{C}_3 \times \mathsf{C}_3$ of the diagram D. Let W_l be the left wing of the upper left boundary of B and let W_r be the right wing of the upper right boundary of B. Assume that W_l and W_r terminate on the boundary of D (that is, the last C_3-chains are on the boundary of D). In this case, the collection of elements of $S = B \cup W_l \cup W_r$ is called a C_3*-scheme* of D, see Figure 1 for an example. The elements of W_l and W_r form the *left wing* and the *right wing* of this C_3-scheme, respectively, while B is the *base.* The middle element of S is the *anchor* of the scheme. A C_3-scheme is uniquely determined by its anchor. Of course, D may have cover-preserving C_3^2's that cannot be extended to C_3-schemes. For example, the slim semimodular diagrams in Figure 4 have cover-preserving C_3^2 sublattices but no C_3-schemes.

The concept of a C_2-scheme and the related terminology are analogous, see Figures 2 and 7 for two examples. The base of a C_2-scheme is a cover-preserving N_7, and its wings are in C_2-trajectories. The middle element of the base is again called the anchor, and it determines the C_2-scheme. Since C_2-trajectories always reach

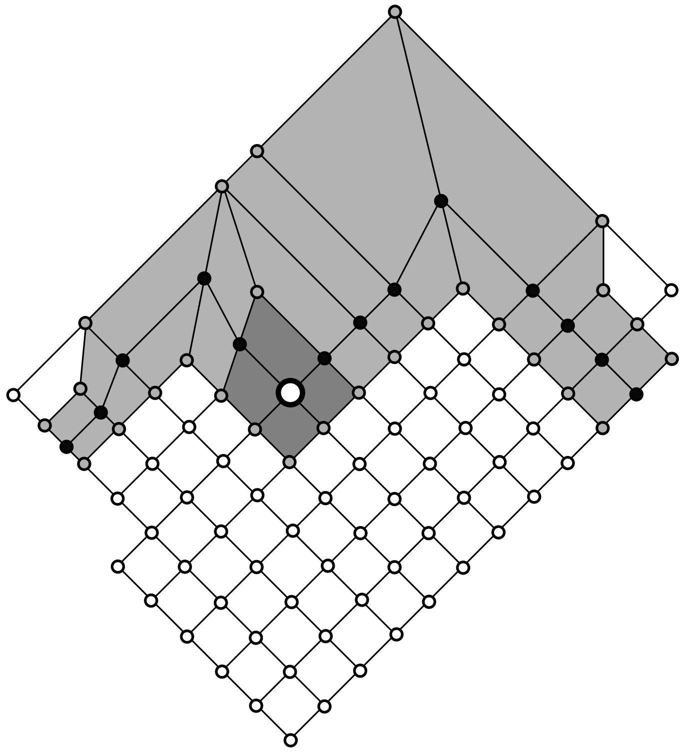

FIGURE 1. Resect this diagram at the element marked by the big circle by deleting the black-filled elements

