Lattice tolerances and congruences

Gábor Czédli and George Grätzer

Abstract. We prove that a tolerance relation of a lattice is a homomorphic image of a congruence relation.

Let K and L be lattices. Let $\varphi\colon K \to L$ be a homomorphism of K onto L. Let $\boldsymbol{\theta}$ be a congruence relation on K. Then we can define a binary relation $\varphi(\boldsymbol{\theta})$ in the obvious way:

$$\varphi(\boldsymbol{\theta}) = \bigl\{ \bigl(\varphi(x), \varphi(y)\bigr) \mid (x,y) \in \boldsymbol{\theta} \bigr\}.$$

It belongs to the folklore (see, for instance, E. Fried and G. Grätzer [3] or G. Grätzer [5]), that $\varphi(\boldsymbol{\theta})$ is a tolerance relation, that is, $\varphi(\boldsymbol{\theta})$ is a binary relation on L with the following properties: reflexivity, symmetry, and the Substitution Properties.

We prove the converse.

Theorem 1. *Let $\boldsymbol{\rho}$ be a tolerance relation of a lattice L. Then there exists a lattice K, a congruence $\boldsymbol{\theta}$ of K, and a lattice homomorphism $\varphi\colon K \to L$ such that $\boldsymbol{\rho} = \varphi(\boldsymbol{\theta})$.*

Proof. By a *block* of $\boldsymbol{\rho}$ we mean a maximal subset X of L such that $X^2 \subseteq \boldsymbol{\rho}$. According to [1], the set of all blocks of $\boldsymbol{\rho}$ form a lattice $L/\boldsymbol{\rho}$ such that, for $A, B \in S$, the join $A \vee B$ is the *unique* block of $\boldsymbol{\rho}$ that includes the set

$$\{\, a \vee b \mid a \in A,\ b \in B \,\},$$

and similarly for the meet. This allows us to define a *lattice* K:

$$\{\, (A, x) \mid A \in L/\boldsymbol{\rho},\ x \in A \,\},$$

with the operations

$$(A,x) \vee (B,y) = (A \vee B, x \vee y),$$

and dually. Then

$$\boldsymbol{\theta} = \bigl\{ \bigl((A,x),(B,y)\bigr) \mid A = B \bigr\}$$

is a congruence on K. Clearly, $(A,x) \mapsto x$ defines a homomorphism φ of K onto L. From Zorn's Lemma, we infer that $(x,y) \in \boldsymbol{\rho}$ iff $\{x,y\} \subseteq A$, for some $A \in L/\boldsymbol{\rho}$. Hence $\varphi(\boldsymbol{\theta}) = \boldsymbol{\rho}$. $\square$

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Note that the lattice K in the proof is the *sum* of the lattices A, for $A \in L/\boldsymbol{\rho}$, in the sense of E. Graczyńska and G. Grätzer [4]; for the finite case this is quite easy to see via G. Czédli [2].

For a second look at Theorem 1, let φ be a homomorphism from the lattice L onto the lattice K, with congruence kernel $\boldsymbol{\gamma}$, so $K \equiv L/\boldsymbol{\gamma}$. Let $\boldsymbol{\alpha}$ be a congruence on L. We define a binary relation $\boldsymbol{\rho}$ on K as follows: $x \equiv y\ (\boldsymbol{\rho})$ if there are elements $u, v \in L$ such that $x \equiv y\ (\boldsymbol{\alpha})$ and $\varphi(x) = u$, $\varphi(y) = v$. We use the notation: $\boldsymbol{\rho} = \boldsymbol{\alpha}/\boldsymbol{\gamma}$.

Now we rephrase Theorem 1 with this notation.

Theorem 2. *Let K and L be lattices and let $K = L/\boldsymbol{\gamma}$. Then for every congruence relation $\boldsymbol{\alpha}$ of L, the relation $\boldsymbol{\alpha}/\boldsymbol{\gamma}$ is a tolerance on K.*

Conversely, for a lattice K and a tolerance $\boldsymbol{\tau}$ of K, there is a lattice L, congruences $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ on L, and a lattice isomorphism $\varphi\colon L/\boldsymbol{\gamma} \to K$ such that $\boldsymbol{\tau} = \varphi(\boldsymbol{\alpha}/\boldsymbol{\gamma})$.

References

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Gábor Czédli
University of Szeged, Bolyai Institute, Szeged, Aradi vértanúk tere 1, Hungary 6720
e-mail: czedli@math.u-szeged.hu
URL: http://www.math.u-szeged.hu/∼czedli/

George Grätzer
Department of Mathematics, University of Manitoba, Winnipeg, MB R3T 2N2, Canada
e-mail: gratzer@me.com
URL: http://server.math.umanitoba.ca/homepages/gratzer/