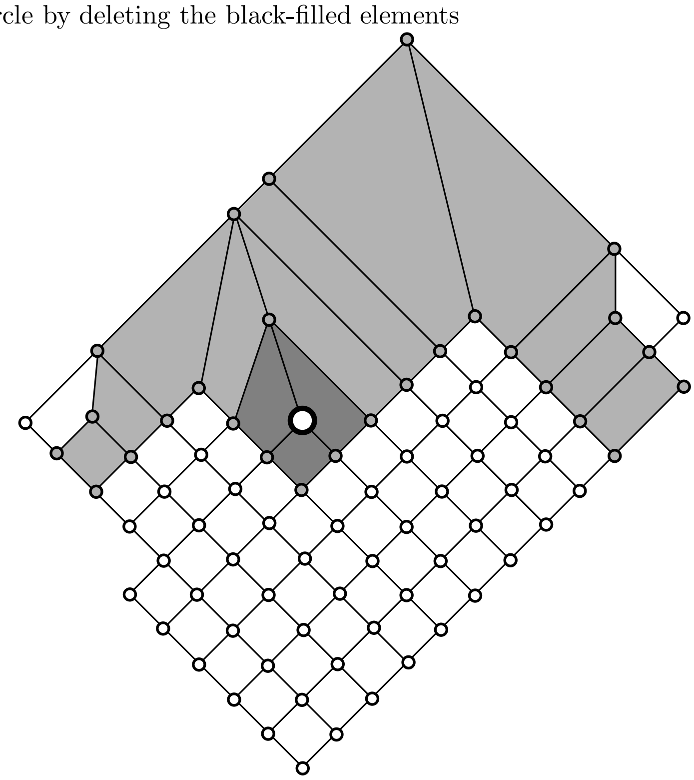

FIGURE 2. to obtain this diagram

the boundary of D, each cover-preserving N_7 sublattice is the base of a unique C_2-scheme.

For $i \in \{2, 3\}$ and a C_i-scheme S, we define the *upper boundary*, the *lower boundary*, and the *interior* of S as expected.

Let S be a C_3-scheme of a slim semimodular diagram D. By removing all the interior elements of S but its anchor, we obtain a new slim semimodular diagram, D', and S turns into a C_2-scheme of D'. We say that D' is obtained from D by a *resection*; this is illustrated in Figures 1 and 2. The reverse procedure, transforming a C_2-scheme to a C_3-scheme by adding new interior elements, is called an *insertion*.

3. The results

Following D. Kelly and I. Rival [18], we call two planar diagrams *similar* if there is a bijection φ between them such that φ preserves the left-right order of the upper covers and of the lower covers of an element. We are interested in diagrams only up to similarity.

A *grid* is a planar diagram of the form $\mathsf{C}_m \times \mathsf{C}_n$ for $m, n \geq 2$. We obtain a slim distributive diagram from a grid by a sequence of steps; each step omits a doubly irreducible element from a boundary chain. Our main result generalizes this to slim semimodular lattice diagrams.

Theorem 1. *Slim semimodular lattice diagrams are characterized as diagrams obtained from slim distributive lattice diagrams by a sequence of resections.*

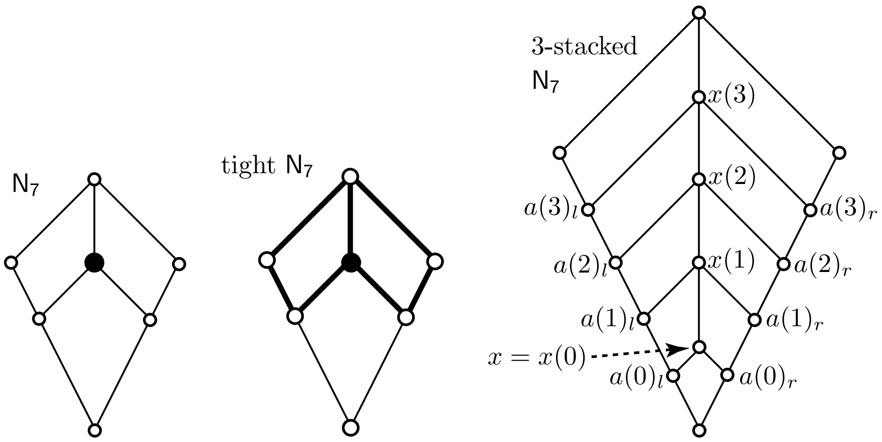

Figure 3. N_7 and its variants

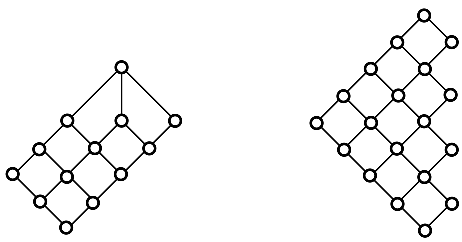

Figure 4. Some slim semimodular diagrams

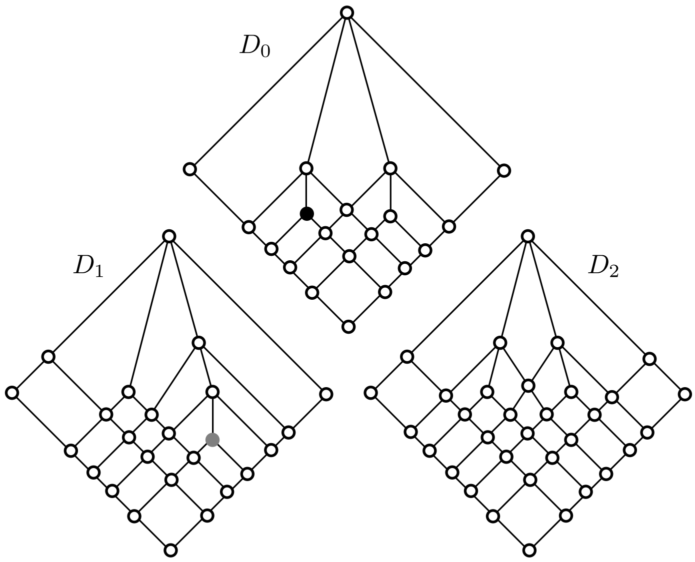

FIGURE 5. The process does not stop

The proof of this theorem now appears clear. Let D be a slim semimodular lattice diagram. Find in it a covering N_7 as in Figure 2. Perform an insertion to obtain the diagram of Figure 1. The diagram of Figure 1 has one fewer covering N_7. Proceed this way until we obtain a diagram without covering N_7-s.

Remark 2. The argument of the last paragraph does not necessarily work. Start with the first diagram in Figure 5. Apply an insertion at the black-filled element, to obtain the second diagram. Apply an insertion at the gray-filled element of the second diagram, to obtain the third diagram. And so on. It is clear that the number of covering N_7-s is not diminishing.

We define a *weak corner* of a planar semimodular diagram D as an element x on the boundary of D with the properties:

(i) x is doubly irreducible;
(ii) x is not comparable to some $y \in D$.

If x is a weak corner such that its lower cover, x_*, has exactly two covers and its upper cover, x^*, has exactly two lower covers, then we call x a *corner*. As defined in G. Grätzer and E. Knapp [14], a planar diagram (and the corresponding lattice) is *rectangular*, if it has exactly one left weak corner and exactly one right weak corner, and these two elements are complementary. Slim semimodular diagrams can be obtained from slim rectangular diagrams by removing corners, one-by-one. Moreover, only slim semimodular diagrams can be obtained this way. So we get:

Corollary 3. *Slim rectangular diagrams are characterized as diagrams obtained from grids by a sequence of resections.*

4. SCHEMES

Let D be a slim semimodular diagram. By G. Grätzer and E. Knapp [12] and G. Czédli and E. T. Schmidt [7, Lemma 2], an element of D has at most two covers.

We also know from [7, Lemma 6] that a join-irreducible element is on the boundary of D.

Let $a < b$ in a planar diagram D, and assume that C_1 and C_2 are maximal chains in the interval $[a, b]$ such that $C_1 - \{a, b\}$ is strictly on the left of C_2, $C_2 - \{a, b\}$ is strictly on the right of C_1, and $C_1 \cap C_2 = \{a, b\}$. Then, following D. Kelly and I. Rival [18], the intersection of the right of C_1 and the left of C_2 is called a *region* of D. A region of D is a planar subdiagram of D. Minimal regions are called *cells*, and cells with four vertices (and four edges) are 4-*cells*. For a slim semimodular diagram D,

(1) the 4-cells and the covering squares of D are the same.

Our proof relies heavily on the following two lemmas, see G. Grätzer and E. Knapp [12, Lemma 7] for the first and G. Grätzer and E. Knapp [12, Lemma 6] and G. Czédli and E. T. Schmidt [7, Lemma 15], for the second.

Lemma 4. *Let D be a planar lattice diagram. Then D is slim and semimodular iff its cells are 4-cells and no two distinct 4-cells have the same bottom.*

Lemma 5. *A slim semimodular diagram is distributive iff it has no cover-preserving N_7.*

Let $\mathrm{Anchor}_i(D)$ denote the set of anchors of C_i-schemes of D for $i \in \{2, 3\}$. The set of *interior elements* of D, that is, the set of those elements that are not on the boundary of D, is denoted by $\mathrm{Inter}(D)$. Clearly,

$$\mathrm{Anchor}_2(D) \subseteq \mathrm{Inter}(D) \cap \mathrm{Mi}\, D. \tag{2}$$

As in G. Grätzer and E. Knapp [13], an N_7 sublattice of D is a *tight* N_7 if the thick edges in the middle diagram of Figure 3 represent coverings. A tight N_7 sublattice is always determined by its *inner dual atom*, we call it the *centre of* N_7, see the black-filled element in the lattice in the middle of Figure 3.

Lemma 6. *Let D be a slim semimodular diagram and let $u \in \mathrm{Inter}(D) \cap \mathrm{Mi}\, D$. Then there exists a unique tight N_7 sublattice of D with u as the anchor. Moreover, if $[a_l, b_l]$, $[u, u^*]$, and $[a_r, b_r]$ are consecutive prime intervals of this sublattice, then this sublattice is $\{u, u^*, a_l, b_l, a_r, b_r, a_l \wedge a_r\}$. Conversely, the center of a tight N_7 sublattice always belongs to $\mathrm{Inter}(D) \cap \mathrm{Mi}\, D$.*

Proof. Assume that $u \in \mathrm{Inter}(D) \cap \mathrm{Mi}\, D$. Consider the C_2-trajectory T containing $[u, u^*]$. Since $[u, u^*]$ is not on the boundary of D, this trajectory makes at least one step to the right, to a prime interval $[a_r, b_r]$. This step is a down-perspectivity since $u \in \mathrm{Mi}\, D$. Similarly, T makes a down-perspective step to the left, to $[a_l, b_l]$. By D. Kelly and I. Rival [18, Lemma 1.2], we obtain easily that $b_l \wedge b_r \leq u$. Thus $b_l \wedge b_r = b_l \wedge u \wedge b_r \wedge u = a_l \wedge a_r$, and we conclude that $\{u, u^*, a_l, b_l, a_r, b_r, a_l \wedge a_r\}$ is a tight N_7 sublattice of D.

Observe that a tight N_7 sublattice with center u is determined by those covering squares (that is, cover-preserving C_2^2 sublattices) of this sublattice that contain $[u, u^*]$ as an upper prime interval. But these covering squares are 4-cells by (1), and $[u, u^*]$ is the upper edge of at most two 4-cells. Hence, apart from left-right symmetry, there is only one tight N_7 sublattice with center u, as described in the last paragraph. This proves the first two parts of the statement. The last part is trivial. □

As in G. Grätzer and E. Knapp [14], a *cover-preserving m-stacked* N_7 (*sublattice*) of D is a cover-preserving sublattice isomorphic to the $(7+3m)$-element diagram given, for $m=3$, on the right of Figure 3. A cover-preserving 0-stacked N_7 is a cover-preserving N_7.

Lemma 7. *Let R be a cover-preserving m-stacked N_7 sublattice of D. Then R is a region of D. Furthermore, if R' is a cover-preserving m-stacked N_7 sublattice of D such that $\operatorname{Inter}(R)\cap\operatorname{Inter}(R')\neq\varnothing$, then $R'=R$.*

Proof. Since R consists of adjacent covering squares, which are 4-cells by (1), it follows easily that R is a region. Let $x_0\prec\cdots\prec x_m$ be the interior of R. Assume that $t\in\operatorname{Inter}(R)$. Then $t=x_j$, and this j is recognized as follows: there is a sequence $t=t_0\succ\cdots\succ t_j$ such that

(a) t_j has only two lower covers;
(b) the t_i have three lower covers for $i\in\{1,\dots,j-1\}$;
(c) t_i is the middle lower cover of t_{i-1}, for $i\in\{1,\dots,j\}$.

It follows that t determines $\operatorname{Inter}(R)$, which clearly determines the whole R via adjacent 4-cells. Hence if $t\in\operatorname{Inter}(R)\cap\operatorname{Inter}(R')$, then $R=R'$. ☐

In view of Lemma 7, cover-preserving m-stacked N_7 sublattices of D are also called *m-stacked N_7 regions*. For a meet-irreducible element $x\in D$ in the interior of D define $x(0)=x$. If the meet-irreducible element $x(i)$ is already defined and $x(i)^*$

(a) is meet-irreducible,
(b) is in the interior of D,
(c) covers exactly three elements,

then define $x(i+1)=x(i)^*$. The *rank* of x, $\operatorname{rank}_D(x)$, is the largest m such that $x(m)$ is defined. By (2), each $x\in\operatorname{Anchor}_2(D)$ has a rank. For another description of $\operatorname{rank}_D(x)$, where $x\in\operatorname{Anchor}_2(D)$, see Corollary 9.

Lemma 8. *Let D be a slim semimodular diagram. Let $x\in\operatorname{Anchor}_2(D)$ and $\operatorname{rank}_D(x)=m$. Then the following statements hold:*

(i) *The element x has exactly two lower covers.*
(ii) *For $i\in\{0,\dots,m\}$, there exists a unique i-stacked N_7 region R_i of D such that $\operatorname{Inter}(R_i)=\{x=x(0)\prec\cdots\prec x(i)\}$.*
(iii) *The interior of the C_2-scheme anchored by x is $\operatorname{Inter}(R_m)$.*

Proof. (i) is trivial.

To prove (ii), let $H(i)$ denote the condition "there exists a unique i-stacked N_7 region R_i of D such that $\operatorname{Inter}(R_i)=\{x=x(0)\prec\cdots\prec x(i)\}$". Observe that x is the center of a cover-preserving N_7 sublattice R_0 by definition. It is a 0-stacked N_7 region. Since R_0 is also a tight N_7 sublattice, R_0 is uniquely determined by Lemma 6. This proves $H(0)$.

Next, let $1\le i\le m$ and assume that $H(i-1)$ holds. Since $x(i-1)$, the anchor of R_{i-1}, has only one cover, it follows that $x(i)$ is the top of R_{i-1}; for an illustration, see Figure 6. Since $x(i)$ is defined, it satisfies (a)–(c). Hence the lower covers of $x(i)$ in D are exactly the same as the dual atoms of R_{i-1}, namely, the left dual atom $a(i)_l$, the anchor $x(i-1)$, and the right dual atom $a(i)_r$ of R_{i-1}. Since $x(i)\in\operatorname{Inter}(D)\cap\operatorname{Mi}D$, the right wing starting from $[x(i),x(i)^*]$ has to make its

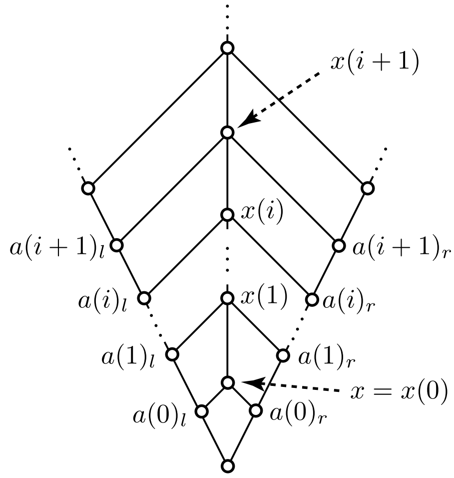

FIGURE 6. Illustrating the proof of Lemma 8

first step downwards to $[a(i)_r, a(i+1)_r]$, where $a(i+1)_r$ is a uniquely determined element of D because

$$\{a(i)_r, x(i), a(i+1)_r, x(i) \vee a(i+1)_r\}$$

is a 4-cell of D. By left-right symmetry, we also obtain a unique 4-cell

$$\{a(i)_l, x(i), a(i+1)_l, x(i) \vee a(i+1)_l\}.$$

Since

$$\{a(i)_l, a(i+1)_l, x(i), a(i)_r, a(i+1)_r\}$$

generates a (unique) tight N_7 sublattice by Lemma 6, it follows that

$$a(i+1)_l \wedge a(i+1)_r = a(i)_l \wedge a(i)_r.$$

This together with the fact that R_{i-1} is a cover-preserving $(i-1)$-stacked N_7 sublattice implies that

$$R_i = R_{i-1} \cup \{a(i+1)_l, x(i)^*, a(i+1)_r\}$$

is a cover-preserving i-stacked N_7 region with interior $\{x = x(0) \prec \cdots \prec x(i)\}$. The uniqueness of this region follows from Lemma 7. Hence $H(i)$ holds for all $i \in \{0, \ldots, m\}$, proving part (ii) of the lemma.

Finally, (iii) is obvious. □

Corollary 9. *Let D be a slim semimodular diagram, and let $x \in \mathrm{Anchor}_2(D)$. Then $\mathrm{rank}_D(x)$ is the largest number in the set*

$$\{\, k \mid x \text{ is the middle atom of a } k\text{-stacked } \mathsf{N}_7\text{-region}\}.$$

5. The proof of the main result

We start with a simple consequence of Lemma 4:

Lemma 10.

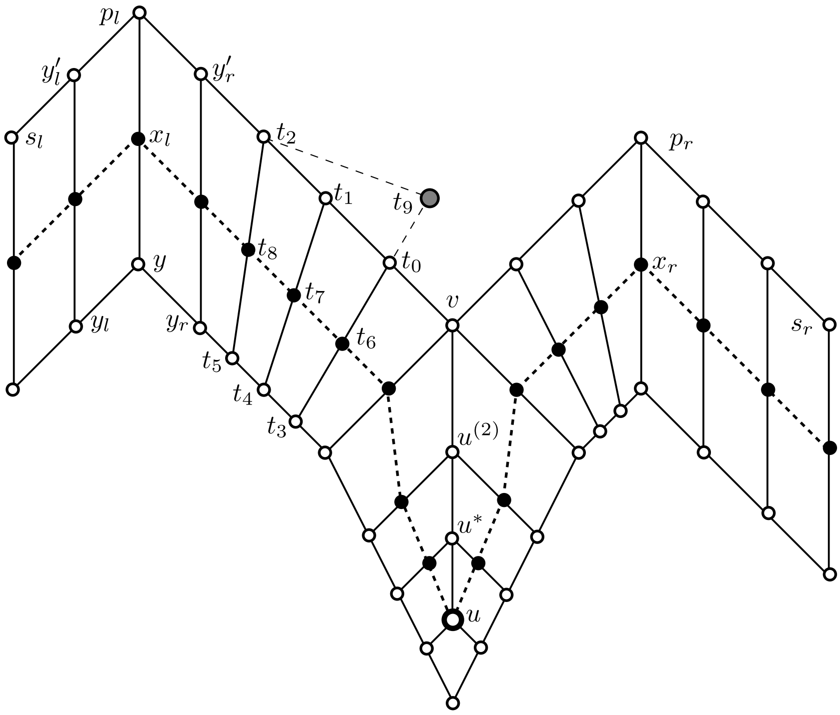

FIGURE 7. Insertion at u (t_9 plays a role only in Case 3)

(i) *Let D be a slim semimodular diagram and let D' be obtained from D by a resection at $u \in \mathrm{Anchor}_3(D)$. Then D' is also a slim semimodular diagram, $u \in \mathrm{Anchor}_2(D')$, and up to similarity, D is obtained from D' by an insertion at u.*

(ii) *Conversely, let D be a slim semimodular diagram and let D' be obtained from D by an insertion at $u \in \mathrm{Anchor}_2(D)$. Then D' is also a slim semimodular diagram, $u \in \mathrm{Anchor}_3(D')$, and up to similarity, D is obtained from D' by a resection at u.*

The following lemma is the major step in the proof of Theorem 1. Note that the inclusions in it are actually equalities, but we do not need—and do not prove—this.

Lemma 11. *Let D be a slim semimodular diagram and assume that $u \in \mathrm{Anchor}_2(D)$. Let D' denote the diagram obtained from D by performing an insertion at u.*

If $\mathrm{rank}_D(u) = 0$, then

$$\mathrm{Anchor}_2(D') \subseteq \mathrm{Anchor}_2(D) - \{u\}.$$

If $\mathrm{rank}_D(u) > 0$, then

$$\mathrm{Anchor}_2(D') \subseteq (\mathrm{Anchor}_2(D) - \{u\}) \cup \{u^*\},$$

and $\mathrm{rank}_{D'}(u^*) = \mathrm{rank}_D(u) - 1$.

Proof. Let S denote the C_2-scheme anchored by u. Let I be the order-ideal of D generated by the lower boundary of S and let F be the order filter generated by the upper boundary of S. Since $I \cap S$ is the lower boundary of S and $F \cap S$ is the upper boundary of S, planarity implies that, for all $x_1, x_2 \in D$,

$$\text{if } x_1 \prec x_2,\ x_2 \in F - S, \text{ and } x_1 \notin F - S, \text{ then } x_1 \in F \cap S. \tag{3}$$

By Lemma 8, u is the inner atom of a unique $\mathrm{rank}_D(u)$-stacked N_7 region, whose top we denote by v, see Figure 7. (Note that we utilize that t_9 exists and is placed in the diagram as shown only in Case 3; in general, t_9 is not in S, and it may not be in D.) Let p_l and p_r denote the top elements of the wings. Let s_l and s_r denote the largest elements of the wings on the boundary of D. It is possible that $s_l = p_l$, or $p_l = v$, or $s_r = p_r$, or $p_r = v$.

If x_l in Figure 7 (the element of $D' - D$ covered by p_l) belongs to $\mathrm{Inter}(D')$, then it has at least three lower covers in D'; similarly for x_r. All the other elements of $D' - D$ are meet-reducible. Thus $\mathrm{Anchor}_2(D') \subseteq D$. So we have to show that every element w of $D \cap \mathrm{Anchor}_2(D') = \mathrm{Anchor}_2(D')$ is in $\mathrm{Anchor}_2(D)$.

Since D can be partitioned into

$$I \cup (F - S) \cup (F \cap S) \cup (D - (I \cup F)),$$

the condition $w \in D$ splits into four cases as to which block in this partition w belongs to.

Case 1: $w \in I$. If $w \notin S$, then $w^* \in I \subseteq D$ by the dual of (3), the unique cover-preserving N_7 sublattice is in $I \subseteq D$, and $w \in \mathrm{Anchor}_2(D)$, as required by the lemma. Therefore, we can assume that w belongs to the lower boundary of S. Since $w \in \mathrm{Inter}(D') \cap \mathrm{Mi}\, D'$, it has to be where a wing (properly) turns down, $w = y$ in Figure 7 (or symmetrically, on the right). It has exactly two lower covers by Lemma 8. Thus these lower covers, y_l and y_r in Figure 7, also belong to the lower boundary of S. We use the notation y_l' and y_r' as in Figure 7. Lemma 6 yields that $\{y, p_l, y_l, y_l', y_r, y_r', y_l \wedge y_r\}$ is a tight N_7 sublattice of D. Since $y \in \mathrm{Anchor}_2(D')$ yields that $y_l \wedge y_r \prec y_l$ and $y_l \wedge y_r \prec y_r$, this tight N_7 sublattice is a cover-preserving N_7 sublattice. Hence $y \in \mathrm{Anchor}_2(D)$, as required.

Case 2: $w \in F - S$. The element w has exactly two lower covers, w_l and w_r, by Lemma 8. They belong to F, and we have that $w_l \wedge w_r \prec w_l$ and $w_l \wedge w_r \prec w_r$. If at least one of w_l and w_r does not belong to S (equivalently, to its upper boundary, $F \cap S$), then $w_l \wedge w_r \in F$ by (3), whence the cover-preserving N_7 sublattice determined by w belongs to F and $w \in \mathrm{Anchor}_2(D)$, as required. Hence we can assume that w_l and w_r are on the upper boundary of S but $w_l \wedge w_r \notin F$. Since $w_l \parallel w_r$, the only possibility, up to left-right symmetry, is that $w = w_l \vee w_r$ equals p_r. However, this case is excluded by Lemma 8 since p_r has at least three lower covers.

Case 3: $w \in F \cap S$. Let $w = t_1$ be on the upper boundary of S, as in Figure 7. Since it has only two lower covers by Lemma 8 and belongs to $\mathrm{Inter}(D')$, we conclude that $t_1 \notin \{v, p_l, p_r, s_l, s_r\}$. Hence there are elements t_0, t_2 in the upper boundary of S, in the same wing as t_1, such that $t_0 \prec t_1 \prec t_2$. We use the notation $t_4, \dots, t_8$ as in Figure 7. Consider the cover-preserving N_7 sublattice in D' that is anchored by t_1. Since this sublattice is also a tight N_7 sublattice in D', it is $\{t_0, t_1, t_2, t_6, t_7, t_8, t_9\}$ with rightmost dual atom t_9 by Lemma 6. Therefore, applying Lemma 6 again, the tight N_7 sublattice determined by t_1 in D is $\{t_0, \dots, t_5, t_9\}$. It is a cover-preserving N_7 sublattice since $t_3 \prec t_6$ and $t_3 \prec t_4$. Thus $t_1 \in \mathrm{Anchor}_2(D)$, as required.

Case 4: $w \in D - (I \cup F)$. Notice that $w \in \mathrm{Inter}(S)$. By Lemma 8, w belongs to the interior of the unique m-stacked N_7 region R_m with centre u, where $m =$

$\mathrm{rank}_D(u)$. Assume first that $m = 0$. Then $w = u$, whence it does not belong to $\mathrm{Anchor}_2(D')$ since it has two upper covers in D'. Secondly, assume that $m > 0$. Then, clearly again, $u \notin \mathrm{Anchor}_2(D')$. Moreover, of the other elements in the interior of S, that is, in

$$\mathrm{Inter}(S) = \mathrm{Inter}(R_m) = \{\, u^{(i)} \mid 1 \le i \le m \,\},$$

the element $u^* = u^{(1)}$ is the only one in $\mathrm{Anchor}_2(D')$. □

Proof of Theorem 1. Let D be a diagram. If it is obtained from a slim distributive diagram by a sequence of restrictions, then D is a slim semimodular diagram by Lemma 10. Conversely, assume that D is a slim semimodular diagram. By virtue of Lemma 10, it suffices to show that we can obtain a slim distributive diagram from D by a finite sequence of insertions. That is, we want a finite sequence $D = D_0, D_1, \dots$ of diagrams such that D_{i+1} is obtained from D_i by an insertion, and the last member of the sequence is distributive. If D_i is distributive, then it is the last member of the sequence, and we are ready. If it is non-distributive, then $\mathrm{Anchor}_2(D_i)$ is non-empty by Lemma 5. Pick an element $u_i \in \mathrm{Anchor}_2(D_i)$ such that $\mathrm{rank}_{D_i}(u_i)$ is the smallest member of $\{\, \mathrm{rank}_{D_i}(x) \mid x \in \mathrm{Anchor}_2(D_i) \,\}$, and perform an insertion at u_i to obtain D_{i+1}. This procedure terminates in finitely many steps by Lemma 11. □

Proof of Corollary 3. If we perform an insertion to obtain D' from D, then the weak corners of D' are the same as those of D, $0_{D'} = 0_D$, and $1_{D'} = 1_D$. Hence our statement follows from G. Grätzer and E. Knapp [14, Lemma 6] and the argument used in the proof of Theorem 1. □

There are some efficient ways to check whether a planar diagram is a slim semimodular lattice diagram; in addition to Lemma 4, see [7, Theorems 11 and 12]. The following test follows trivially from the proof of Theorem 1.

Lemma 12. *Let D be a planar diagram. Construct the sequence*

$$D = D_0, D_1, \dots$$

as in the proof of Theorem 1. Then D is a slim semimodular lattice iff the sequence terminates with a planar distributive lattice.

Remark 2 points out that the clause "as in the proof of Theorem 1" in Lemma 12 cannot be dropped.

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